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A Variable-Free Logic for Mass Terms

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Abstract This paper presents a logic appropriate for mass terms, that is, a logic that does not presuppose interpretation in discrete models. Models may range from atomistic to atomless. This logic is a generalization of the author's work on natural language reasoning. The following claims are made for this logic. First, absence of variables makes it simpler than more conventional formalizations based on predicate logic. Second, the capability to deal effectively with discrete terms, and in particular with singular terms, can be added to the logic, making it possible to reason about discrete entities and mass entities in a uniform manner. Third, this logic is similar to surface English in that the formal language and English are "well-translatable," making it particularly suitable for natural language applications. Fourth, deduction performed in this logic is similar to syllogistic and therefore captures an essential characteristic of human reasoning.

1 Introduction This paper presents a logic appropriate for mass terms, that is, a logic that does not presuppose interpretation in discrete models. Models may range from atomistic to atomless. This logic is a generalization of the logic for reasoning in natural language presented in Purdy [5]. It is also related, in its objectives, to the generalization of first order logic defined by Roeper [8].

Claims made for this logic are the following. First, the absence of variables makes it simpler than more conventional predicate logics such as [8]. Second, the capability to deal effectively with discrete terms, and in particular with singular terms, can be added to the logic, making it possible to reason about discrete entities and mass entities in a uniform manner. Third, this logic is similar to surface English, in that the formal language and English are "well-translatable" (see Čulík [3]), making it particularly suitable for natural language applications. Fourth, deduction performed in this logic is similar to syllogistic, and it therefore captures an essential characteristic of human reasoning.

The first claim is supported by the body of this paper. The definition of the language, its semantics, its axiomatization, and the proofs of soundness and com-

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pleteness are simpler and more straightforward than the more conventional formulation given in [8]. Support for the second claim can be found in Section 4. The third and fourth claims are essentially those made for the discrete version of this logic. Support for these claims can be found in Purdy [5] and [6]. No claims are made for solving the many linguistic and philosophical problems related to mass terms.

2 Definition of the language The language described in this section is the same as $\mathcal{L}_N$, presented in [5], but without singular predicates. The semantics of $\mathcal{L}_N$ is suitably generalized to permit nonatomic interpretations.

2.1 Syntax The alphabet of $\mathcal{L}_N$ consists of the following. (Define $\omega_+ := \omega - \{0\}$.)

1. Predicate symbols $\mathcal{R} = \bigcup_{j \in \omega_+} \mathcal{R}_j$, where $\mathcal{R}_j = \{R_i^j : i \in \omega\}$.
2. Selection operators $\{\langle k_1, \ldots, k_n \rangle : n \in \omega_+, k_i \in \omega_+,\ 1 \leq i \leq n\}$.
3. Boolean operators $\cap$ and $^-$.
4. Parentheses (and).

$\mathcal{L}_N$ is partitioned into sets of n-ary expressions for $n \in \omega$. These sets are defined to be the smallest satisfying the following conditions.

1. For each $n \in \omega_+$, each $R_i^n \in \mathcal{R}_n$ is an n-ary expression.
2. For each $m \in \omega_+$, for each $R_i^m \in \mathcal{R}_m$, $\langle k_1, \ldots, k_m \rangle R_i^m$ is an n-ary expression where $n = \max(k_i)_{1 \leq i \leq m}$.
3. If X is an n-ary expression then $\overline{(X)}$ is an n-ary expression.
4. If X is an m-ary expression and Y is an l-ary expression then $(X \cap Y)$ is an n-ary expression where $n = \max(l, m)$.
5. If X is a unary expression and Y is an $(n + 1)$-ary expression then (XY) is an n-ary expression.

In the sequel, superscripts and parentheses are dropped whenever no confusion can result. Metavariables are used as follows: R^n ranges over $\mathcal{R}_n$; R ranges over $\mathcal{R}_1$; X, Y, Z, W, V range over $\mathcal{L}_N$; and X^n, Y^n, Z^n, W^n, V^n range over n-ary expressions of $\mathcal{L}_N$. Applying subscripts to these symbols does not change their ranges.

2.2 Semantics An *interpretation* of $\mathcal{L}_N$ is a pair $\mathcal{I} = \langle \mathcal{A}, \mathcal{F} \rangle$ where $\mathcal{A} = \langle A, \subseteq \rangle$ is a nonempty set partially ordered by inclusion, possibly having a least element 0, and $\mathcal{F}$ is a mapping defined on $\mathcal{R}$. For each $R^n \in \mathcal{R}_n$, $\mathcal{F}(R^n) \subseteq A^n$ and satisfies:

1. if $\langle a_1, \ldots, a_n \rangle \in \mathcal{F}(R^n)$, then $\langle a_1, \ldots, a_n \rangle$ is a nonzero element of A^n;
2. if $\langle a_1, \ldots, a_n \rangle \in \mathcal{F}(R^n)$ then for all nonzero $\langle b_1, \ldots, b_n \rangle \subseteq \langle a_1, \ldots, a_n \rangle : \langle b_1, \ldots, b_n \rangle \in \mathcal{F}(R^n)$;
3. if for all $\langle b_1, \ldots, b_n \rangle \subseteq \langle a_1, \ldots, a_n \rangle$, there exists $\langle c_1, \ldots, c_n \rangle \subseteq \langle b_1, \ldots, b_n \rangle$ such that $\langle c_1, \ldots, c_n \rangle \in \mathcal{F}(R^n)$, then $\langle a_1, \ldots, a_n \rangle \in \mathcal{F}(R^n)$.

Here $\langle a_1, \ldots, a_n \rangle$ is nonzero $:\Leftrightarrow$ for $1 \leq i \leq n : a_i \neq 0$, and $\langle b_1, \ldots, b_n \rangle \subseteq \langle a_1, \ldots, a_n \rangle :\Leftrightarrow$ for $1 \leq i \leq n : b_i \subseteq a_i$.

If $\alpha = \langle a_1, a_2, \dots \rangle \in A^\omega$, then α is nonzero if for all $n \in \omega_+$, $\langle a_1, \dots, a_n \rangle$ is nonzero. If $\beta = \langle b_1, b_2, \dots \rangle \in A^\omega$ also, then $\beta \subseteq \alpha$ if for all $n \in \omega_+$, $\langle b_1, \dots, b_n \rangle \subseteq \langle a_1, \dots, a_n \rangle$.

Let $\alpha = \langle a_1, a_2, \dots \rangle \in A^\omega$ be a sequence of elements of A. Then $X \in \mathcal{L}_N$ *is satisfied by* α *in* $\mathcal{I}$ (written $\mathcal{I} \models_\alpha X$) iff α is nonzero and one of the following holds:

1. $X = R^n$ and $\langle a_1, \dots, a_n \rangle \in \mathcal{T}(X)$;
2. $X = \langle k_1, \dots, k_m \rangle R^m$ and for all nonzero $\beta \subseteq \alpha$, there exists nonzero $\gamma \subseteq \beta : \langle c_{k_1}, \dots, c_{k_m} \rangle \models_\gamma R^m$;
3. $X = \bar{Y}$ and for all nonzero $\beta \subseteq \alpha : \mathcal{I} \not\models_\beta Y$;
4. $X = Y \cap Z$ and $\mathcal{I} \models_\alpha Y$ and $\mathcal{I} \models_\alpha Z$;
5. $X = Y^1 Z^{n+1}$ and for some nonzero $a \in A : \langle a \rangle \models_\alpha Y^1$ and $\langle a \rangle \models_\alpha Z^{n+1}$.

