442
Notre Dame Journal of Formal Logic
Volume 33, Number 3, Summer 1992

Real Reduced Models for Relevant Logics without WI

STEVE GIAMBRONE

Abstract Slaney has provided reduced models (ones in which there is but one "real" world) for a number of relevant logics via certain kinds of frames, as opposed to the conventional Routley-Meyer model structures. This paper does three things: it corrects Slaney's paper, extends his results in a different direction, and draws a moral from the errors it corrects.

The corrections to Slaney's paper are very minor, the errors having been more in the nature of "slips" than of outright mistakes. The semantic structures of Slaney's paper are criticized for not being "semantical enough". It is then shown that Slaney's basic results can be used to provide reduced models for most of the same logics (the system **E** being a notable exception) using the Routley-Meyer model structures which do not suffer from this defect. The basic slip in the original paper was not to close the worlds of the canonical models of some of the systems under *all* of the primitive rules of inference of that system. The paper ends with a brief discussion of the philosophical significance of insisting that theories (worlds) be closed under certain rules of inference as well as under provable implication. That discussion insists upon the importance of a distinction between primitive/derivable rules of inference and merely admissible rules along the lines of Anderson and Belnap.

1 Introduction Slaney [8] discusses the motivationally important matter of reduced modeling for relevant logics, duly notes that many important weaker relevant logics have not been provided with reduced modeling, and goes on to offer such for them in terms of frames, as opposed to the conventional model structures of Routley and Meyer [5],[6] for instance. In addition to their motivational importance, reduced models are technically and practically important for the practicing logician. They are simpler and hence easier to use. However we find [8] lacking in some very important respects.

In the first place, there are some minor (i.e., easily fixed) technical inaccuracies in the paper. Some of the claims made therein are false as stated, and some

Received June 24, 1991; revised September 12, 1991

of the proffered proofs are unsound. So in what follows we will revise the claims of [8] as necessary and at least indicate how the incorrect proofs can be made right. (It is very unlikely that the author of [8] was actually mistaken on any of these points. The errors corrected herein were surely just "slips".)

But our discontent with [8] runs deeper: the models of [8] are not in general *really* models—in a very clear sense, they are not semantical enough! As the reader will see when we get around to introducing the technical definitions, for a given logic **L** an **L**-frame is *defined* in terms of the crucial syntactic notions of 'theorem of **L**' and 'primitive rule of **L**'. Now there may be those who are not purists in these matters and whose sensibilities are not offended by such wanton self-indulgence. We ourselves are very open-minded on this point and would not care to judge them to be moral reprobates. However, it is our duty to warn them that they will pay for their sins immediately in this case. For this illicit conjoining of syntax and semantics begets here a degenerate offspring: frames so defined are useless for many of the tasks to which logicians are wont to appoint them, e.g., proving that a given formula (neither already known to be a theorem nor known to be a nontheorem of **L**) is valid or invalid as the case may be.[1]

It is not *seriously* being suggested that the results of [8] are unworthy. The extension of metavaluation techniques given there is particularly admirable. But such upright works should be put at the service of holier tasks. So in addition to correcting [8], we will extend it. In particular, we prove that the basic results of [8] can be used to show that almost all of the logics treated there are characterized by *real* reduced models, i.e., ones based on the traditional Routley-Meyer relational model structures. So without further ado, the technical details.

2 Syntactic and semantic preliminaries The systems to be considered can be fitted with Hilbert-style axiomatizations from the following list of axioms and rules (the side notation designates the corresponding semantic postulate as listed further below):

(A1)	$A \to A$	p1
(A2)	$A \mathbin{\&} B \to A$	---
(A3)	$A \mathbin{\&} B \to B$	---
(A4)	$(A \to B) \mathbin{\&} (A \to C) \to .A \to B \mathbin{\&} C$	p2.i
(A5)	$A \to A \vee B$	---
(A6)	$B \to A \vee B$	---
(A7)	$(A \to C) \mathbin{\&} (B \to C) \to .A \vee B \to C$	p2.i
(A8)	$A \mathbin{\&} (B \vee C) \to (A \mathbin{\&} B) \vee C$	---
(A9)	$\neg\neg A \to A$	p4
(A10)	$A \to \neg B \to .B \to \neg A$	p7
(A11)	$A \to B \to .C \to A \to .C \to B$	p6.i
(A12)	$A \to B \to .B \to C \to .A \to C$	p6.ii
(A13)	$(A \to B) \mathbin{\&} (B \to C) \to .A \to C$	p9
(A14)	$A \to .A \to A$	p12
(A15)	$A \to .B \to A$	p13
(A16)	$(A \to B \vee C) \mathbin{\&} (A \mathbin{\&} B \to C) \to .A \to C$	?
(A17)	$A \to .A \to B \to B$	p10
(A18)	$(A \to .B \to C) \to .B \to .A \to C$	p11

