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Cut-Free Modal Sequents for Normal Modal Logics

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Abstract We develop cut-free calculi of sequents for normal modal logics by using semantic modal sequents that are trees of usual sequents plus an accessibility relation, and by introducing modal operators when moving formulas along the branches of such trees. Those calculi are a natural improvement of modal tree-sequents, introduced in a previous work, are cut-free, and work well for all of the main normal modal logics.

0 Introduction We introduce a variant of sequents to develop cut-free calculi for normal modal logics (NLs). Namely, we enhance the modal tree-sequents introduced in Cerrato [3] (as the counterpart of systems of natural deduction based on strict implication developed in Cerrato [2]) by considering trees of usual sequents instead of trees of sequences of formulas, and by adding the Kripkean accessibility relation to those trees (so we call our modal sequents "semantic").

We use only two general modal rules for all NLs $\Box\vdash$ and $\vdash\Box$ (with a technical exception for systems containing the axiom schema D): we vary the first rule when varying the system depending on the accessibility, while we fix the second rule for all NLs.

We prove the completeness of our calculi for the normal modal logics **K**, **KB**, **KD**, **KT** (=**T**), **K**4, **K**5, **KBD**, **KBT** (=**B**), **KB**4, **KD**4, **KD**5, **KD**45, **K**45, **KT**4 (=**S4**), **KT**5 (=**S5**) (see Chellas [4]), giving also a semantic proof of cut-elimination, and, as a corollary, of the subformula property.

Thus we obtain a uniform treatment of calculi of sequents the normal modal logics that work well in every case. Namely, our calculi work better than the usual ones (Fitting [6], Ohnishi [8], [9]) that are not cut-free for some systems (e.g., for "symmetric" systems, those containing the axiom schema B) even when the subformula property holds (Takano [12]). Furthermore, our calculi work also better than those using higher level sequents (Došen [5]), that are not cut-free, and are developed only for **S4** and **S5**. Finally, our calculi work better than those

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using extra metalinguistic signs (Cerrato [1]), that are proved to be cut-free only for **K**.

1 Semantic modal sequents To develop cut-free calculi for normal modal logics, we introduce semantic modal sequents that are trees of usual sequents plus an accessibility relation (the same used in Kripke models). Trees of sequents are an improvement of "tree-sequents" introduced in [3]: namely, the former are trees of usual sequents, i.e., of ordered pairs of sequences of formulas, that are the left and the right part of the sequent, while the latter are trees of single sequences of formulas.

Furthermore, the use of the accessibility relation directly into the structure of modal sequents allows us to fix only two modal rules (a left and a right one) and to obtain rules for any system by suitably changing only that accessibility relation. So, semantic modal sequents really characterize modal behaviors by the structure of the calculus alone, instead of by specifically arranged rules.

Our language is $\mathbf{L} = \{\mathcal{P}, \wedge, \vee, \neg, \rightarrow, \Box\}$; we define the other operators as the following abbreviations:

equivalence	$A \leftrightarrow B =_{df} (A \rightarrow B) \wedge (B \rightarrow A)$
possibility	$\Diamond A =_{df} \neg\Box\neg A$
strict implication	$A \Rightarrow B =_{df} \Box(A \rightarrow B)$
strict equivalence	$A \Leftrightarrow B =_{df} (A \Rightarrow B) \wedge (B \Rightarrow A)$

Furthermore, $\top$ and $\bot$ denote a generic theorem and a generic contradiction, respectively.

A *semantic modal sequent* is a triple $\langle W, \rightarrow, R\rangle$ where W is a non-empty set of (occurrences of) usual sequents called worlds (i.e., $\Gamma \vdash \Delta$, where Γ, Δ are sequences of formulas of **L**), $\rightarrow$ is a strict tree-ordering on W, and R is a binary relation on W that extends $\rightarrow$, called accessibility. We often use "modal sequent" to refer to a semantic modal sequent, and "sequent" to refer to a usual sequent; furthermore, the locution "occurrences of" indicates that several instances of the same sequent can occur as different worlds in a modal sequent.

The first two components of semantic modal sequents, W and $\rightarrow$, give rise to *trees of sequents*, that we can also inductively define by:

$\Gamma \vdash \Delta$ is a tree of sequents, where Γ, Δ are sequences of formulas of **L**;

$$\begin{array}{c} \Gamma \vdash \Delta \\ \swarrow \cdots \searrow \\ \lambda_0 \ \cdots \ \lambda_n \end{array}$$

is a tree of sequents, where Γ, Δ are sequences of formulas of **L** and $\lambda_0, \ldots, \lambda_n$ $(n \geq 0)$ are trees of sequents.

We use capital latin letters for formulas, capital greek letters for sequences of formulas, small latin letters for worlds, and small greek letters for trees.

The other component of semantic modal sequents, the relation R, puts the corresponding semantic accessibility relation into sequents: namely, for a system having as modal axioms $\mathrm{Ax}_1, \ldots, \mathrm{Ax}_n$ the accessibility R is the minimal relation containing $\rightarrow$, $R(\mathrm{Ax}_1), \ldots, R(\mathrm{Ax}_n)$, where the correspondence between axioms and relations is given by the following table:

Ax	R(Ax)
K	$\rightarrow$
T	the reflexive closure of $\rightarrow$
4	the transitive closure of $\rightarrow$
5	the euclidean closure of $\rightarrow$
B	the symmetric closure of $\rightarrow$.

In the case of the axiom D, since we cannot univocally determine R as the serial closure of $\rightarrow$, we consider a new rule, namely the empty rule; when proving completeness, that rule (read upward) really allows us only to add unessential accessible worlds that maintain "serial" the constructing countermodel.

The rules for propositional connectives are $\neg\vdash$, $\vdash\neg$, $\wedge\vdash$, $\vdash\wedge$, $\vee\vdash$, $\vdash\vee$, $\rightarrow\vdash$, $\vdash\rightarrow$ (see Takeuti [11]) with suitable modifications for the use of trees. First, as to notation, two semantic modal sequents $\alpha = \langle W, \rightarrow, R\rangle$ and $\beta = \langle W', \rightarrow', R'\rangle$ are similar by (w, w'), where $w \in W$ and $w' \in W'$, (written as $\alpha \;{}_{w}\!\sim_{w'} \beta$) iff they become the same by simply replacing w with w', i.e., iff $W - \{w\} = W' - \{w'\}$, and, given the bijection f between W and W' that is the identity but $f(w) = w'$, we have both $w_0 \rightarrow w_1$ iff $f(w_0) \rightarrow' f(w_1)$ and $w_0 R w_1$ iff $f(w_0) R' f(w_1)$, for every $w_0, w_1 \in W$. Then, given a usual **PC**-rule $a'a''/b$, where a', a'', b are usual sequents, the corresponding tree version of that rule is $\alpha'\alpha''/\beta$, where α', α'', β are semantic modal sequents and $\alpha' \;{}_{a'}\!\sim_{a''} \alpha'' \;{}_{a''}\!\sim_{b} \beta \;{}_{b}\!\sim_{a'} \alpha'$ (we can apply a similar reasoning to rules with only one premise). So, the tree version of a **PC**-rule leaves unaltered both the structures of the tree and of all the unaffected usual sequents.