Here $\mathcal{I} \not\models_\alpha Y$ is an abbreviation for not ($\mathcal{I} \models_\alpha Y$), and $\langle b_1, \dots, b_n \rangle \models_\alpha Y$ (or $\langle b_1, \dots, b_n \rangle \models Y$ when there is no possibility of confusion) is an abbrevation for $\mathcal{I} \models_{\langle b_1, \dots, b_n, a_1, a_2, \dots \rangle} Y$.

X is true in $\mathcal{I}$ (written $\mathcal{I} \models X$) iff $\mathcal{I} \models_\alpha X$ for every nonzero $\alpha \in A^\omega$. *X is valid* (written $\models X$) iff X is true in every interpretation of $\mathcal{L}_N$. A 0-ary expression of $\mathcal{L}_N$ is called a *sentence*. A set Γ of sentences *is satisfied in* $\mathcal{I}$ iff each $X \in \Gamma$ is true in $\mathcal{I}$.

The intuitive notion is that $X \in \mathcal{L}_N$ is satisfied by α in $\mathcal{I}$ if and only if $\langle a_1, \dots, a_n \rangle$ is nonzero and is *included in* the denotation of X. This notion implies certain properties of mass terms, which in turn motivate the semantics. First, if $\mathcal{I} \models_\alpha X$ then $\mathcal{I} \models_\beta X$ for any nonzero $\beta \subseteq \alpha$. Second, whereas it is possible that $\mathcal{I} \models_\alpha X$ and $\mathcal{I} \not\models_\alpha \bar{X}$, or that $\mathcal{I} \models_\alpha \bar{X}$ and $\mathcal{I} \not\models_\alpha X$, or that $\mathcal{I} \not\models_\alpha X$ and $\mathcal{I} \not\models_\alpha \bar{X}$, it is not possible that $\mathcal{I} \models_\alpha X$ and $\mathcal{I} \models_\alpha \bar{X}$. Third, $\mathcal{I} \models_\alpha X$ iff $\mathcal{I} \models_\alpha \bar{\bar{X}}$.

The first property is known as the *distributive* property of mass terms (see Bunt [2] and Roeper [7]). It is imposed on basic expressions by the second restriction on the denotation function $\mathcal{T}$. The first and second properties together motivate the definition of satisfaction for $X = \bar{Y}$. As a consequence of the definition of satisfaction for $X = \bar{Y}$, $\mathcal{I} \models_\alpha \bar{\bar{X}}$ iff $\forall\beta \subseteq \alpha : \mathcal{I} \not\models_\beta \bar{X}$ iff $\forall\beta \subseteq \alpha : \exists\gamma \subseteq \beta : \mathcal{I} \models_\gamma X$. This together with the third property motivates the so-called *cumulative* property of mass terms (see [2],[7]) which is assured for basic expressions by the third restriction on the denotation function $\mathcal{T}$. For more on mass terms, see [2] and [7]. Roeper [7] gives a clear and concise presentation of the necessary background for a logic of mass terms. Bunt [2] provides a comprehensive review of the lingusitic and philosophical issues as well as a logic of mass terms.

The following lemma and corollary establish the distributive, cumulative, and complement properties in the general case.

Lemma 1 (schema) (*i*) *If* $\mathcal{I} \models_\alpha X$ *then* $\forall$ *nonzero* $\beta \subseteq \alpha : \mathcal{I} \models_\beta X$; (*ii*) *if* $\forall\beta \subseteq \alpha : \exists\gamma \subseteq \beta : \mathcal{I} \models_\gamma X$ *then* $\mathcal{I} \models_\alpha X$.

Proof: Proof is by induction on the structure of X. The basis follows directly from the definition of satisfaction and the definition of $\mathcal{T}$. The induction step involves four cases.

Case 1. $X = \langle k_1, \dots, k_m \rangle R^m$. (i) and (ii) follow directly from the definition of satisfaction (2) and the transitivity of inclusion.

Case 2. $X = \bar{Y}$. (i) follows directly from the definition of satisfaction (3) and the transitivity of inclusion. (ii) $\forall\beta \subseteq \alpha : \exists\gamma \subseteq \beta : \mathcal{I} \vDash_\gamma X$ implies $\forall\beta \subseteq \alpha : \exists\gamma \subseteq \beta : \forall\delta \subseteq \gamma : \mathcal{I} \nvDash_\delta Y$ (definition of satisfaction (3)). Now suppose $\mathcal{I} \nvDash_\alpha X$. This implies $\exists$ nonzero $\beta' \subseteq \alpha : \mathcal{I} \vDash_{\beta'} Y$ (definition of satisfaction (3)), which implies $\exists$ nonzero $\beta' \subseteq \alpha : \forall$ nonzero $\gamma' \subseteq \beta' : \mathcal{I} \vDash_{\gamma'} Y$ (induction hypothesis (i)), i.e., $\neg(\forall$ nonzero $\beta \subseteq \alpha : \exists$ nonzero $\gamma \subseteq \beta : \mathcal{I} \nvDash_\gamma Y)$. But this contradicts the preceding result for $\delta = \gamma$. Hence $\mathcal{I} \vDash_\alpha X$.

Case 3. $X = Y \cap Z$. (i) and (ii) follow directly from the definition of satisfaction (4) and the induction hypothesis.

Case 4. $X = YZ$. (i) follows directly from the definition of satisfaction (5) and the induction hypothesis. (ii) $\forall\beta \subseteq \alpha : \exists\gamma \subseteq \beta : \mathcal{I} \vDash_\gamma X$ implies $\forall\beta \subseteq \alpha : \exists\gamma \subseteq \beta : \exists c \in A : \langle c\rangle \vDash_\gamma Y$ and $\langle c\rangle \vDash_\gamma Z$ (definition of satisfaction (5)). This implies $\forall\beta \subseteq \alpha : \exists\gamma \subseteq \beta : \exists c \in A : \forall$ nonzero $\delta \subseteq \gamma \cdot \forall$ nonzero $d \subseteq c : \langle d\rangle \vDash_\delta Y$ and $\langle d\rangle \vDash_\delta Z$ (induction hypothesis (i)). Hence $\forall\beta \subseteq \alpha : \forall d \subseteq c : \exists\delta \subseteq \beta : \exists d \subseteq d : \langle d\rangle \vDash_\delta Y$ and $\langle d\rangle \vDash_\delta Z$. This implies $\langle c\rangle \vDash_\alpha Z$ (induction hypothesis (ii)), which implies $\mathcal{I} \vDash_\alpha YZ$ (definition of satisfaction (5)).