(A19)	$(A \to B)\ \&\ A \to B$	p8
(R1)	$\vdash A \to B, \vdash A \Rightarrow \vdash B$	p1 (viz., R000)
(R2)	$\vdash A, \vdash B \Rightarrow \vdash A\ \&\ B$	---
(R3)	$\vdash A \to B, \vdash C \to D \Rightarrow \vdash B \to C \to .A \to D$	p2.i (and p1)
(R4)	$\vdash A \to \neg B \Rightarrow \vdash B \to \neg A$	p4
(R5)	$\vdash A \Rightarrow \vdash B \to A$	p14
(R6)	$\vdash A \Rightarrow \vdash A \to B \to B$	p5.

Some of the better known systems to be dealt with here include:

B	(A1) through (A9) + (R1) through (R4)
DW	**B** + (A10)
TW	**DW** + (A11) + (A12)
DJ	**DW** + (A13)
TJ	**TW** + (A13)
EW	**TW** + (R6)
RW	**TW** + (A17) or TW + (A18)
R	**RW** + (A19)
BCK	**RW** + (A15) or RW + (R5)
RM	**R** + (A14).

An *M1 logic* is any system axiomatizable as **B** plus zero or more of (A10) through (A16) with or without (R5). An *M2 logic* is one axiomatizable as **B** plus (R6) plus any of (A10), (A11), (A12), (A14), (A15), (A17), (A18), (R5). Where **L** is a logic containing **B**, an *L-theory T* is a set of sentences in the language of **L** closed under adjunction (R2) and provable **L**-implication. For **L**-theories a and b, $ab = \{C: B \to C$ is in a for some B in $b\}$. It is *regular* iff it contains **L**. It is *consistent* iff it is not the case that some wff and its negation are both elements of it; *prime* iff it contains at least one disjunct of any disjunction in it; *ordinary* iff regular and closed under (R1), (R3), and (R4); *normal* iff ordinary, prime, and consistent; and *A-consistent* iff it does not contain A. We add that it is *really regular* iff regular and closed under the primitive rules of **L**, and *really normal* iff really regular, prime and consistent.

A *frame* is a structure $\langle K, P, O, R, * \rangle$ where K is a set, P is a subset of K, O a member of P, R a ternary relation on K, and $*$ a unary operation on K, such that for all members a, b, c, d of K:

(D1) $a \leq b =_{df} Rxab$ for some x in P
(D2) $R^2abcd =_{df} Rabx$ and $Rxcd$ for some x in K
(D3) $R^2a(bc)d =_{df} Rbcx$ and $Raxd$ for some x in K

(p1)	$a \leq a$	(p1)
(p2)	$\forall x \in P$, if (i) R^2xabc or (ii) $R^2x(ab)c$ or (iii) $R^2a(xb)c$ then $Rabc$	(p2),(p6) (p7)[2]
(p3)	$a** = a$	(p3)
(p4)	If $a \leq b$ then $b* \leq a*$.	(p4)

The notation of [6] (chapter 4) for each postulate, where such exists, is given to its right.

With SL the set of sentential letters, an *f*-model is a pair $\langle F, v \rangle$, where F is a frame and v is a function from $SL \times K$ into {T,F} such that for all p in SL and all a, b in K, $(v(p, a) = \mathrm{T}$ and $a \leq b) \Rightarrow v(p, b) = \mathrm{T}$. As usual, each model determines an interpretation **I** satisfying:

Ip $\mathbf{I}(p, a) = v(p, a)$ for all p in SL;
I$\neg$ $\mathbf{I}(A, a) = \mathrm{T}$ iff $\mathbf{I}(A, a*) = \mathrm{F}$;
I& $\mathbf{I}(A \;\&\; B, a) = \mathrm{T}$ iff $\mathbf{I}(A, a) = \mathbf{I}(B, a) = \mathrm{T}$;
I$\vee$ $\mathbf{I}(A \vee B, a) = \mathrm{T}$ iff either $\mathbf{I}(A, a) = \mathrm{T}$ or $\mathbf{I}(B, a) = \mathrm{T}$;
I$\rightarrow$ $\mathbf{I}(A \rightarrow B, a) = \mathrm{T}$ iff for all b, c in K, if $\mathbf{I}(A, b) = \mathrm{T}$ and $Rabc$ then $\mathbf{I}(B, c) = \mathrm{T}$.