For example, now the rule $\rightarrow\vdash$ appears as:

$$\frac{\begin{array}{c}\gamma\\ \downarrow\\ \Gamma \vdash \Delta, C\\ \swarrow\cdots\searrow\\ \delta_1 \;\ldots\; \delta_n\end{array}\qquad\begin{array}{c}\gamma\\ \downarrow\\ D, \Pi \vdash \Lambda\\ \swarrow\cdots\searrow\\ \delta_1 \;\ldots\; \delta_n\end{array}}{\begin{array}{c}\gamma\\ \downarrow\\ C \rightarrow D, \Gamma, \Pi \vdash \Delta, \Lambda\\ \swarrow\cdots\searrow\\ \delta_1 \;\ldots\; \delta_n\end{array}}$$

where $\gamma, \delta_1, \ldots, \delta_n$ are trees of sequents (when $n = 0$ no δ_i appears), $\Gamma, \Delta, \Pi, \Lambda$ are sequences of formulas, and C, D are formulas of **L**.

The structural rules are the tree adaptation of the usual weakening $\vdash$, $\vdash$ weakening, exchange $\vdash$, $\vdash$ exchange, contraction $\vdash$, $\vdash$ contraction (see [11]).

The modal rules are $\Box\vdash$ and $\vdash\Box$, and move formulas along trees. Rule $\Box\vdash$ varies when varying modal systems, depending on the accessibility relation R:

$\Box\vdash$: from a modal sequent α having two worlds $\Gamma \vdash \Delta$ and $A, \Pi \vdash \Lambda$ with $(\Gamma \vdash \Delta) R (A, \Pi \vdash \Lambda)$ infer the modal sequent β obtained from α by substituting (both in the domain and in the relations $\rightarrow$ and R) those worlds with $\Box A, \Gamma \vdash \Delta$ and $\Pi \vdash \Lambda$, respectively.

As an example, when R is the transitive closure of $\rightarrow$ that rule appears as (when $n = 0$ no δ_i appears):

$$\frac{\begin{array}{ccc} & \gamma & \\ & \downarrow & \\ & \Gamma \vdash \Delta & \\ & \swarrow \cdots \downarrow \cdots \searrow & \\ \cdots & \vdots & \cdots \\ & \downarrow & \\ & A, \Pi \vdash \Lambda & \\ & \swarrow \cdots \searrow & \\ & \delta_1 \quad \cdots \quad \delta_n & \end{array}}{\begin{array}{ccc} & \gamma & \\ & \downarrow & \\ & \Box A, \Gamma \vdash \Delta & \\ & \swarrow \cdots \downarrow \cdots \searrow & \\ \cdots & \vdots & \cdots \\ & \downarrow & \\ & \Pi \vdash \Lambda & \\ & \swarrow \cdots \searrow & \\ & \delta_1 \cdots \delta_n & \end{array}} \; .$$

Rule $\vdash\Box$ is the same for all modal systems, since it depends on $\rightarrow$ instead of on R:

$\vdash\Box$: from a modal sequent α having two worlds $\Gamma \vdash \Delta$ and $\vdash A$ with $(\Gamma \vdash \Delta) \rightarrow (\vdash A)$ infer the modal sequent β obtained from α by substituting (both in the domain and in the relations $\rightarrow$ and R) the former world with $\Gamma \vdash \Delta, \Box A$ and by completely deleting both the latter world from the domain and any occurrence of it from $\rightarrow$, and consequently modifying R.

That rule appears as (when $n = 0$ no δ_i appears):

$$\frac{\begin{array}{ccc} & \gamma & \\ & \downarrow & \\ & \Gamma \vdash \Delta & \\ & \swarrow \cdots \downarrow \cdots \searrow & \\ \cdots & \vdash A & \cdots \\ & \swarrow \cdots \searrow & \\ & \delta_1 \cdots \delta_n & \end{array}}{\begin{array}{ccc} & \gamma & \\ & \downarrow & \\ & \Gamma \vdash \Delta, \Box A & \\ & \swarrow \cdots \quad \cdots \searrow & \\ \cdots & & \cdots \\ & & \\ & \delta_1 \ldots \delta_n & \end{array}}$$

Note that when the sequent $\vdash A$ is not a terminal leaf that rule should introduce a disconnection in the tree of sequents. We can afford this problem in two different ways: first by requiring the relation $\rightarrow$ to be connected in every case, limiting the applicability of the rules (e.g., requiring the fact that the sequent $\vdash A$ is a terminal leaf as a necessary condition to apply the rule $\vdash\Box$); otherwise, by allowing $\rightarrow$ also to be disconnected, only restricting the end-sequents to be connected: since any rule neither reconnects sequents nor eliminates disconnected sequents, we really must consider only those proofs where all sequents are connected. In this work we follow the first strategy, since it allows us to restrict the attention only to significant proofs.

Finally, we must specify the empty rule that appears only in calculi of sequents for "serial" systems, i.e., for those systems containing the axiom D:

empty rule: from a modal sequent α having a world $\vdash$ infer the modal sequent β obtained from α by deleting that world (when not occurring as top node of α) from W and from both the relations R and $\rightarrow$.

Clearly, the empty rule does not introduce any disconnection only when $\vdash$ is a terminal leaf, so that it appears as:

$$\frac{\begin{array}{c}\gamma\\ \downarrow\\ \vdash\end{array}}{\gamma}$$

The notions *proof* and *end-sequent* are used as in standard calculi of sequents (e.g., see [11]). An *initial sequent* is one of the form $\langle W,\rightarrow,R\rangle$ where $A \vdash A \in W$ for some formula A, a usual sequent $\Gamma \vdash \Delta$ is *provable* when $\langle W,\rightarrow,R\rangle$ is an end-sequent with $W = \{\Gamma \vdash \Delta\}$, and a formula A is *provable* when $\vdash A$ is provable.

2 Soundness Now we prove the calculi we have introduced are sound for the 15 normal modal logics, **K**, **KB**, **KD**, **KT** (=**T**), **K4**, **K5**, **KBD**, **KBT** (=**B**), **KB4**, **KD4**, **KD5**, **KD45**, **K45**, **KT4** (=**S4**), **KT5** (=**S5**); namely, using the same technique used in [3], we really prove that for any NL Λ if there exists a sequent-style Λ-proof of a formula A (written, "A is Λ-provable") then there exists a Hilbert-style Λ-proof of it (written, "A is a Λ-theorem"), so that (by soundness of usual Hilbert-style systems) A is true in any Λ-model (written, "A is Λ-valid"). This section is really a technical adaptation to semantic modal sequents of those lemmas and theorems used in [3] for the same purposes.

We recall what the system **K** is in the Hilbert style (see [4]):

1. it is a modal system, i.e., a system closed under the rule of inference RPL: $(A_1,A_2,\ldots,A_n)/A$ where A is a tautological consequence of $A_1,A_2,\ldots,A_n$
2. it contains the axiom schema K: $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
3. it is closed under the rule of necessitation RN: $A/\Box A$.

We do not consider the axiom schema Df$\Diamond$: $\Diamond A \leftrightarrow \neg\Box\neg A$ (see [4]), since our language does not contain an explicit possibility symbol.

Also, recall that the other 14 NLs have, beyond the axioms of **K**, suitable combinations of the following formulas as proper axioms:

T: $\Box A \to A$
4: $\Box A \to \Box\Box A$
5: $\Diamond A \to \Box\Diamond A$
B: $A \to \Box\Diamond A$
D: $\Box A \to \Diamond A$.

$\vdash_\Lambda A$ means that the formula A is a theorem of Λ, Λ being one of the above systems.