Corollary 2 (schema) $\quad \mathcal{I} \vDash_\alpha \bar{\bar{X}}$ *iff* $\mathcal{I} \vDash_\alpha X$.

2.3 A Boolean structure The semantics of the previous subsection defines a Boolean structure for $\mathcal{L}_N$. Use of this structure simplifies the soundness argument to be presented in the next section. Define $|X| := \{\alpha : \mathcal{I} \vDash_\alpha X\}$. Then $|X \cap Y| = |X| \sqcap |Y|$, where $\sqcap$ is set intersection. Further define $|X|^* := \{\alpha : \forall\beta \subseteq \alpha(\mathcal{I} \nvDash_\beta X)\}$. Then $|X|^* = |\bar{X}|$. Now let $\mathbf{L}$ be the image of $\mathcal{L}_N$ under $|\cdot|$. It is straightforward to verify that $\mathbf{L}$ is a pseudocomplemented meet-semilattice with lower bound $\varnothing$. It follows from lattice theory (see Gratzer [4], Thm. I.6.4) that $S(\mathbf{L}) = \{|X|^* : |X| \in \mathbf{L}\}$, the so-called "skeleton" of $\mathbf{L}$, is a Boolean lattice with meet $\sqcap$, complement *, and join $\sqcup$, defined $|X| \sqcup |Y| := (|X|^* \sqcap |Y|^*)^*$. But by Corollary 2, $|\bar{\bar{X}}| = |X|$. Hence $|X|^{**} = |X|$ and so $S(\mathbf{L}) = \mathbf{L}$. Thus $\mathbf{L}$ is itself a Boolean lattice.

The following abbreviations in $\mathcal{L}_N$ are motivated by this Boolean structure.

1. $X \cup Y := \overline{(\bar{X} \cap \bar{Y})}$
2. $X \subseteq Y := \overline{X \cap \bar{Y}}$
3. $X \equiv Y := (X \subseteq Y) \cap (Y \subseteq X)$
4. $T := (R_0^1 \subseteq R_0^1)$.

The situation can be summarized as follows. $\mathbf{L}$ is a Boolean lattice with meet $\sqcap$ such that $|X| \sqcap |Y| = |X \cap Y|$, complement * such that $|X|^* = |\bar{X}|$, join $\sqcup$ such that $|X| \sqcup |Y| = |X \cup Y|$, bounds $|T|$ and $|\bar{T}|$, and ordered by inclusion such that $|X| \subseteq |Y|$ iff $|X \subseteq Y| = |T|$. The expression XY has the Boolean property: $|XY| = |\bar{T}|$ iff $|X| \subseteq |Y|^*$. It follows immediately that:

1. $\forall\alpha : \mathcal{I} \vDash_\alpha X \subseteq Y$ iff $\forall\alpha : (\mathcal{I} \vDash_\alpha X$ implies $\mathcal{I} \vDash_\alpha Y)$
2. $\forall\alpha : \mathcal{I} \vDash_\alpha X \equiv Y$ iff $\forall\alpha : (\mathcal{I} \vDash_\alpha X$ iff $\mathcal{I} \vDash_\alpha Y)$
3. $\forall\alpha : \mathcal{I} \vDash_\alpha \overline{XY}$ iff $\forall\alpha : \mathcal{I} \vDash_\alpha X \subseteq \bar{Y}$.

2.4 Additional abbreviations The following abbreviations are introduced to improve readability.

1. $\wedge X^1 Y := \overline{X^1 \overline{Y}}$
2. $X_n X_{n-1} \cdots X_1 Y := (X_n(X_{n-1} \cdots (X_1 Y) \cdots)$
3. $X^1 Y_n^2 \circ Y_{n-1}^2 \circ \cdots \circ Y_1^2 := (\cdots (X^1 Y_n^2) Y_{n-1}^2) \cdots Y_1^2)$
4. $\breve{R}^n := \langle n, \ldots, 1 \rangle R^n$.

Using the previously stated results for **L**, it is easy to see that:

1. $\mathcal{I} \models X_n \cdots X_1 Y^n$ iff for some $\langle d_1, \ldots, d_n \rangle \in A^n : \langle d_1 \rangle \models X_1$ and $\cdots$ and $\langle d_n \rangle \models X_n$ and $\langle d_1, \ldots, d_n \rangle \models Y^n$
2. $\mathcal{I} \models \wedge X_n \cdots \wedge X_1 Y^n$ iff for all $\langle d_1, \ldots, d_n \rangle \in A^n : (\langle d_1 \rangle \models X_1$ and $\cdots$ and $\langle d_n \rangle \models X_n)$ implies $\langle d_1, \ldots, d_n \rangle \models Y^n$
3. $\mathcal{I} \models X_2 X_1 Y_n^2 \circ \cdots \circ Y_1^2$ iff for some $\langle d_0, d_1, \ldots, d_n \rangle \in A^{n+1} : \langle d_1, d_0 \rangle \models Y_1^2$ and $\langle d_2, d_1 \rangle \models Y_2^2$ and $\cdots$ and $\langle d_n, d_{n-1} \rangle \models Y_n^2$ and $\langle d_n \rangle \models X_1$ and $\langle d_0 \rangle \models X_2$.

Intuitively then, ZXY^2 renders "some X is Y to some Z"; $\wedge Z \wedge XY^2$ renders "all X is Y to all Z"; and $ZXY_2^2 \circ Y_1^2$ renders "some X is Y_2^2 composed with Y_1^2 to some Z".

3 Axiomatization of $\mathcal{L}_N$ The *universal closure* of an n-ary expression $X \in \mathcal{L}_N$ is defined to be the nullary expression $(\wedge T)^n X$. The axiom schemas of $\mathcal{L}_N$ are the following:

BT *The universal closure of every schema that can be obtained from a tautologous Boolean wff by uniform substitution of metavariables of $\mathcal{L}_N$ for sentential variables, $\cap$ for $\wedge$, and $^-$ for $\neg$.*

C1 $X_n \cdots X_1 \langle k_1, \ldots, k_m \rangle R^m \subseteq X_{k_m} \cdots X_{k_1} R^m$ *where* $n = \max(k_j)_{1 \le j \le m}$.

C2 $X_n \cdots X_1 \overline{\langle k_1, \ldots, k_m \rangle R^m} \subseteq X_{k_m} \cdots X_{k_1} \overline{R^m}$ *where* $n = \max(k_j)_{1 \le j \le m}$.

EG $(ZT \cap \wedge ZX \cap \wedge X_n \cdots \wedge X_1 \wedge ZY^{n+1}) \subseteq \wedge X_n \cdots \wedge X_1 XY^{n+1}$.

SS $(\wedge ZX \cap \wedge X_n \cdots \wedge X_1 \wedge XY^{n+1}) \subseteq \wedge X_n \cdots \wedge X_1 \wedge ZY^{n+1}$.