A is *verified* by $\langle F, v \rangle$ iff $\mathbf{I}(A, O) = \mathrm{T}$. *A holds strongly* in an f-model just in case it is true at every member of P in that model. With **L** a logic as before, an *L-frame* is a frame F such that in every f-model of F: (i) every theorem of **L** holds strongly; and (ii) every primitive rule of **L** preserves strong holding. A frame is *reduced* iff $P = \{O\}$.

A *B model structure* (a B m.s.) is just a frame as defined above. Then where **L** is an extension of B via one or more of the above axioms or rules, an **L** m.s. is a B m.s. satisfying the further appropriate postulate(s) from those given below. (Which postulate(s) are appropriate is indicated along with the axioms and rules above.) Also notice that we know of no appropriate postulate for (A16).

(p5)	For some x in P, $Raxa$ (dr1)	
(p6)	If R^2abcd, then (i) $R^2a(bc)d$	(q4)
	and (ii) $R^2b(ac)d$	(q3)
(p7)	If $Rabc$, then $Rac{*}b{*}$	(s4)
(p8)	$Raaa$	(q1)
(p9)	If $Rabc$, then $R^2a(ab)c$	(q2)
(p10)	If $Rabc$, then $Rbac$	(q6)
(p11)	If R^2abcd, then R^2acbd	(q7)
(p12)	If $Rabc$, then $a \leq c$ or $b \leq c$	(q18)
(p13)	If $Rabc$, then $a \leq c$	(q11)
(p14)	For all x in P, if $Rxbc$, then $x \leq c$.	---

A model structure is *reduced* iff P $= \{O\}$. A *valuation* and its associated *interpretation* are defined as above. An **L**-*model* is an **L** m.s. with an interpretation. *True in a model* is defined in the usual way.

3 Real reduced models With all of these technical details in place, we proceed to criticize, correct, and expand [8] as promised. Unless otherwise indicated, page numbers below refer to [8]. Since we have no real modeling condition for (A16), all claims about **L** m.s. and real models thereon are to be understood to exclude logics having it as an axiom.

Observation 1 (p. 398) is false for any **L** which has (R5) or (R6) as a primitive rule but does not contain (A15) or (A17), respectively, as a theorem. (**EW** is conspicuous among these.) In such a case, T need not and will not in general be closed under all the primitive rules, and hence $\langle P_T, \{T\}, T, R_T, *_T \rangle$ will fail the second condition of the definition of an **L** frame. So we repair and expand the claim as follows:

Observation 1′ *Let T be a prime, really regular* **L**-*theory.* $\langle P_T, \{T\}, T, R_T, *_T \rangle$ *is an* **L**-*frame and* (*for* **L** *not containing* (A16)) *an* **L** *m.s., where* $P_T = \{$*all prime T theories*$\}$, $R_T abc$ *iff* $ab \subseteq c$, $a*_T = \{A : \neg A$ *is not in* $a\}$.

Proof: As in [8], see chapter 4 of [6] for most of the logics. Since (R5) and its corresponding postulate p14 are not considered there, we add the following. (Note that for such systems, **L**- and T-theories must be nonempty. Again, see [6].) It will suffice to show that if $b \leq c$, then $T \subseteq c$. But T is a subset of a, for any a in P_T, as we now show. Assume A is in T. Choose B in c. Since T is really regular, $B \rightarrow A$ is in T. Whence by the definition of an **L**-theory, A is in T.

Observation 2 (p. 398) is now correct as stated:

Observation 2 *There is an interpretation* **I** *on* $\langle P_T, \{T\}, T, R_T, *_T \rangle$ *such that* $\mathbf{I}(A, a) = \mathrm{T}$ *for all wff* A *and all* a *in* P_T.

Observation 3 (p. 399) is true as stated, though its proof was unsound since Observation 1 was false. But it and Observation 4 (p. 399) are irrelevant in any event. Observation 5 (p. 400) is also true as stated but is less than what is needed when **L** has (R5) or (R6) without the corresponding theorem. The reason for this is that a really regular A-consistent **L**-theory rather than a merely ordinary A-consistent **L**-theory is required once Lemmas 5 and 6 are corrected to 5′ and 6′ below, as they must be. So we beef up Observation 5 to:

Observation 5′ *For any A, a nontheorem of* **L**, *there is an A-consistent, really regular* **L**-*theory.*

Proof: More or less as before. Simply redefine *immediate consequence* in **L** (p. 399) as: The conclusion of an instance of a primitive rule of **L** is an *immediate consequence* of the premises of that instance. Then a *derivation* of formula B from set X in logic **L** is a finite sequence of formulas, the last of which is B and each of which is either a theorem of **L**, a member of X, or an immediate consequence of one or more earlier members of the sequence. Then with T defined as on p. 399, it is straightforward to show that it is an A-consistent, really regular **L**-theory.