Theorem 1 (Soundness of the NLs-calculi) *For any* NL Λ, *if A is Λ-provable then A is a Λ-theorem, and so it is Λ-valid.*

Proof: We translate semantic modal sequents into formulas by suitably modifying the Schütte translation (see Schütte [10], [2], and [3]). Namely, we define the translation $*$ by an induction on the complexity of trees:

$$(\Gamma \vdash \Delta)^* = \bigwedge \Gamma \to \bigvee \Delta$$

$$\begin{pmatrix} \Gamma \vdash \Delta \\ \swarrow \ldots \searrow \\ \lambda_0 \;\ldots\; \lambda_n \end{pmatrix}^* = \bigwedge \Gamma \to (\bigvee \Delta \vee \Box\lambda_0^* \vee \ldots \vee \Box\lambda_n^*)$$

where A is a formula of **L**, Γ and Δ are sequences of formulas, $\lambda_0, \ldots, \lambda_n$ are modal sequents and, by definition, $\bigwedge \varnothing = \top$ and $\bigvee \varnothing = \bot$.

Note that only the relation $\to$ influences the translation $*$; in fact, there is no mention of R, that will be reintroduced in this theorem indirectly, by using the Hilbert-style modal axioms.

We extend the translation to sequent-style inferences into Hilbert-style inferences by:

$$\left(\frac{\text{premise}_0, \ldots, \text{premise}_n}{\text{conclusion}}\right)^* = \frac{(\text{premise}_0)^*, \ldots, (\text{premise}_n)^*}{(\text{conclusion})^*}.$$

This translation is well defined, i.e., for any NL Λ, each inference is really an Hilbert-style Λ-inference, as we shall prove in Theorem 5.

By $*$, initial sequents are translated to **K**-theorems (as we shall prove in Theorem 4), and sequent-style inferences are translated to Hilbert-style inferences, so that the translation of a Λ-provable sequent is also Hilbert-style provable in Λ; so, if a formula A is Λ-provable, i.e., if $\vdash A$ is an end-sequent, then $(\vdash A)^*$ is a Λ-theorem, i.e., (by the definition of $*$) A is a Λ-theorem.

So we must prove the translations of initial sequents are **K**-theorems and the translations of sequent-style inferences are really Hilbert-style inferences. As in [3], to prove that the translation $*$ of a rule is well-founded, we isolate the usual sequent affected by that rule and prove a suitable implication restricted to that sequent; then, we transport that implication up along all of the trees of sequents obtaining $\vdash_{\mathbf{K}}$ ((premise$_0$)* $\wedge \ldots \wedge$ (premise$_n$)*) $\rightarrow$ (conclusion)*, so that, by Modus Ponens, we immediately have the thesis.

To transport implications along trees we need two technical lemmas:

Lemma 2 *For any* NL Λ, *given the semantic modal sequents*:

$$\alpha = \begin{array}{c} a \\ \swarrow \ldots \searrow \\ \alpha_1 \; \ldots \; \alpha_n \end{array} \qquad \beta = \begin{array}{c} b \\ \swarrow \ldots \searrow \\ \beta_1 \; \ldots \; \beta_n \end{array} \qquad \gamma = \begin{array}{c} c \\ \swarrow \ldots \searrow \\ \gamma_1 \; \ldots \; \gamma_n \end{array}$$

where a, b, c *are usual sequents, and* $\alpha_i, \beta_i, \gamma_i$ $(1 \leq i \leq n)$ *are trees of sequents* (*when* $n = 0$, *no one of them appears*)

$$\textit{if } \vdash_\Lambda \alpha_i^* \rightarrow \gamma_i^* \ (1 \leq i \leq n) \textit{ and } \vdash_\Lambda a^* \rightarrow c^* \textit{ then}$$
$$\vdash_\Lambda \alpha^* \rightarrow \gamma^*;$$

furthermore,

$$\textit{if } \vdash_\Lambda (\alpha_i^* \vee \beta_i^*) \rightarrow \gamma_i^* (1 \leq i \leq n) \textit{ and } \vdash_\Lambda (a^* \wedge b^*) \rightarrow c^* \textit{ then}$$
$$\vdash_\Lambda (\alpha^* \wedge \beta^*) \rightarrow \gamma^*.$$

Proof: We prove the latter, since the former is a subcase of it. By the hypotheses $\vdash_\Lambda (\alpha_i^* \vee \beta_i^*) \rightarrow \gamma_i^*$ $(1 \leq i \leq n)$ and by necessitation we have $\vdash_\Lambda \Box((\alpha_i^* \vee \beta_i^*) \rightarrow \gamma_i^*)$, so that, by axiom K we obtain $\vdash_\Lambda \Box(\alpha_i^* \vee \beta_i^*) \rightarrow \Box\gamma_i^*$, and by the **K**-theorem $(\Box A \vee \Box B) \rightarrow \Box(A \vee B)$ we have $\vdash_\Lambda (\Box\alpha_i^* \vee \Box\beta_i^*) \rightarrow \Box\gamma_i^*$. By that, by the hypothesis $\vdash_\Lambda ((\Gamma_a \rightarrow \Delta_a) \wedge (\Gamma_b \rightarrow \Delta_b)) \rightarrow (\Gamma_c \rightarrow \Delta_c)$ (where $a = \Gamma_a \vdash \Delta_a$, $b = \Gamma_b \vdash \Delta_b$, $c = \Gamma_c \vdash \Delta_c$), and by the **PC**-tautology

$$\left(\bigwedge_1^N ((A_i \vee B_i) \rightarrow C_i) \wedge (((A' \rightarrow A'') \wedge (B' \rightarrow B'')) \rightarrow (C' \rightarrow C''))\right)$$
$$\rightarrow \left(\left(\left(A' \rightarrow \left(A'' \vee \bigvee_1^N A_i\right)\right)\right.\right.$$
$$\left.\left.\wedge \left(B' \rightarrow \left(B'' \vee \bigvee_1^N B_i\right)\right)\right) \rightarrow \left(C' \rightarrow \left(C'' \vee \bigvee_1^N C_i\right)\right)\right),$$

we obtain

$$\vdash_\Lambda \left(\left(\left(\Gamma_a \rightarrow \left(\Delta_a \vee \bigvee_1^N \Box\alpha_i^*\right)\right) \wedge \left(\Gamma_b \rightarrow \left(\Delta_b \vee \bigvee_1^N \Box\beta_i^*\right)\right)\right)\right.$$
$$\left.\rightarrow \left(\Gamma_c \rightarrow \left(\Delta_c \vee \bigvee_1^N \Box\gamma_i^*\right)\right)\right),$$

that is, recalling the definition of *, the thesis.

Lemma 3 *For any* NL Δ, *given the semantic modal sequents* $\gamma, \alpha, \beta, \delta$

$$\textit{if } \vdash_\Lambda \alpha^* \rightarrow \delta^* \textit{ then } \vdash_\Lambda \begin{pmatrix} \gamma \\ \downarrow \\ \alpha \end{pmatrix}^* \rightarrow \begin{pmatrix} \gamma \\ \downarrow \\ \delta \end{pmatrix}^*$$

and

$$\textit{if } \vdash_\Lambda (\alpha^* \wedge \beta^*) \rightarrow \delta^* \textit{ then } \vdash_\Lambda \left(\left(\begin{pmatrix} \gamma \\ \downarrow \\ \alpha \end{pmatrix}^* \wedge \begin{pmatrix} \gamma \\ \downarrow \\ \beta \end{pmatrix}^* \right) \rightarrow \begin{pmatrix} \gamma \\ \downarrow \\ \delta \end{pmatrix}^* \right).$$