D1 $(X_j T \cap \cdots \cap X_n T \cap \wedge X_n \cdots \wedge X_1 (Y^m \cap Z^1)) \subseteq (\wedge X_m \cdots \wedge X_1 Y^m \cap \wedge X_l \cdots \wedge X_1 Z^l)$ *where* $n = \max(l, m)$ and $j = \min(l, m) + 1$.

D2 $(\wedge X_m \cdots \wedge X_1 Y^m \cap \wedge X_l \cdots \wedge X_1 Z^l) \subseteq \wedge X_n \cdots \wedge X_1 (Y^m \cap Z^l)$ *where* $n = \max(l, m)$.

N $\wedge X_n \cdots \wedge X_1 \overline{Y^n} \equiv \overline{X_n \cdots X_1 Y^n}$.

The inference rules of $\mathcal{L}_N$ are the following.

MP From X^0 and $X^0 \subseteq Y^0$ infer Y^0

EI From $\overline{(V^0 \cap RT \cap \wedge RX \cap X_n \cdots X_1 \wedge RY^{n+1})}$, where $R \in \mathcal{R}_1$ does not occur in $X, X_1, \ldots, X_n$, Y^{n+1}, or V^0, infer $\overline{(V^0 \cap X_n \cdots X_1 XY^{n+1})}$.

The restriction imposed on the unary predicate R by inference rule EI is abbreviated by the phrase *R is fresh*.

The set $\mathcal{T}$ of theorems of $\mathcal{L}_N$ is the smallest set containing the axioms and closed under MP and EI.

Theorem 3 (Soundness) $X \in \mathcal{T}$ *only if* $\models X$.

Proof: It suffices to prove that the axioms are valid and that validity is preserved by the inference rules. Proofs will be given for Axioms C2 and D1 and Inference Rule EI. The others are similar.

(i) Axiom C2 is valid.

$\mathcal{I} \models X_n \cdots X_1\overline{\langle k_1, \ldots, k_m\rangle R^m}$ iff $\exists\langle d_1, \ldots, d_n\rangle \in A^n: (\langle d_1\rangle \models X_1 \wedge \cdots \wedge \langle d_n\rangle \models X_n) \wedge \langle d_1, \ldots, d_n\rangle \models \overline{\langle k_1, \ldots, k_m\rangle R^m}$ (Section 2.4) iff $\exists\langle d_1, \ldots, d_n\rangle \in A^n: (\langle d_1\rangle \models X_1 \wedge \cdots \wedge \langle d_n\rangle \models X_n) \wedge \forall\langle e_1, \ldots, e_n\rangle \subseteq \langle d_1, \ldots, d_n\rangle : \langle e_1, \ldots, e_n\rangle \not\models \langle k_1, \ldots, k_m\rangle R^m$ (definition of satisfaction (3)) iff $\exists\langle d_1, \ldots, d_n\rangle \in A^n: (\langle d_1\rangle \models X_1 \wedge \cdots \wedge \langle d_n\rangle \models X_n) \wedge \forall\langle e_1, \ldots, e_n\rangle \subseteq \langle d_1, \ldots, d_n\rangle : \exists$ nonzero $\langle f_1, \ldots, f_n\rangle \subseteq \langle e_1, \ldots, e_n\rangle : \forall$ nonzero $\langle g_1, \ldots, g_n\rangle \subseteq \langle f_1, \ldots, f_n\rangle : \langle g_{k_1}, \ldots, g_{k_m}\rangle \not\models R^m$ (definition of satisfaction (2)) implies $\exists\langle d_{k_1}, \ldots, d_{k_m}\rangle \in A^m: (\langle d_{k_1}\rangle \models X_{k_1} \wedge \cdots \wedge \langle d_{k_m}\rangle \models X_{k_m}) \wedge \forall\langle e_{k_1}, \ldots, e_{k_m}\rangle \subseteq \langle d_{k_1}, \ldots, d_{k_m}\rangle : \exists$ nonzero $\langle f_{k_1}, \ldots, f_{k_m}\rangle \subseteq \langle e_{k_1}, \ldots, e_{k_m}\rangle : \langle f_{k_1}, \ldots, f_{k_m}\rangle \models \overline{R^m}$ (definition of satisfaction (3)) implies $\exists\langle d_{k_1}, \ldots, d_{k_m}\rangle \in A^m: (\langle d_{k_1}\rangle \models X_{k_1} \wedge \cdots \wedge \langle d_{k_m}\rangle \models X_{k_m}) \wedge \langle d_{k_1}, \ldots, d_{k_m}\rangle \models \overline{R^m}$ (Lemma 1) iff $\mathcal{I} \models X_{k_m} \cdots X_{k_1}\overline{R^m}$ (Section 2.4). Thus $\mathcal{I} \models X_n \cdots X_1\overline{\langle k_1, \ldots, k_m\rangle R^m}$ implies $\mathcal{I} \models X_{k_m} \cdots X_{k_1}\overline{R^m}$ whence by Section 2.3, $\mathcal{I} \models X_n \cdots X_1\langle k_1, \ldots, k_m\rangle R^m \subseteq X_{k_m} \cdots X_{k_1}\overline{R^m}$. Since $\mathcal{I}$ is arbitrary, Axiom C2 is valid.

(ii) Axiom D1 is valid.

$\mathcal{I} \models X_jT \cap \cdots \cap X_nT \cap \wedge X_n \cdots \wedge X_1(Y^m \cap Z^l)$ iff $\mathcal{I} \models X_jT \wedge \cdots \wedge \mathcal{I} \models X_nT \wedge \mathcal{I} \models \wedge X_n \cdots \wedge X_1(Y^m \cap Z^l)$ (definition of satisfaction (4)). $\mathcal{I} \models \wedge X_n \cdots \wedge X_1(Y^m \cap Z^l)$ iff $\forall\langle d_1, \ldots, d_n\rangle \in A^n: (\langle d_1\rangle \models X_1 \wedge \cdots \wedge \langle d_n\rangle \models X_n) \rightarrow \langle d_1, \ldots, d_n\rangle \models Y^m \cap Z^l$ (Section 2.4) iff $\forall\langle d_1, \ldots, d_n\rangle \in A^n: (\langle d_1\rangle \models X_1 \wedge \cdots \wedge \langle d_n\rangle \models X_n) \rightarrow (\langle d_1, \ldots, d_m\rangle \models Y^m \wedge \langle d_1, \ldots, d_l\rangle \models Z^l)$ (definition of satisfaction (4)), which implies $(\forall\langle d_1, \ldots, d_m\rangle \in A^m: (\langle d_1\rangle \models X_1 \wedge \cdots \wedge \langle d_m\rangle \models X_m) \rightarrow \langle d_1, \ldots, d_m\rangle \models Y^m) \wedge (\forall\langle d_1, \ldots, d_l\rangle \in A^l: (\langle d_1\rangle \models X_1 \wedge \cdots \wedge \langle d_l\rangle \models X_l) \rightarrow \langle d_1, \ldots, d_l\rangle \models Z^l)$ iff $\mathcal{I} \models \wedge X_m \cdots \wedge X_1Y^m \wedge \mathcal{I} \models \wedge X_l \cdots \wedge X_1Z^l$ (Section 2.4) iff $\mathcal{I} \models \wedge X_m \cdots \wedge X_1Y^m \cap \wedge X_l \cdots \wedge X_1Z^l$ (definition of satisfaction (4)). Thus $\mathcal{I} \models X_jT \cap \cdots \cap X_nT \cap \wedge X_n \cdots \wedge X_1(Y^m \cap Z^l)$ implies $\mathcal{I} \models \wedge X_m \cdots \wedge X_1Y^m \cap \wedge X_l \cdots \wedge X_1Z^l$ whence by Section 2.3, $\mathcal{I} \models (X_jT \cap \cdots \cap X_nT \cap \wedge X_n \cdots \wedge X_1(Y^m \cap Z^l)) \subseteq (\wedge X_m \cdots \wedge X_1Y^m \cap \wedge X_l \cdots \wedge X_1Z^l)$. Since $\mathcal{I}$ is arbitrary, Axiom D1 is valid.

