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Book Review

Richard Kaye. *Models of Peano Arithmetic.* Oxford Logic Guides, Oxford University Press, Oxford. 1991.

Our vocabulary lacks a term to denote a person whose calling is the study of models of arithmetic. Model theorists, topologists, even functional analysts can identify themselves succinctly, but we have to resort to such locutions as "I'm in models of arithmetic". And the name of the field itself—"models of arithmetic"—also seems to bespeak an insecurity about whether it is a field at all: the objects of study are baldly named without any pretensions to a grand theory or -ology.

Models of arithmetic certainly is a bona fide field. It has its own meetings, folklore, and stars. It has built up a coherent body of knowledge relevant to some of the central problems of modern logic. But it has never sat comfortably within the traditional fourfold division of logic, it is sparsely populated and has been known to lie dormant for decades, and it has never had a "bible". Access to this difficult terrain has been daunting to outsiders.

Under these circumstances the appearance of this volume by a leading young modelofarithmetist is a significant event, perhaps marking an overdue coming-of-age of the field. With no previous books on the subject (except conference proceedings), Kaye has bravely stepped into the void, setting himself the difficult task of writing a comprehensible introduction to the subject while including a good amount of material of more advanced interest. He has achieved this admirably well. Although space limitations preclude an encyclopedic work, Kaye presents all the mainstream material straightforwardly and interestingly, and he goes on to reach far into the depths of the subject in certain directions. He finishes with an excellent survey of further reading. Throughout he provides stimulating and challenging exercises.

Kaye's no-frills approach is reflected in the overall plan of the book, which is clear and uncluttered. I shall sketch what he does and offer minor comments.

After presenting the background material that the reader should know—this book will be accessible to a first-year graduate student—Kaye begins with a brief look at the standard model of arithmetic: this is a nice touch. He then introduces the base theory PA^- (essentially the theory of discretely ordered rings), and gives a traditional enough proof of the first incompleteness theorem for PA^- (with the second sketched as an exercise). There are a few reasons why I would not have covered incompleteness in this way: this kind of thing is of course widely available and probably known to anyone who has covered the background in model theory and recursion theory. Since the remainder of the book will concentrate mainly on PA rather than the weaker base theory, it would have made more sense to wait to present incompleteness in a way that is directly "arithmetizable" and stress the aspects that are most relevant to nonstandard models of PA. In fact Kaye does delay the proof of Gödel's lemma until two chapters later for reasons something like these.

After introducing PA (first order Peano arithmetic), Kaye shows that it is robust under minor variations of axiomatization and extensions of the language. He provides illustrations of working in PA by developing a little number theory: coding sequences, exponentiation, the infinitude of primes. (I think it would have been illuminating to include at some point an elementary discussion of how bounded induction seems to be insufficient to capture these arguments. Likewise perhaps an exercise on the power of $I\Sigma_1$.)

The model theory proper begins with the fundamental overspill lemma, the order-type of a nonstandard model, cofinal extensions (Gaifman's theorem), prime models, and end-extensions (the MacDowell–Specker theorem). Then comes the chapter that, as Kaye says, "no one wanted to have to write": the details of the arithmetization of syntax and semantics. The trouble is that probably no one will want to have to read it either. Although Kaye has wisely delayed it until unavoidable, I think he could have made it easier on the sedulous reader by presenting just enough to give the spirit of the enterprise and then relying on some intuitive comments to motivate the rest, with details left as exercises or appendices. (Isn't that how this stuff has been learned traditionally? Only a few dusty dissertations have all the details.) It would be a pity if any readers stumble away here, their eyes glazed over, and never reach the rich payoffs that follow. An alternative might be to motivate the material by testing its limits, as in the Paris–Wilkie work on bounded induction.

Next comes a presentation of two of the basic model-theoretic techniques for PA, used here to sort out the infinite hierarchy of subsystems of PA. Then the coded sets; Σ_n-recursive saturation, a powerful tool for proving souped-up overspill results, with a nice application to Tennenbaum's theorem that nonstandard models of PA are not recursive (with the traditional proof thrown in for good measure); and Friedman's embedding theorem, a fundamental structural result.

These ideas are presented clearly and live up to Kaye's aim of avoiding fancy footwork in order to make the book accessible. But I'd have liked to see the words "back-and-forth construction" mentioned in the proof of Friedman's theorem; this is another of several places in the book where the presentation is rather dry and an occasional intuitive remark might have illuminated the technical passages. While on the subject, I'll mention some other such places: the discussion of expansions of the language (pp. 47–50); the proof of the embeddability of prime models (Theorem 8.2); and Lemma 15.1.

Returning to the scheme of the book: Scott sets are studied, in a wider model theoretic context, and lead very naturally to the arithmetized completeness theorem (though the statement of the latter seems unnecessarily messy). A chapter on indicators culminates in the Paris–Harrington incompleteness theorem, with the Kanamori–McAloon version integrated into the proof. I have a couple of quibbles: the proof of Theorem 14.7, which gives the existence of indicators for recursively axiomatized theories, calls on both Friedman's theorem and the Wilmers machinery of $\mathfrak{A}$-saturation. It would have been relatively easy to give a more self-contained proof. And the proof of the crucial lemma in the Paris-Harrington theorem (Lemma 14.12) is slick rather than revealing. It is one of those combinatorial arguments that seem to work by magic. The sophisticated reader will appreciate its elegance, but a more prosaic argument might have brought out better the links with the structure of models of arithmetic.

The final area covered, in the longest chapter, is recursive saturation. I am glad that Kaye chose, as the most advanced excursion in his book, the results of Lachlan, Kotlarski, and Krajewski on satisfaction classes. But I am at a loss to know why he covers this demanding material before going into some of the basic results on recursive saturation. A feeling for the special nature of recursively saturated models—their homogeneity, their quasi-uniqueness (for countable models, up to complete theory and standard system)—should surely be part of the groundwork that would motivate the advanced stuff. And the material on resplendency, given here in a very general context, is available elsewhere: something more specific to models of arithmetic might have been a better use of space.

One more grouch of a general nature: Kaye is haphazard with his historical remarks and attributions, only occasionally including them. It is true that these are unusually difficult for a subject whose history is often obscure and folklore-laden. But that is all the more reason why this ground-breaking book, likely to become a standard reference, has a responsibility to give credit where it is known to be due, whether for something as well-known as the theory Q (Exercise 2.14), or as arcane as the definition of a tall model (Exercise 11.7), or the construction of recursively saturated ω_1-like models (Exercise 15.6).

Typographical and minor editing errors are refreshingly sparse. In only one place did they throw me off balance: Exercise 15.14.

Let me stress again that the complaints I offer above are minor ones. Kaye's book is highly recommended to beginners and old hands alike as a solid, workmanlike introduction to the field that covers an amazing amount of ground and can hold the interest of experts. It fills a gap and fills it well.

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