Proof: We prove the latter, since the former is a subcase of it. We proceed by an induction on the complexity of the modal sequent γ: when γ is empty the thesis is immediate. Now suppose we have proved the lemma for γ: we must prove it for

$$\begin{array}{c} \Gamma \ \vdash \ \Delta \\ \swarrow \cdots \swarrow \downarrow \searrow \cdots \searrow \\ \sigma_1 \ldots \sigma_r \quad \gamma \quad \sigma_{r+1} \ldots \sigma_m \end{array} \quad ;$$

by the inductive hypothesis, necessitation, and axiom K we easily obtain

$$\vdash_\Lambda \left(\left(\Box \begin{pmatrix} \gamma \\ \downarrow \\ \alpha \end{pmatrix}^* \wedge \Box \begin{pmatrix} \gamma \\ \downarrow \\ \beta \end{pmatrix}^* \right) \rightarrow \Box \begin{pmatrix} \gamma \\ \downarrow \\ \delta \end{pmatrix}^* \right),$$

so that, by the **PC**-tautology $((A \wedge B) \rightarrow C) \rightarrow (((G \rightarrow (S_0 \vee \ldots \vee S_r \vee A \vee S_{r+1} \vee \ldots \vee S_m)) \wedge (G \rightarrow (S_0 \vee \ldots \vee S_r \vee B \vee S_{r+1} \vee \ldots \vee S_m))) \rightarrow (G \rightarrow (S_0 \vee \ldots \vee S_r \vee C \vee S_{r+1} \vee \ldots \vee S_m)))$ applied with $G = \bigwedge \Gamma$, $S_0 = \bigvee \Delta$, $S_i = \Box \sigma_i^*$ $(1 \leq i \leq m)$, recalling the definition of *, we easily obtain the thesis.

Now we prove the initial sequents are **K**-theorems and the translations of the rules are well-founded for any NL:

Theorem 4 *The translation * of any initial sequent is a* **K**-*theorem.*

Proof: Let

$$\alpha = \begin{array}{c} \gamma \\ \downarrow \\ A \vdash A \\ \swarrow \cdots \searrow \\ \delta_1 \ \ldots \ \delta_n \end{array}$$

be an initial sequent; we prove the theorem by an induction on the complexity of γ: when γ is empty, by the **PC**-tautologies $A \rightarrow A = (A \vdash A)^*$ and $(B \rightarrow C) \rightarrow (B \rightarrow (C \vee D_1 \vee \ldots \vee D_n))$ we have $\vdash_{\mathbf{PC}} A \rightarrow (A \vee \Box\delta_1^* \vee \ldots \vee \Box\delta_n^*) = \alpha^*$. Suppose we have proved the thesis for a given γ: we must prove it for

$$\begin{array}{c} \Gamma \vdash \Delta \\ \swarrow \cdots \swarrow \ \downarrow \ \searrow \cdots \searrow \\ \sigma_1 \ldots \sigma_r \quad \gamma \quad \sigma_{r+1} \ldots \sigma_m \end{array}$$

By the inductive hypothesis and necessitation we have

$$\vdash_{\mathbf{K}} \Box \begin{pmatrix} \gamma \\ \downarrow \\ A \vdash A \\ \swarrow \cdots \searrow \\ \delta_1 \ \ldots \ \delta_n \end{pmatrix}^* ;$$

so, by the **PC**-tautology $A \to (B \to (S_0 \vee \ldots \vee S_r \vee A \vee S_{r+1} \vee \ldots \vee S_m))$ we obtain

$$\vdash_{\mathbf{K}} \bigwedge \Gamma \to \left(\bigvee \Delta \vee \Box\sigma_1^* \vee \ldots \vee \Box\sigma_r^* \vee \Box \begin{pmatrix} \gamma \\ \downarrow \\ A \vdash A \\ \swarrow \ldots \searrow \\ \delta_1 \ \ldots \ \delta_n \end{pmatrix}^* \vee \Box\sigma_{r+1}^* \vee \ldots \vee \Box\sigma_m^* \right) = \alpha^*,$$

i.e., the thesis.

Theorem 5 *For any* NL Λ, *the translation* $*$ *of any rule is well-defined.*

Proof: We must only prove for any NL $\Lambda \vdash_\Lambda ((\text{premise}_0)^* \wedge \ldots \wedge (\text{premise}_n)^*) \to (\text{conclusion})^*$, so that the thesis follows immediately by Modus Ponens.

The thesis is immediate for structural and **PC** rules: in fact all of the corresponding formulas are **K**-theorems. As an example, for the rule $\to\vdash$ we show that

$$\left(\begin{pmatrix} \gamma \\ \downarrow \\ \Gamma \vdash \Delta, C \\ \swarrow \ldots \searrow \\ \delta_1 \ \ldots \ \delta_n \end{pmatrix}^* \wedge \begin{pmatrix} \gamma \\ \downarrow \\ D, \Pi \vdash \Lambda \\ \swarrow \ldots \searrow \\ \delta_1 \ \ldots \ \delta_n \end{pmatrix}^* \right) \to \begin{pmatrix} \gamma \\ \downarrow \\ C \to D, \Gamma, \Pi \vdash \Delta, \Lambda \\ \swarrow \ldots \searrow \\ \delta_1 \ \ldots \ \delta_n \end{pmatrix}^*$$

is really a **K**-theorem. In fact, by the **PC**-tautologies $((\bigwedge\Gamma \to (\bigvee\Delta \vee C)) \wedge ((D \wedge \bigwedge\Pi) \to \bigvee\Lambda)) \to (((C \to D) \wedge \bigwedge\Gamma \wedge \bigwedge\Pi) \to (\bigvee\Delta \vee \bigvee\Lambda))$ and $(\delta_i^* \vee \delta_i^*) \to \delta_i^*$ $(1 \leq i \leq n)$, by applying Lemma 2 we obtain

$$\vdash_{\mathbf{K}} \left(\begin{pmatrix} \Gamma \vdash \Delta, C \\ \swarrow \ldots \searrow \\ \delta_1 \ \ldots \ \delta_n \end{pmatrix}^* \wedge \begin{pmatrix} D, \Pi \vdash \Lambda \\ \swarrow \ldots \searrow \\ \delta_1 \ \ldots \ \delta_n \end{pmatrix}^* \right) \to \begin{pmatrix} C \to D, \Gamma, \Pi \vdash \Delta, \Lambda \\ \swarrow \ldots \searrow \\ \delta_1 \ \ldots \ \delta_n \end{pmatrix}^*.$$

So transporting that implication along all of γ by Lemma 3, we obtain the thesis.

As to rule $\vdash\Box$, recalling that $\to$ must be connected, the affected sequent is in a terminal leaf; we have:

$$\begin{pmatrix} \Gamma \vdash \Delta \\ \swarrow \ldots \swarrow \ \downarrow \ \searrow \ldots \searrow \\ \sigma_1 \ldots \sigma_r \quad \vdash A \quad \sigma_{r+1} \ldots \sigma_m \end{pmatrix}^*$$

$$= \bigwedge \Gamma \to (\bigvee \Delta \vee \Box\sigma_1^* \vee \ldots \vee \Box\sigma_r^* \vee \Box(\vdash A)^* \vee \Box\sigma_{r+1}^* \vee \ldots \vee \Box\sigma_m^*) \qquad \text{def of } *$$

$$\leftrightarrow \bigwedge \Gamma \to (\bigvee \Delta \vee \Box A \vee \Box\sigma_1^* \vee \ldots \vee \Box\sigma_m^*) \qquad \mathbf{PC}, (1)$$

$$= \begin{pmatrix} \Gamma \vdash \Delta, \Box A \\ \swarrow \ldots \searrow \\ \sigma_1 \ \ldots \ \sigma_m \end{pmatrix}^* \qquad \text{def of } *$$

where (1) $(\vdash A)^* = (\bigwedge \varnothing \to A) = (\top \to A) \leftrightarrow A$; so transporting that equivalence along γ by Lemma 3, we have that the required implication is a **K**-theorem, obtaining the thesis.