(iii) Rule EI preserves validity.

Suppose $\models\overline{(V^0 \cap RT \cap \wedge RX \cap X_n \cdots X_1\wedge RY^{n+1})}$, where R is fresh, but there exist interpretations $\mathcal{I}$ such that $\mathcal{I} \models V^0 \cap X_n \cdots X_1XY^{n+1}$. In such interpretations, $\exists\langle d, d_1, \ldots, d_n\rangle \in A^{n+1}: \langle d\rangle \models X$ and $\langle d_1\rangle \models X_1$ and $\cdots$ and $\langle d_n\rangle \models X_n$ and $\langle d, d_1, \ldots, d_n\rangle \models Y^{n+1}$. Since R is fresh, among the interpretations $\mathcal{I}$ there are interpretations $\mathcal{I}'$ such that $\mathcal{T}'(R) = \{\langle d\rangle\}$. But then $\mathcal{I}' \models V^0 \cap RT \cap \wedge RX \cap X_n \cdots X_1\wedge RY^{n+1}$, which contradicts the assumption of validity.

Next, completeness of the axiomatization is shown. The proof is in the style of Henkin. But because of the absence of atomicity, the construction of an interpretation is not the standard one. Therefore the proof of the satisfiability theorem is given in full. First some definitions are needed. An expression of the form Y^1Z^{n+1} is an *image*. For example, $Y_3^1Y_2^1(Y_1^1Z^5 \cap \overline{V^2})$, $Y_2^1(Y_1^1Z^5 \cap \overline{V^2})$, and $Y_1^1Z^5$ are images, whereas $Y_1^1Z^5 \cap \overline{V^2}$ and $\overline{V^2}$ are not. Let $\Gamma \subseteq \mathcal{L}_N$ be a set of sentences. Γ is *consistent* iff it does not contain $X_1, \ldots, X_n$ such that $\overline{X_1 \cap \cdots \cap X_n}$ is in $\mathcal{T}$. Γ is *complete* iff for every sentence $X \in \mathcal{L}_N$, either X or $\overline{X}$ is in Γ. Γ is *saturated* iff it is complete, consistent and contains RT, $\wedge RX$ and $X_n \cdots X_1 \wedge RY^{n+1}$ for some $R \in \mathcal{R}_1$ whenever it contains $X_n \cdots X_1 XY^{n+1}$ where Y^{n+1} is not an image. Γ^* is the set of sentences obtained from Γ by uniform substitution of R^1_{2i} for R^1_i in each $X \in \Gamma$. Thus only unary predicate symbols with even index occur in Γ^*, leaving a denumerably infinite number of fresh unary predicate symbols. Notice that the axioms do not reference any *particular* predicate symbol except R^1_0. Therefore any uniform substitution of distinct unary predicate symbols for distinct unary predicate symbols that leaves R^1_0 fixed preserves consistency and inconsistency.

Lemma 4 *Let $\Gamma \subseteq \mathcal{L}_N$ be a set of sentences. If Γ^* is consistent it can be extended to a saturated set of sentences $\Gamma^+ \subseteq \mathcal{L}_N$.*

Proof: Let $W_1, W_2, \ldots$ be an enumeration of the sentences of $\mathcal{L}_N$ such that if $W_i = X_n \cdots X_1 XY^{n+1}$, where Y^{n+1} is not an image, then $W_{i+1} = R^1_j T \cap \wedge R^1_j X \cap X_n \cdots X_1 \wedge R^1_j Y^{n+1}$ for some j such that j is odd and R^1_j does not occur in W_k for $k \leq i$. Let $\Gamma_0 = \Gamma^*$ and $\Gamma_{i+1} = \Gamma_i \cup \{W_{i+1}\}$ if it is consistent and $\Gamma_{i+1} = \Gamma_i$ otherwise. Let $\Gamma^+ = \bigcup_{i \in \omega} \Gamma_i$.

(1) Γ^+ is consistent since each Γ_i is.

(2) Γ^+ is complete, for suppose $X \notin \Gamma^+$ and $\overline{X} \notin \Gamma^+$. Then for some i, $W_{j_1}, \ldots, W_{j_n} \in \Gamma_i$ such that $\overline{W_{j_1} \cap \cdots \cap W_{j_n} \cap X} \in \mathcal{T}$ and for some i' (say $i \leq i'$) $W'_{k_1}, \ldots, W'_{k_m} \in \Gamma_{i'}$ such that $\overline{W'_{k_1} \cap \cdots \cap W'_{k_m} \cap \overline{X}} \in \mathcal{T}$. But then by Axiom BT and Rule MP, $\overline{W_{j_1} \cap \cdots \cap W_{j_n} \cap W'_{k_1} \cap \cdots \cap W'_{k_m}} \in \mathcal{T}$, contradicting the consistency of $\Gamma_{i'}$.

(3) Γ^+ is saturated, for suppose $W_i = X_n \cdots X_1 XY^{n+1} \in \Gamma_i$ for Y^{n+1} not an image. Then Γ_{i+1} contains $RT \cap \wedge RX \cap X_n \cdots X_1 \wedge RY^{n+1}$ for some fresh R unless there are $W_{j_1}, \ldots, W_{j_m} \in \Gamma_i$ such that $\overline{W_{j_1} \cap \cdots \cap W_{j_m} \cap RT \cap \wedge RX \cap X_n \cdots X_1 \wedge RY^{n+1}} \in \mathcal{T}$. But by Rule EI, this implies $\overline{W_{j_1} \cap \cdots \cap W_{j_m} \cap X_n \cdots X_1 XY^{n+1}} \in \mathcal{T}$, contradicting the consistency of Γ_i.