Lemmas 5 and 6 (pp. 402 and 405) are true as stated, but again are not up to their task when **L** has (R5) or (R6) as a primitive rule but lacks the corresponding theorem. Though [8] claims to show that m_1T (m_2T) (these are defined by Slaney's extension of metavaluations in [7]) is closed under the appropriate rule in those cases, the proof is unsound. For there is no guarantee that $C \rightarrow B$ is in T ($B \rightarrow C \rightarrow C$ is in T) as required, even though B is in m_1T (m_2T) and hence an element of T itself. Again we beef them up to:

Lemma 5′ *Where* **L** *is an M1 logic and T is a really regular* **L**-*theory,* m_1T *is a really normal* **L**-*theory.*

Lemma 6′ *Where* **L** *is an M2 logic and T is a really regular* **L**-*theory,* m_2T *is a really normal* **L**-*theory.*

Proofs: The original proofs now work. For good measure we show m_1T closed under (R5) when appropriate. Assume B is in m_1T. By Lemma 1, B is in T; hence so is $C \rightarrow B$, since T is really regular and thus closed under (R5) in this case. Using classical reasoning, we see that if C is in m_1T, then B is, by truth of consequent. And by Lemma 2 and the initial assumption, B is in $m_1 * T$. So again by truth of consequent, if C is in m_1T, so is B.

Of course the normalization theorems (pp. 404–405) must also be strengthened.

M1 Real Normalization Theorem *Let* **L** *be any M1 logic and let A be a non-theorem of* **L**. *Then there is a really normal, A-consistent* **L**-*theory.*

Proof: Observation 5′, Lemma 1 (p. 402) and Lemma 5′.

M2 Real Normalization Theorem *Let* **L** *be any M2 logic and let A be a non-theorem of* **L**. *Then there is a really normal A-consistent* **L**-*theory.*

Proof: Observation 5′, Lemma 1 (p. 402) and Lemma 6′.

Reduced models (and frames) are finally in hand.

M1 Reduced Model Theorem *Every M1 logic formulated without* (A16) *is characterized by its reduced relational models. Further, all M1 logics are characterized by their reduced frames.*

Proof: Observation 1′, Observation 2 and the M1 Real Normalization Theorem.

M2 Reduced Model Theorem *Every M2 logic is characterized by its reduced relational models and by its reduced frames.*

Proof: Observation 1′, Observation 2, and the M2 Real Normalization Theorem.

4 Conclusion and philosophical remarks A few comments are in order before closing. The reader will note that the results of [8] really do hold more or less as stated, and that the new results are completely in line with the original ones. In particular, the real worlds of our "real" models are large in that appropriate if somewhat obscure sense of [8]: there really are "models looking like reality, in which there are many truths beyond what is given by Pure reason".

And our use of really regular theories deserves remark. Formally, of course, they are just what was required. But is there also some underlying philosophical point to be seen? Anderson and Belnap [1] insist that there is an important distinction to be made between primitive/derivable rules of inference and those that are merely admissible. (Belnap [2] refers to them as object and meta rules.) The former mark valid inference just as well as a true implication, if I take their point aright. But the latter are simply facts about a particular set of statements, namely the supposed truths of logic. Given that 'A entails B' is supposed to mean 'there is a valid deduction (inference) of B from A', the insistence of [1] and [2] that a system of entailment should have theorems "corresponding" in some appropriate way to every derivable rule seems reasonable.

[6] takes issue with this criterion (pp. 256–257) and seems to doubt that there is a viable and/or worthwhile distinction to be made. However, it is worth noting that the theory of **L**-theories originally set forth in [5] and used in [6] makes a *de facto* distinction between two different kinds of rules. This distinction seems to agree with Belnap's, at least in spirit, though the authors do not even note it much less afford it any philosophical weight. As far as the motivational story of [5] goes, there is no requirement that an **L**-theory contain all of the theorems

of **L** (i.e., be regular), only that it be closed under **L**-implication: the physicist need not assert the truths of logic, but she must reason properly. It is Logic's job to govern our reasoning, not to tell us what is true. Accordingly, only the "logical" worlds (members of *P*) are required to be regular. But in [5], and for the stronger logics in general, all theories are in fact required to be closed under the real (derivable) rules. (Note the insistence, by definition no less, that all **L**-theories be closed under adjunction (R2).) And only some theories, if any, will be closed under the admissible rules of **L**. Note particularly that even the logical worlds need not be and are not in general closed under those rules.