As to rule $\Box\vdash$, we really move a formula by using the relation R, that depends on the system; first we examine the case of system **K**:

$$\left(\begin{array}{ccc} & \Gamma \vdash \Delta & \\ \swarrow \dots \swarrow & \downarrow & \searrow \dots \searrow \\ \sigma_1 \dots \sigma_r & A, \Pi \vdash \Lambda & \sigma_{r+1} \dots \sigma_m \\ & \swarrow \dots \searrow & \\ & \delta_1 \ \dots \ \delta_n & \end{array}\right)^*$$

$$= \bigwedge \Gamma \to (\bigvee \Delta \vee \Box\Sigma_1^* \vee \Box((A \wedge \bigwedge \Pi) \to (\bigvee \Lambda \vee \Box\Psi^*)) \vee \Box\Sigma_2^*) \qquad (1)$$

$$\to \bigwedge \Gamma \to (\bigvee \Delta \vee \Box\Sigma_1^* \vee \Box(A \to (\bigwedge \Pi \to (\bigvee \Lambda \vee \Box\Psi^*))) \vee \Box\Sigma_2^*) \qquad (2)$$

$$\to \bigwedge \Gamma \to (\bigvee \Delta \vee \Box\Sigma_1^* \vee (\Box A \to \Box(\bigwedge \Pi \to (\bigvee \Lambda \vee \Box\Psi^*))) \vee \Box\Sigma_2^*) \qquad \text{axiom K}$$

$$\to (\Box A \wedge \bigwedge \Gamma) \to (\bigvee \Delta \vee \Box\Sigma_1^* \vee \Box(\bigwedge \Pi \to (\bigvee \Lambda \vee \Box\Psi^*)) \vee \Box\Sigma_2^*) \qquad (3)$$

$$= (\Box A \wedge \bigwedge \Gamma) \to \left(\bigvee \Delta \vee \Box\Sigma_1^* \vee \Box \left(\begin{array}{c} \Pi \vdash \Lambda \\ \swarrow \dots \searrow \\ \delta_1 \ \dots \ \delta_n \end{array}\right)^* \vee \Box\Sigma_2^*\right) \qquad \text{def of } ^*$$

$$= \left(\begin{array}{ccc} & \Box A, \Gamma \vdash \Delta & \\ \swarrow \dots \swarrow & \downarrow & \searrow \dots \searrow \\ \sigma_1 \dots \sigma_r & \Pi \vdash \Lambda & \sigma_{r+1} \dots \sigma_m \\ & \swarrow \dots \searrow & \\ & \delta_1 \ \dots \ \delta_n & \end{array}\right)^* \qquad \text{def of } ^*$$

where

(1) $\Box\Sigma_1^* =_{df} \Box\sigma_1^* \vee \dots \vee \Box\sigma_r^*$;
$\Box\Sigma_2^* =_{df} \Box\sigma_{r+1}^* \vee \dots \vee \Box\sigma_m^*$;
$\Box\Psi^* =_{df} \Box\delta_1^* \vee \dots \vee \Box\delta_n^*$.
(2) $\vdash_{\mathbf{PC}} ((A \wedge B) \to C) \to (A \to (B \to C))$
$\vdash_{\mathbf{PC}} ((A \to (B \vee C)) \wedge (C \to D)) \to (A \to (B \vee D))$
(3) $\vdash_{\mathbf{PC}} (A \to (B_1 \vee (C \to D) \vee B_2)) \to ((C \wedge A) \to (B_1 \vee D \vee B_2))$;

so, transporting that implication along γ by Lemma 3, we obtain the thesis.

As to the other systems, since the rule $\Box\vdash$ really involves the relation R, we must move a formula A from (the left part of) a sequent "a" to (the left part of) a sequent "b" when aRb.

In the previous case (axiom K) we have proved that we can change

$$\begin{array}{c} \Gamma \vdash \Delta \\ \downarrow \\ A, \Pi \vdash \Lambda \end{array} \quad \text{into} \quad \begin{array}{c} \Box A, \Gamma \vdash \Delta \\ \downarrow \\ \Pi \vdash \Lambda \end{array}$$

without changing the context.

By axiom $B: A \to \Box\Diamond A$ we can change

$$\begin{array}{c} A,\Gamma \vdash \Delta \\ \downarrow \\ \Pi \vdash \Lambda \end{array} \quad \text{into} \quad \begin{array}{c} \Gamma \vdash \Delta \\ \downarrow \\ \Box A,\Pi \vdash \Lambda \end{array}$$

without changing the context; in fact:

$$\left(\begin{array}{ccc} & A,\Gamma\vdash\Delta & \\ \swarrow\cdots\swarrow & \downarrow & \searrow\cdots\searrow \\ \sigma_1\ldots\sigma_r & \Pi\vdash\Lambda & \sigma_{r+1}\ldots\sigma_m \\ & \swarrow\cdots\searrow & \\ & \delta_1\ \ldots\ \delta_n & \end{array}\right)^*$$

$$= (A \wedge \bigwedge\Gamma) \to (\bigvee\Delta \vee \Box\Sigma_1^* \vee \Box(\bigwedge\Pi \to (\bigvee\Lambda \vee \Box\Psi^*)) \vee \Box\Sigma_2^*) \qquad (1)$$

$$\to \bigwedge\Gamma \to (\bigvee\Delta \vee \Box\Sigma_1^* \vee \neg A \vee \Box(\bigwedge\Pi \to (\bigvee\Lambda \vee \Box\Psi^*)) \vee \Box\Sigma_2^*) \qquad (2)$$

$$\to \bigwedge\Gamma \to (\bigvee\Delta \vee \Box\Sigma_1^* \vee \Box\neg\Box A \vee \Box(\bigwedge\Pi \to (\bigvee\Lambda \vee \Box\Psi^*)) \vee \Box\Sigma_2^*) \qquad \text{PC, axiom } B$$

$$\to \bigwedge\Gamma \to (\bigvee\Delta \vee \Box\Sigma_1^* \vee \Box(\neg\Box A \vee (\bigwedge\Pi \to (\bigvee\Lambda \vee \Box\Psi^*))) \vee \Box\Sigma_2^*) \qquad (3), \text{PC}$$

$$\leftrightarrow \bigwedge\Gamma \to (\bigvee\Delta \vee \Box\Sigma_1^* \vee \Box((\Box A \wedge \bigwedge\Pi) \to (\bigvee\Lambda \vee \Box\Psi^*)) \vee \Box\Sigma_2^*) \qquad (4)$$

$$= (\Box A \wedge \bigwedge\Gamma) \to \left(\bigvee\Delta \vee \Box\Sigma_1^* \vee \Box\left(\begin{array}{c} \Box A,\Pi\vdash\Lambda \\ \swarrow\cdots\searrow \\ \delta_1\ \ldots\ \delta_n \end{array}\right)^* \vee \Box\Sigma_2^*\right) \qquad \text{def of } *$$

$$= \left(\begin{array}{ccc} & \Gamma\vdash\Delta & \\ \swarrow\cdots\swarrow & \downarrow & \searrow\cdots\searrow \\ \sigma_1\ldots\sigma_r & \Box A,\Pi\vdash\Lambda & \sigma_{r+1}\ldots\sigma_m \\ & \swarrow\cdots\searrow & \\ & \delta_1\ \ldots\ \delta_n & \end{array}\right)^* \qquad \text{def of } *$$

where

(1) $\Box\Sigma_1^* =_{df} \Box\sigma_1^* \vee \ldots \vee \Box\sigma_r^*$;
$\Box\Sigma_2^* =_{df} \Box\sigma_{r+1}^* \vee \ldots \vee \Box\sigma_m^*$;
$\Box\Psi^* =_{df} \Box\delta_1^* \vee \ldots \vee \Box\delta_n^*$.
(2) $\vdash_{\mathbf{PC}}((A \wedge B) \to (C \vee D)) \to (B \to (C \vee \neg A \vee D))$
(3) $\vdash_{\mathbf{PC}}(\Box A \vee \Box B) \to \Box(A \vee B)$
(4) $\vdash_{\mathbf{PC}}(\neg A \vee (B \to C)) \leftrightarrow ((A \wedge B) \to C)$

so transporting that implication along γ by Lemma 3, we obtain the thesis.