Theorem 5 (Satisfiability) *Let $\Gamma \subseteq \mathcal{L}_N$ be a set of sentences. If Γ^* is consistent there is an interpretation $\mathcal{I} = \langle \mathcal{A}, \mathcal{T} \rangle$ of $\mathcal{L}_N$ satisfying Γ^*.*

Proof: Let Γ^+ be a saturated set of sentences extending Γ^*. It suffices to show that $\mathcal{I}$ satisfies Γ^+. Let $\mathcal{A}$ be the subalgebra of unary expressions of the Lindenbaum algebra of Γ^+ (see Bell and Machover [1]). Then $\mathcal{A}$ is a Boolean algebra whose universe is the set of equivalence classes of unary expressions of $\mathcal{L}_N$ defined: $X \approx Y$ iff $\wedge T(X \equiv Y) \in \Gamma^+$. Let $|X|$ be the equivalence class of X.

The partial order of $\mathcal{A}$ is defined: $|X| \subseteq |Y|$ iff $\wedge T(X \subseteq Y) \in \Gamma^+$. Some simple properties of this partial order are the following. These properties are based on the theorem schemas $\wedge XT$ and $\wedge XY \equiv \wedge T(X \subseteq Y)$, which follow directly from the axiomatization.

(i) $\wedge XT$ and $\wedge XT \equiv \wedge T(X \subseteq T)$ imply $\wedge T(X \subseteq T)$. Hence $|T|$ is the upper bound of $\mathcal{A}$.

(ii) From (i) and Axiom BT, $\wedge T(\bar{T} \subseteq X)$. Hence $|\bar{T}|$ is the lower bound of $\mathfrak{A}$.

(iii) $XT \in \Gamma^+$ iff $\overline{XT} \notin \Gamma^+$ iff $\wedge X\bar{T} \notin \Gamma^+$ iff $\wedge T(X \subseteq \bar{T}) \notin \Gamma^+$ iff $|X|$ is nonzero in $\mathfrak{A}$.

(iv) $\wedge XY \in \Gamma^+$ iff $\wedge T(X \subseteq Y) \in \Gamma^+$ iff $|X| \subseteq |Y|$ in $\mathfrak{A}$.

For each $R^n \in \mathcal{R}_n$ define $\mathcal{F}(R^n) := \{\langle |X_1|, \ldots, |X_n|\rangle : X_1T \cap \cdots \cap X_nT \cap \wedge X_n \ldots \wedge X_1 R^n \in \Gamma^+\}$. $\mathcal{F}$ satisfies the requirements for a denotation function (see Section 2.2). That the first requirement is satisfied follows from the definition of $\mathcal{F}$ and property (iii) above. Satisfaction of the second requirement follows from Axiom SS and property (iv). That the third requirement is satisfied can be seen as follows. Suppose $\forall$ nonzero $\langle |W_1|, \ldots, |W_n|\rangle \subseteq \langle |V_1|, \ldots, |V_n|\rangle : \exists$ nonzero $\langle |U_1|, \ldots, |U_n|\rangle \subseteq (|W_1|, \ldots, |W_n|\rangle : \langle |U_1|, \ldots, |U_n|\rangle \in \mathcal{F}(R^n)$ but $\langle |V_1|, \ldots, |V_n|\rangle \notin \mathcal{F}(R^n)$. Then $U_1T, \ldots, U_nT, \wedge U_1V_1, \ldots, \wedge U_nV_n, \wedge U_n \cdots \wedge U_1R^n \in \Gamma^+$ and $\overline{\wedge V_n \cdots \wedge V_1 R^n} \in \Gamma^+$ (Γ^+ is complete). Hence $V_n \cdots V_1\overline{R^n} \in \Gamma^+$ and $\exists R_1, \ldots, R_n \in \mathcal{R}_1$ such that $R_1T, \ldots, R_nT, \wedge R_1V_1, \ldots, \wedge R_nV_n, \wedge R_n \cdots \wedge R_1\overline{R^n} \in \Gamma^+$ (Γ^+ is saturated). By the initial assumption and properties (iii) and (iv) above, $\exists Q_1, \ldots, Q_n : Q_1T, \ldots, Q_nT, \wedge Q_1R_1, \ldots, \wedge Q_nR_n, \wedge Q_n \cdots \wedge Q_1R^n \in \Gamma^+$, and so by Axiom EG, $Q_n \cdots Q_1R^n \in \Gamma^+$. But because $\mathcal{F}$ satisfies the second requirement, $\wedge Q_n \cdots \wedge Q_1\overline{R^n} \in \Gamma^+$ whence by Axiom N, $\overline{Q_n \cdots Q_1R^n} \in \Gamma^+$, contradicting the consistency of Γ^+.

The proof will actually establish the more general claim: for each $X^n \in \mathcal{L}_N$, $\langle |V_1|, \ldots, |V_n|\rangle \vDash X^n$ iff $\forall$ nonzero $\langle |W_1|, \ldots, |W_n|\rangle \subseteq \langle |V_1|, \ldots, |V_n|\rangle : \exists$ nonzero $\langle |U_1|, \ldots, |U_n|\rangle \subseteq \langle |W_1|, \ldots, |W_n|\rangle : \wedge U_n \cdots \wedge U_1X^n \in \Gamma^+$. Proof is by induction on the structure of X^n. The basis follows directly from the definition of satisfaction, the definition of $\mathcal{F}$, and the requirements for a denotation function. The induction step involves four cases. Axiom BT and Rule MP are used implicitly.

Case 1. $X^n = \langle k_1, \ldots, k_m\rangle R^m$, where $n = \max(k_j)_{1 \leq j \leq m}$.

$\langle |V_1|, \ldots, |V_n|\rangle \vDash X^n$ iff $\forall$ nonzero $\langle |W_1|, \ldots, |W_n|\rangle \subseteq \langle |V_1|, \ldots, |V_n|\rangle : \exists$ nonzero $\langle |U_1|, \ldots, |U_n|\rangle \subseteq \langle |W_1|, \ldots, |W_n|\rangle : \langle |U_{k_1}|, \ldots, |U_{k_m}|\rangle \vDash R^m$ (definition of satisfaction) iff $\forall$ nonzero $\langle |W_1|, \ldots, |W_n|\rangle \subseteq \langle |V_1|, \ldots, |V_n|\rangle : \exists$ nonzero $\langle |U_1|, \ldots, |U_n|\rangle \subseteq \langle |W_1|, \ldots, |W_n|\rangle : \forall$ nonzero $\langle |Q_{k_1}|, \ldots, |Q_{k_m}|\rangle \subseteq \langle |U_{k_1}|, \ldots, |U_{k_m}|\rangle : \exists$ nonzero $\langle |P_{k_1}|, \ldots, |P_{k_m}|\rangle \subseteq \langle |Q_{k_1}|, \ldots, |Q_{k_m}|\rangle : \wedge P_{k_m} \cdots \wedge P_{k_1}R^m \in \Gamma^+$ (induction hypothesis) iff $\forall$ nonzero $\langle |W_1|, \ldots, |W_n|\rangle \subseteq \langle |V_1|, \ldots, |V_n|\rangle : \exists$ nonzero $\langle |P_1|, \ldots, |P_n|\rangle \subseteq \langle |W_1|, \ldots, |W_n|\rangle : \wedge P_{k_m} \cdots \wedge P_{k_1}R^m \in \Gamma^+$ (transitivity of $\subseteq$). The proof for this case is completed by proving the following claim.