This is as one would have expected if (i) the distinction between derivable and merely admissible rules is genuine and (ii) the semantics is to give us some sort of interpretation of the syntactic formalism. But the situation is much less clear for the weaker systems dealt with here. For not all **L**-theories must be closed under the primitive rules of **L**, not even under modus ponens (R1). Does this indicate that these rules are merely admissible from the point of view of **L**, and hence that the systems in question are poorly formulated? Maybe so. But notice that in the reduced models the logical world must be really regular and hence closed under at least the primitive rules. Does this indicate that there is some intermediate kind of rule? We think not.

We suggest (at least tentatively) that this apparent confusion is simply the fault of the systems themselves. They are not good candidates for a system of genuine entailment. (R5) and (R6) are not valid rules of inference, and no logic worth its salt will claim that they are. A full discussion of (R3) would be long and complicated, and hence is best left to another time. Suffice it to say here that we think that conjunctive transitivity (A13) is a truth about entailment. With respect to (R4), we think that it should be dropped, though we are still of two minds over adding its theorem form (A10). And as far as modus ponens (R1) goes, the solution is simple. Although we have and will continue to advocate contractionless relevant logics, i.e., ones lacking $(A \rightarrow .A \rightarrow B) \rightarrow .A \rightarrow B$ (**W**)), conjunctive modus ponens (**WI**, i.e., A19) is also a truth about entailment, and any adequate logic will say so.[3] So the lack of respect for the distinction reflects a basic fault of (at least some of) the contractionless systems.

Finally, after all of our somewhat disparaging remarks about frames, it is only fair to emphasize that they go beyond the relational model structures that we favor here. As noted above, (A16) escapes our semantic net so far. Further, even though **EW** behaves well for reduced relational modeling, **E** does not—so far as anyone knows. But, as [8] notes (p. 399), **E** is characterized by reduced models defined in terms of frames. We confess to not being particularly fond of **E** and think it a rather odd system. From this point of view, the fact that its real models are messy is neither surprising nor displeasing. Still, it was originally presented as the centerpiece of relevant logics, so it is Idealogically, if not Philosophically, Good that it should answer to reduced models of some sort.[4]

NOTES

1. Of course the models of [8] really are models in the straightforward and useful sense of formal models that we all employ. Moreover, the reduced models of [8] are iden-

tical to the reduced models of the relational semantics given below. But the point still holds against relying on syntactic notions in *defining* semantic structures—and would continue to hold even if the whole class of **L**-frames should turn out to be identical to the class of **L** model structures when **L** is not **E**.

2. As [6] notes, this postulate is stronger than required for completeness. But as noted in Giambrone [3] and [4], the additional strength is already there in the canonical models and seems to be needed for the normalization argument for systems such as *T*.
3. This is a devastating fact for **RW**, since **W** and **WI** are equivalent therein. But this is not so for better systems such as **TW**.
4. We (this author *and* the reader) owe a debt to the referee for several corrections and suggestions for clarification of this paper.

REFERENCES

[1] Anderson, A.R. and N.D. Belnap Jr., *Entailment*, vol. 1, Princeton University Press, Princeton, 1975.

[2] Belnap, N.D. Jr., *The Formalization of Entailment*, Yale University Doctoral Dissertation, New Haven, 1959.

[3] Giambrone, S., *Gentzen Systems and Decision Procedures for Relevant Logics*, Australian National University Doctoral Dissertation, Canberra, 1982.

[4] Giambrone, S. and R.K. Meyer, "Completeness and conservative extension results for some Boolean relevant logics," *Studia Logica*, vol. 48 (1989), pp. 1–14.

[5] Routley, F.R. and R.K. Meyer, "Semantics of entailment I," pp. 199–243 in *Truth, Syntax and Modality*, edited by H. Leblanc, North-Holland, Amsterdam, 1973.

[6] Routley, F.R. and R.K. Meyer, V. Plumwood, R.T. Brady, *Relevant Logics And Their Rivals*, vol. 1, Ridgeview, Atascadero, 1983.

[7] Slaney, J., "A metacompleteness theorem for contraction-free relevant logics," *Studia Logica*, vol. 43 (1985), pp. 159–168.

[8] Slaney, J., "Reduced models for relevant logics without WI," *Notre Dame Journal of Formal Logic*, vol. 28 (1987), pp. 395–407.

Université des Acadiens
University of Southwestern Louisiana
P.O. Box 42531
Lafayette, Louisiana 70504-2531