By axiom $\mathrm{T}: \Box A \to A$ we can immediately change

$$A,\Gamma \vdash \Delta \text{ into } \Box A,\Gamma \vdash \Delta$$

without changing the context.

By axiom 5: $\Diamond A \to \Box\Diamond A$ we can change

$$\begin{array}{ccc} & \Sigma\vdash\Theta & \\ \swarrow & & \searrow \\ A,\Gamma\vdash\Delta & & \Pi\vdash\Lambda \end{array} \quad \text{into} \quad \begin{array}{ccc} & \Sigma\vdash\Theta & \\ \swarrow & & \searrow \\ \Gamma\vdash\Delta & & \Box A,\Pi\vdash\Lambda \end{array}$$

without changing the context; in fact:

$$\begin{pmatrix} & & \Sigma \vdash \Theta & & \\ \swarrow\dots\swarrow & \swarrow & & \searrow & \searrow\dots\searrow \\ \sigma_1\dots\sigma_r & A,\Gamma\vdash\Delta & & \Pi\vdash\Lambda & \sigma_{r+1}\dots\sigma_m \\ & \swarrow\dots\searrow & & \swarrow\dots\searrow & \\ & \delta_1\ \dots\ \delta_n & & \lambda_1\ \dots\ \lambda_t & \end{pmatrix}^*$$

$$= \bigwedge \Sigma \to (\bigvee \Theta \vee \Box\Sigma_1^* \vee \Box((A \wedge \bigwedge \Gamma) \to (\bigvee \Delta \vee \Box\Psi^*)) \vee \Box(\bigwedge \Pi \to (\bigvee \Lambda \vee \Box\Phi^*)) \vee \Box\Sigma_2^*) \quad (1)$$

$$\to \bigwedge \Sigma \to (\bigvee \Theta \vee \Box\Sigma_1^* \vee (\Box A \to \Box(\bigwedge \Gamma \to (\bigvee \Delta \vee \Box\Psi^*))) \vee \Box(\bigwedge \Pi \to (\bigvee \Lambda \vee \Box\Phi^*)) \vee \Box\Sigma_2^*) \quad (2)$$

$$\leftrightarrow \bigwedge \Sigma \to (\bigvee \Theta \vee \Box\Sigma_1^* \vee \neg\Box A \vee \Box(\bigwedge \Gamma \to (\bigvee \Delta \vee \Box\Psi^*)) \vee \Box(\bigwedge \Pi \to (\bigvee \Lambda \vee \Box\Phi^*)) \vee \Box\Sigma_2^*) \quad (3)$$

$$\to \bigwedge \Sigma \to (\bigvee \Theta \vee \Box\Sigma_1^* \vee \Box\neg\Box A \vee \Box(\bigwedge \Gamma \to (\bigvee \Delta \vee \Box\Psi^*)) \vee \Box(\bigwedge \Pi \to (\bigvee \Lambda \vee \Box\Phi^*)) \vee \Box\Sigma_2^*) \quad (4)$$

$$\to \bigwedge \Sigma \to (\bigvee \Theta \vee \Box\Sigma_1^* \vee \Box(\bigwedge \Gamma \to (\bigvee \Delta \vee \Box\Psi^*)) \vee \Box(\neg\Box A \vee (\bigwedge \Pi \to (\bigvee \Lambda \vee \Box\Phi^*))) \vee \Box\Sigma_2^*) \quad (5)$$

$$\leftrightarrow \bigwedge \Sigma \to (\bigvee \Theta \vee \Box\Sigma_1^* \vee \Box(\bigwedge \Gamma \to (\bigvee \Delta \vee \Box\Psi^*)) \vee \Box((\Box A \wedge \bigwedge \Pi) \to (\bigvee \Lambda \vee \Box\Phi^*)) \vee \Box\Sigma_2^*) \quad (6)$$

$$= \bigwedge \Sigma \to \left(\bigvee \Theta \vee \Box\Sigma_1^* \vee \Box \begin{pmatrix} \Gamma \vdash \Delta \\ \swarrow\dots\searrow \\ \delta_1\ \dots\ \delta_n \end{pmatrix}^* \vee \Box \begin{pmatrix} \Box A,\Pi \vdash \Lambda \\ \swarrow\dots\searrow \\ \lambda_1\ \dots\ \lambda_t \end{pmatrix}^* \vee \Box\Sigma_2^*\right)$$

def of *

$$= \begin{pmatrix} & & \Sigma \vdash \Theta & & \\ \swarrow\dots\swarrow & \swarrow & & \searrow & \searrow\dots\searrow \\ \sigma_1\dots\sigma_r & \Gamma\vdash\Delta & & \Box A,\Pi\vdash\Lambda & \sigma_{r+1}\dots\sigma_m \\ & \swarrow\dots\searrow & & \swarrow\dots\searrow & \\ & \delta_1\ \dots\ \delta_n & & \lambda_2\ \dots\ \lambda_t & \end{pmatrix}^*$$

def of *

where

(1) $\Box\Sigma_1^* =_{df} \Box\sigma_1^* \vee \dots \vee \Box\sigma_r^*$;
$\Box\Sigma_2^* =_{df} \Box\sigma_{r+1}^* \vee \dots \vee \Box\sigma_m^*$;
$\Box\Psi^* =_{df} \Box\delta_1^* \vee \dots \vee \Box\delta_n^*$;
$\Box\Phi^* =_{df} \Box\lambda_1^* \vee \dots \vee \Box\lambda_t^*$.

(2) $\vdash_{\mathbf{PC}}((A \wedge B) \to C) \leftrightarrow (A \to (B \to C))$ and Necessitation

(3) $\vdash_{\mathbf{PC}}(A \to B) \leftrightarrow (\neg A \vee B)$

(4) Axiom 5

(5) $\vdash_{\mathbf{PC}}(\Box A \vee \Box B) \to \Box(A \vee B)$

(6) $\vdash_{\mathbf{PC}}(\neg A \vee (B \to C)) \leftrightarrow ((A \wedge B) \to C)$

so transporting that implication along γ by Lemma 3, we obtain the thesis.

By axiom 4: $\Box A \rightarrow \Box\Box A$ we can immediately change

$$\Box\Box A, \Gamma \vdash \Delta \text{ into } \Box A, \Gamma \vdash \Delta$$

without changing the context.

Finally the axiom D affects neither the relation R nor the rule $\Box\vdash$ (and in fact, is related to the specific empty rule).

Suitably combining these properties, we can move from a formula A of a sequent "a" to a formula $\Box A$ of a sequent "b", when aRb, proving for any NL that the translation of the rule $\Box\vdash$ is well-defined.