Claim $\wedge P_{k_m} \cdots \wedge P_{k_1}R^m \in \Gamma^+$ *iff* $\wedge P_n \cdots \wedge P_1\langle k_1, \ldots, k_m\rangle R^m \in \Gamma^+$.

The only if direction follows directly from Axiom C2. For the if direction, suppose $\wedge P_{k_m} \cdots \wedge P_{k_1}R^m \notin \Gamma^+$. Then $P_{k_m} \cdots P_{k_1}\overline{R^m} \in \Gamma^+$ (Γ^+ is complete) and therefore $R_{k_1}T \cap \cdots \cap R_{k_m}T \cap \wedge R_{k_1}P_{k_1} \cap \cdots \cap \wedge R_{k_m}P_{k_m} \cap \wedge R_{k_m} \cdots \wedge R_{k_1}\overline{R^m} \in \Gamma^+$ for some $R_{k_1}, \ldots, R_{k_m} \in \mathcal{R}_1$ (Γ^+ is saturated). Hence $\wedge R_n \cdots \wedge R_1\overline{\langle k_1, \ldots, k_m\rangle R^m} \in \Gamma^+$, where $R_j = P_j$ if $j \notin \{k_1, \ldots, k_m\}$ (Axioms C1 and N), and by Axiom EG, $P_n \cdots P_1\overline{\langle k_1, \ldots, k_m\rangle R^m} \in \Gamma^+$. That is, $\overline{\wedge P_n \cdots}$

$\overline{\wedge P_1\langle k_1,\ldots,k_m\rangle R^m} \in \Gamma^+$ and so $\wedge P_n \cdots \wedge P_1\langle k_1,\ldots,k_m\rangle R^m \notin \Gamma^+$ (Γ^+ is complete).

Case 2. $X = \bar{Y}$.

$\langle|V_1|,\ldots,|V_n|\rangle \vDash X^n$ iff $\forall$ nonzero $\langle|W_1|,\ldots,|W_n|\rangle \subseteq \langle|V_1|,\ldots,|V_n|\rangle$: $\langle|W_1|,\ldots,|W_n|\rangle \nvDash Y$ (definition of satisfaction) iff $\forall$ nonzero $\langle|W_1|,\ldots,|W_n|\rangle \subseteq \langle|V_1|,\ldots,|V_n|\rangle$: $\exists$ nonzero $\langle|U_1|,\ldots,|U_n|\rangle \subseteq \langle|W_1|,\ldots,|W_n|\rangle$: $\forall$ nonzero $\langle|Q_1|,\ldots,|Q_n|\rangle \subseteq \langle|U_1|,\ldots,|U_n|\rangle$: $\wedge Q_n \cdots \wedge Q_1 Y \notin \Gamma^+$ (induction hypothesis). But $\wedge Q_n \cdots \wedge Q_1 Y \notin \Gamma^+$ iff $Q_n \cdots Q_1 \bar{Y} \in \Gamma^+$ (Γ^+ is complete) iff for some $R_1,\ldots,R_n \in \mathfrak{R}_1 : R_1 T \cap \cdots \cap R_n T \cap \wedge R_1 Q_1 \cap \cdots \cap \wedge R_n Q_n \cap \wedge R_n \cdots \wedge R_1 \bar{Y} \in \Gamma^+$ (Γ^+ is saturated). It follows from properties (iii) and (iv) above that $|R_i|$ is nonzero and $|R_i| \subseteq |Q_i|$ in $\mathfrak{A}$. By transitivity of $\subseteq$, $\forall$ nonzero $\langle|W_1|,\ldots,|W_n|\rangle \subseteq \langle|V_1|,\ldots,|V_n|\rangle$: $\exists$ nonzero $\langle|R_1|,\ldots,|R_n|\rangle \subseteq \langle|W_1|,\ldots,|W_n|\rangle$: $\wedge R_n \cdots \wedge R_1 \bar{Y} \in \Gamma^+$, which supports the claim.

Case 3. $X^n = Y^m \cap Z^l$ where $n = \max(l, m)$.

$\langle|V_1|,\ldots,|V_n|\rangle \vDash X^n$ iff $\langle|V_1|,\ldots,|V_m|\rangle \vDash Y^m$ and $\langle|V_1|,\ldots,|V_l|\rangle \vDash Z^l$ (definition of satisfaction) iff $\forall$ nonzero $\langle|W_1|,\ldots,|W_m|\rangle \subseteq \langle|V_1|,\ldots,|V_m|\rangle$: $\exists$ nonzero $\langle|U_1|,\ldots,|U_m|\rangle \subseteq \langle|W_1|,\ldots,|W_m|\rangle$: $\wedge U_m \cdots \wedge U_1 Y^m \in \Gamma^+$ and $\forall$ nonzero $\langle|W_1|,\ldots,|W_l|\rangle \subseteq \langle|V_1|,\ldots,|V_l|\rangle$: $\exists$ nonzero $\langle|Q_1|,\ldots,|Q_l|\rangle \subseteq \langle|W_1|,\ldots,|W_l|\rangle$: $\wedge Q_l \cdots \wedge Q_1 Z^l \in \Gamma^+$ (induction hypothesis). Now observe that in general, $(\forall\beta \subseteq \alpha : \exists\gamma \subseteq \beta : \phi(\gamma)) \wedge (\forall\beta \subseteq \alpha : \exists\delta \subseteq \beta : \psi(\delta))$ iff $\forall\beta \subseteq \alpha$: $\exists\gamma \subseteq \beta : (\phi(\gamma) \wedge \exists\delta \subseteq \gamma : \psi(\delta))$. Using this observation, the last condition can be modified: iff $\forall$ nonzero $\langle|W_1|,\ldots,|W_n|\rangle \subseteq \langle|V_1|,\ldots,|V_n|\rangle$: $\exists$ nonzero $\langle|U_1|,\ldots,|U_n|\rangle \subseteq \langle|W_1|,\ldots,|W_n|\rangle$: $\wedge U_m \cdots \wedge U_1 Y^m \in \Gamma^+$ and $\exists$ nonzero $\langle|Q_1|,\ldots,|Q_n|\rangle \subseteq \langle|U_1|,\ldots,|U_n|\rangle$: $\wedge Q_l \cdots \wedge Q_1 Z^l \in \Gamma^+$. According to properties (iii) and (iv) above, $Q_i T, \wedge Q_i U_i \in \Gamma^+$. This implies $\wedge Q_m \cdots \wedge Q_1 Y^m \in \Gamma^+$ (Axiom SS) and hence $\wedge Q_n \cdots \wedge Q_1(Y^m \cap Z^l) \in \Gamma^+$ (Axiom D2). Conversely, suppose $\forall$ nonzero $\langle|W_1|,\ldots,|W_n|\rangle \subseteq \langle|V_1|,\ldots,|V_n|\rangle$: $\exists$ nonzero $\langle|Q_1|,\ldots,|Q_n|\rangle \subseteq \langle|W_1|,\ldots,|W_n|\rangle$: $\wedge Q_n \cdots \wedge Q_1(Y^m \cap Z^l) \in \Gamma^+$. Then $\wedge Q_m \cdots \wedge Q_1 Y^m, \wedge Q_l \cdots \wedge Q_1 Z^l \in \Gamma^+$ (Axiom D1) and hence $\langle|V_1|,\ldots,|V_m|\rangle \vDash Y^m$ and $\langle|V_1|,\ldots,|V_l|\rangle \vDash Z^l$ (induction hypothesis) whence $\langle|V_1|,\ldots,|V_n|\rangle \vDash Y^m \cap Z^l$ (definition of satisfaction).