Finally, the empty rule is well-defined for any "serial" systems, i.e., for any system containing the axiom D: $\Box A \rightarrow \Diamond A$:

$$\left(\begin{array}{ccc} & \Gamma \vdash \Delta & \\ \swarrow \dots \swarrow & \downarrow & \searrow \dots \searrow \\ \sigma_1 \dots \sigma_r & \vdash & \sigma_{r+1} \dots \sigma_m \end{array}\right)^*$$

$$= \bigwedge \Gamma \rightarrow (\bigvee \Delta \vee \Box\sigma_1^* \vee \dots \vee \Box\sigma_r^* \vee \Box(\vdash)^* \vee \Box\sigma_{r+1}^* \vee \dots \vee \Box\sigma_m^*) \qquad \text{def of }^*$$

$$\leftrightarrow \bigwedge \Gamma \rightarrow (\bigvee \Delta \vee \Box\sigma_1^* \vee \dots \vee \Box\sigma_r^* \vee \Box\bot \vee \Box\sigma_{r+1}^* \vee \dots \vee \Box\sigma_m^*) \qquad \text{PC, (1)}$$

$$\rightarrow \bigwedge \Gamma \rightarrow (\bigvee \Delta \vee \Box A \vee \Box\sigma_1^* \vee \dots \vee \Box\sigma_m^*) \qquad \text{PC, (2)}$$

$$= \left(\begin{array}{c} \Gamma \vdash \Delta \\ \swarrow \dots \searrow \\ \sigma_1 \quad \dots \quad \sigma_m \end{array}\right)^* \qquad \text{def of }^*$$

where

(1) $(\vdash)^* = (\bigwedge \varnothing \rightarrow \bigvee \varnothing) = (\top \rightarrow \bot) \leftrightarrow \bot$;
(2) $\vdash_{\mathbf{KD}} \Box\bot \rightarrow \bot$

so transporting that equivalence along γ by Lemma 3, we have that the required implication is a **KD**-theorem, obtaining the thesis. This completes the proof of Theorem 5.

3 Completeness and semantic cut-elimination A sequent proof appears as a reverted, at most binary, finite tree, whose root is the end-sequent (at the bottom), and whose terminal leaves are initial sequents (at the top). The way of moving along such reversed trees allows us to prove completeness: in fact, when moving downward we consider that tree as a "proof", otherwise when moving upward we consider it as an attempt at constructing a "countermodel"; we really prove that when a usual sequent is not cut-free provable we can construct a countermodel of it (or, more exactly, of its Schütte translation), so that it is not valid.

Theorem 6 (Completeness of the NLs-calculi) *For any* NL Λ, *if R is Λ-valid then A is Λ-provable (without any use of cut).*

Proof: As usual, we prove the statement by contraposition: if A is not Λ-provable (without any use of cut) then A is not Λ-valid. We proceed by several steps, using a simplified version of the proof used in Girard [7] for **PC**, adapted to modal systems (note we have only propositional symbols, and transform Schütte valuations directly into Kripke valuations). Namely, for any NL Λ:

1. we fix a class of functions on Kripke frames (the Schütte valuations) that preserve the values true **t** and false **f** of formulas when going along a proof-tree as countermodel and when requiring every formula that appears on the left part of a sequent to be true, and any on the right to be false. Really, we define such functions by imposing conditions that must be respected when going along the tree as a proof, so that we can define them by an induction on the complexity of formulas (that increases downward);

2. we prove that for any usual sequent $\Gamma \vdash \Delta$ that is not cut-free provable there is a Schütte valuation on a Kripke frame such that for some world the value of $\bigwedge\Gamma$ is **t** while the value of $\bigvee\Delta$ is **f**;

3. we prove that any Schütte valuation can be reduced to a binary valuation (i.e., a valuation that can take only the values **t** and **f**, not **u**) that preserves the values **t** and **f** (so, there is a Kripke model where $\Gamma \vdash \Delta$ is not valid, proving completeness);

4. we introduce three-valued models, proving both that any Schütte valuation can be reduced to a three-valued model valuation that preserves the values **t** and **f** (so, there is a three-valued model where $\Gamma \vdash \Delta$ is not valid) and that any three-valued model valuation can be reduced to a binary valuation that preserves the values **t** and **f** (so, proving completeness in another way).

Let us examine each step:

1. Given the set of well-founded formulas of the language, wff(**L**), a set of worlds W, a strict tree-ordering on $W, \rightarrow$, a relation R extending $\rightarrow$ and according with the properties of the accessibility for the system Λ, and a set of values $\{\mathbf{t},\mathbf{f},\mathbf{u}\}$, we define a Schütte valuation as a function $S: W \times \text{wff}(\mathbf{L}) \rightarrow \{\mathbf{t},\mathbf{f},\mathbf{u}\}$ such that, for any $w, w' \in W$:

if $S(w,A) \neq \mathbf{f}$ and $S(w,B) \neq \mathbf{f}$	then $S(w,A \wedge B) \neq \mathbf{f}$
if $S(w,A) \neq \mathbf{t}$ or $S(w,B) \neq \mathbf{t}$	then $S(w,A \wedge B) \neq \mathbf{t}$
if $S(w,A) \neq \mathbf{f}$ or $S(w,B) \neq \mathbf{f}$	then $S(w,A \vee B) \neq \mathbf{f}$
if $S(w,A) \neq \mathbf{t}$ and $S(w,B) \neq \mathbf{t}$	then $S(w,A \vee B) \neq \mathbf{t}$
if $S(w,A) \neq \mathbf{t}$	then $S(w,\neg A) \neq \mathbf{f}$
if $S(w,A) \neq \mathbf{f}$	then $S(w,\neg A) \neq \mathbf{t}$
if $S(w,A) \neq \mathbf{t}$ or $S(w,B) \neq \mathbf{f}$	then $S(w,A \rightarrow B) \neq \mathbf{f}$
if $S(w,A) \neq \mathbf{f}$ and $S(w,B) \neq \mathbf{t}$	then $S(w,A \rightarrow B) \neq \mathbf{t}$
if exists w' s.t. wRw' and $S(w',A) \neq \mathbf{t}$	then $S(w,\Box A) \neq \mathbf{t}$
if for every w' s.t. wRw' $S(w',A) \neq \mathbf{f}$	then $S(w,\Box A) \neq \mathbf{f}$.

2. We define the domain W, the relations $\rightarrow$ and R, and the function S that falsifies $\Gamma \vdash \Delta$ on the basis of an infinite branch of an inductively constructed reversed tree of semantic modal sequents. Namely, we start from $\Gamma \vdash \Delta$, and at any level of depth, we examine every sequent of that level, giving rise to the sequents of the next level: since any sequent can construct at most two other sequents, at any level we have a finite number of sequents; for any of those sequents, we examine only one formula of only one world (if any) terminating the examination of that level in a finite number of steps. At the level n, for any sequent we need a list of its worlds, and for any world we need a list of its formulas: we examine the first formula of the first world in those lists, possibly we add new worlds or new formulas at the end of such lists (to not bypass the queue)

and then we put the world just examined and formula at the end of the corresponding lists (rotating them cyclically, so that in a finite number of steps we can reach any formula of any world).

For the sake of simplicity, an index n stresses only the level where a sequent γ_n occurs, without specifying the branch (in fact we are really only interested in one infinite branch of that tree); list_n and $\text{list}_n(v)$ denote the corresponding list of worlds and, for any world v, the list of formulas, respectively.

Finally, since the worlds of a semantic modal sequent are really usual sequents, with a left and a right part, and since those parts can change at any step (actually, they can only increase), we need two other lists, namely $\text{left}_n(v)$ and $\text{right}_n(v)$, the left and the right sequence of formulas of the world v at the step n, respectively.

So, let $\gamma_0 = \langle W_0, \rightarrow_0, R_0 \rangle$ with $W_0 = \{v_0\}$, $\text{left}(v_0) = \Gamma$, $\text{right}(v_0) = \Delta$, $\rightarrow_0 = \varnothing$, and let R_0 be the suitable closure of $\rightarrow_0$; also, let $\text{list}_0 = v_0$ (the listing of the worlds of W_0) and $\text{list}_0(v_0) = \Gamma, \Delta$ (the listing of the formulas of v_0).