Case 4. $X^n = Y^1 Z^m$ where $m = n + 1$.

$\langle|V_1|,\ldots,|V_n|\rangle \vDash X^n$ iff for some nonzero $|V| : \langle|V|\rangle \vDash Y^1$ and $\langle|V|,|V_1|,\ldots,|V_n|\rangle \vDash Z^m$. Proceeding as in Case 3, it can be seen that the preceding statement holds iff $\forall$ nonzero $\langle|W|,|W_1|,\ldots,|W_n|\rangle \subseteq \langle|V|,|V_1|,\ldots,|V_n|\rangle$: $\exists$ nonzero $\langle|U|,|U_1|,\ldots,|U_n|\rangle \subseteq \langle|W|,|W_1|,\ldots|W_n|\rangle$: $\wedge U_n \cdots \wedge U_1 \wedge U Z^m \in \Gamma^+$ and $\exists$ nonzero $\langle|Q|\rangle \subseteq \langle|U|\rangle$: $\wedge Q Y^1 \in \Gamma^+$, which implies $QT \cap \wedge Q Y^1 \cap \wedge U_n \cdots \wedge U_1 \wedge Q Z^m \in \Gamma^+$, whence $\wedge U_n \cdots \wedge U_1 Y^1 Z^m \in \Gamma^+$ (Axiom EG). Conversely, suppose $\forall$ nonzero $\langle|W_1|,\ldots,|W_n|\rangle \subseteq \langle|V_1|,\ldots,|V_n|\rangle$: $\exists$ nonzero $\langle|U_1|,\ldots,|U_n|\rangle \subseteq \langle|W_1|,\ldots,|W_n|\rangle$: $\wedge U_n \cdots \wedge U_1 Y^1 Z^m \in \Gamma^+$. Since $U_i T$ and $\wedge U_i U_i \in \Gamma^+$, $\wedge U_n \cdots \wedge U_1 Y^1 Z^m \in \Gamma^+$ implies $U_n \cdots U_1 Y^1 Z^m \in \Gamma^+$ by Axiom EG. Let $Z^m = M_l \cdots M_1 N^{m+l}$, where $l \geq 0$ and N^{m+l} is not an image. Since $U_n \cdots U_1 Y^1 M_l \cdots M_1 N^{m+l} \in \Gamma^+$ and Γ^+ is saturated, there exist $R, R_1, \ldots, R_n, R_{n+1}, \ldots, R_{n+l} \in \mathfrak{R}_1$ such that $RT \cap R_1 T \cap \cdots \cap R_{n+l} T \cap \wedge R Y^1 \cap \wedge R_1 U_1 \cap \cdots \cap \wedge R_{n+l} M_l \cap \wedge R_n \cdots \wedge R_1 \wedge R \wedge R_{n+l} \cdots \wedge R_{n+1} N^{m+l} \in \Gamma^+$. Hence $\wedge R_n \cdots \wedge R_1 \wedge R M_l \cdots M_1 N^{m+l} \in \Gamma^+$ (Axiom EG), i.e., $\wedge R_n \cdots$

$\wedge R_1 \wedge RZ^m \in \Gamma^+$. By Axiom SS, $\forall$ nonzero $|Q| \subseteq |R| : \wedge QY^1 \cap \wedge R_n \cdots \wedge R^1 \wedge QZ^m \in \Gamma^+$ and hence $\langle |R| \rangle \models Y^1$ and $\langle |R|, |V_1|, \ldots, |V_n| \rangle \models Z^m$ (induction hypothesis) whence $\langle |V_1|, \ldots, |V_n| \rangle \models Y^1 Z^m$ (definition of satisfaction).

Corollary 6 (Completeness) $\models X$ *only if* $X \in \mathcal{T}$.

4 Conclusion In the discrete version of $\mathcal{L}_N$ presented in [5], the absence of variables did not result in loss of expressiveness or increased complexity of proofs. In the generalization of $\mathcal{L}_N$ presented in this paper, the absence of variables enhances expressiveness and reduces the complexity of proofs relative to conventional predicate logic. For a comparison, see the elegant generalization of predicate logic to nonatomic domains presented by Roeper [8]. In a language for mass terms, variables are superfluous if not intrusive. Consider the sentence $\wedge XY\breve{R}$, which with some syntactic sugar is **forall** X **exists** $Y\breve{R}$, and makes the assertion that for all X there exist Y that stand in the relation R. Compare (**all** $X p$) (**some** $Y q$) Rpq, or (**all** p)$(Xp \rightarrow$ (**some** q)$(Yq \wedge Rpq)$), which make the same assertion (see [7],[8]). Far from increasing expressiveness, the variables seem to impede understanding.

Where a logic is desired for models that are nonatomic but not atomless, the present logic can be supplemented by adding singular predicates, $\mathcal{S} = \{S_i : i \in \omega\}$, with semantics:

for each $S \in \mathcal{S}$, $\mathcal{T}(S) = \{\langle a \rangle\}$ for some (not necessarily unique) atom $a \in A$

and axiom schema (S is a metavariable ranging over $\mathcal{S}$):

$$\mathbf{S} \qquad \wedge X_n \cdots \wedge X_1 \overline{(SY^{n+1})} \equiv \wedge X_n \cdots \wedge X_1 S\overline{Y^{n+1}}.$$

In this way, reasoning about mass terms and reasoning about discrete terms can be dealt with uniformly under a single logic.

Having established a sound and complete axiomatization, one can proceed to prove theorems similar to those of [5]. Principal among these is the Monotonicity Theorem, which states that if Y occurs as a subexpression of W such that Y lies in the scopes of an even (respectively, odd) number of complement operators and $(\wedge T)^n(Y \subseteq Z)$ (respectively, $(\wedge T)^n(Z \subseteq Y)$), then $W \subseteq W'$, where W' is obtained from W by substituting Z for that occurrence of Y. (Some of the details have been suppressed to simplify the statement.) These theorems provide an approach to reasoning that is similar to syllogistic and, because of the closeness of the expressions involved to surface English, is termed "surface reasoning" in [6].

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