Given $\gamma_n = \langle W_n, \rightarrow_n, R_n \rangle$ with $\text{list}_n = v_0, \ldots, v_s$ (never empty, since only increasing) and $\text{list}_n(v_0) = A_1, \ldots, A_p$ (possibly empty), let γ_{n+1} (possibly γ'_{n+1} and γ''_{n+1}), list_{n+1} and $\text{list}_{n+1}(v)$ (for $v \in W_{n+1}$) be defined in the following way (to simplify notation, all that we do not name remains unchanged at the step $n+1$):

a. if a formula A occurring both in the left and in the right part of v_0 exists, then we stop the construction of that branch of the tree (we have reached an initial sequent, except some weakening and exchanges); otherwise:
b. if $\text{list}_n(v_0) = \varnothing$ then $\gamma_{n+1} = \gamma_n$, but $\text{list}_{n+1} = v_1, \ldots, v_s, v_0$;
c. if A_1 is an atomic formula and the system Λ does not contain the axiom D, then let $\gamma_{n+1} = \gamma_n$, but $\text{list}_{n+1}(v_0) = A_2, \ldots, A_p, A_1$, and $\text{list}_{n+1} = v_1, \ldots, v_s, v_0$;
d. if A_1 is an atomic formula and the system Λ contains the axiom D, then let $W_{n+1} = W_n \cup \{v_{s+1}\}$, $\rightarrow_{n+1} = \rightarrow_n \cup \{v_0 \rightarrow v_{s+1}\}$, R_{n+1} be the suitable closure of $\rightarrow_{n+1}$ (it is easy to prove that it contains R_n), $\text{list}_{n+1} = v_1, \ldots, v_s, v_{s+1}, v_0$, $\text{list}_{n+1}(v_0) = A_2, \ldots, A_p, A_1$, $\text{left}(v_{s+1}) = \varnothing$, $\text{right}(v_{s+1}) = \varnothing$ and $\text{list}_{n+1}(v_{s+1}) = \varnothing$ (we really only add a serial queue for v_0);
e. if A_1 has a propositional connective as the principal one, let the left and right parts of v_0 and $\text{list}_{n+1}(v_0)$ be modified in the same way indicated in [7], Theorem 3.1.9, in accord with the same cases; furthermore, let $W_{n+1} = W_n$, $\rightarrow_{n+1} = \rightarrow_n$, $R_{n+1} = R_n$, $\text{list}_{n+1} = v_1, \ldots, v_s, v_0$. We recall that here, in some cases, we really obtain two sequents γ'_{n+1} and γ''_{n+1} splitting the tree;
f. otherwise, if A_1 is $\Box A$, and it occurs in the left part of v_0, and $R(v_0) = \{v \in W : v_0 R v\} \neq \varnothing$, then let $\text{left}_{n+1}(v_0) = \text{left}_n(v_0)$, $\text{list}_{n+1}(v_0) = A_2, \ldots, A_p, A_1$, $\text{left}_{n+1}(v) = A, \text{left}_n(v)$, and $\text{list}_{n+1}(v) = \text{list}_n(v), A$ for any $v \in R(v_0)$; let also $W_{n+1} = W_n$, $\rightarrow_{n+1} = \rightarrow_n$, $R_{n+1} = R_n$, $\text{list}_{n+1} = v_1, \ldots, v_s, v_0$;
g. if A_1 is $\Box A$, and it occurs in the left part of v_0, $R(v_0) = \varnothing$, and the system Λ does not contain the axiom D, then let $\gamma_{n+1} = \gamma_n$, but $\text{list}_{n+1}(v_1) = A_2, \ldots, A_p, A_1$, and $\text{list}_{n+1} = v_1, \ldots, v_s, v_0$;
h. if A_1 is $\Box A$, and it occurs in the left part of v_0, $R(v_0) = \varnothing$, and the system Λ contains the axiom D, then let $W_{n+1} = W_n \cup \{v_{s+1}\}$, $\rightarrow_{n+1} = \rightarrow_n \cup \{v_0 \rightarrow v_{s+1}\}$, R_{n+1} be the suitable closure of $\rightarrow_{n+1}$ (it is easy to prove that it contains R_n),

$\text{list}_{n+1} = v_1, \ldots, v_s, v_{s+1}, v_0$, $\text{left}_{n+1}(v_0) = \text{left}_n(v_0)$, $\text{list}_{n+1}(v_0) = A_2, \ldots, A_p, A_1$, $\text{left}(v_{s+1}) = \varnothing$, $\text{right}(v_{s+1}) = \varnothing$ and $\text{list}_{n+1}(v_{s+1}) = \varnothing$ (we really only add a serial queue for v_0);

i. finally, if A_1 is $\Box A$, and it occurs on the right part of v_0, then let $W_{n+1} = W_n \cup \{v_{s+1}\}$, $\rightarrow_{n+1} = \rightarrow_n \cup \{v_0 \rightarrow v_{s+1}\}$, R_{n+1} be the suitable closure of $\rightarrow_{n+1}$ (it contains R_n), $\text{list}_{n+1} = v_1, \ldots, v_s, v_{s+1}, v_0$, $\text{right}_{n+1}(v_0) = \text{right}_n(v_0)$, $\text{list}_{n+1}(v_0) = A_2, \ldots, A_p, A_1$, $\text{right}(v_{s+1}) = A$, $\text{left}(v_{s+1}) = \varnothing$ and $\text{list}_{n+1}(v_{s+1}) = A$.

If γ_{n+1} is (both γ'_{n+1} and γ''_{n+1} are) cut-free provable, then γ_n is cut-free provable too. In fact, in case (a), γ_{n+1} does not exist, since we have stopped the construction on γ_n, an initial sequent; in cases (b), (c), (g) $\gamma_{n+1} = \gamma_n$, cut-free provable by hypothesis; in case (e), we prove the thesis as in [7]; in cases (d), (h) we obtain a cut-free proof by using the empty rule; in case (f), we use the rules $\Box\vdash$ and contraction; finally, in case (i), we use the rules $\vdash\Box$ and contraction, proving the proposition. So, since we stop the construction of the tree on initial sequents, if the tree is finite then it is a cut-free proof of γ_0, against the hypothesis; thus, the tree must be an infinite, denumerable, (at most binary) tree of sequents. By König lemma, an infinite branch exists: let $\gamma_0, \ldots, \gamma_i, \ldots$ $(i < \omega)$ be the list of the sequents on that branch; let $W = \bigcup\{W_i : W_i \in \gamma_i\}$, $\rightarrow = \bigcup\{\rightarrow_i : \rightarrow_i \in \gamma_i\}$, $R = \bigcup\{R_i : R_i \in \gamma_i\}$ (for non-serial systems) or let R be $\bigcup\{R_i : R_i \in \gamma_i\}$ plus the reflexive closure of non-serial worlds (for serial systems); let also $\text{left}(w) = \bigcup\{\text{left}_i(w) : i < \omega\}$ and $\text{right}(w) = \bigcup\{\text{right}_i(w) : i < \omega\}$, for any $w \in W$.

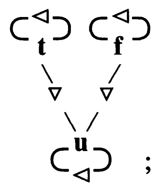
R satisfies the properties required for the accessibility relation of the system Λ: in fact, the union of a chain of reflexive (or symmetric, or transitive, or euclidean) relations is also reflexive (or symmetric, or transitive, or euclidean). In the case of serial systems, by construction, only the empty worlds have no other accessible worlds, so that we can make them accessible from themselves without going against any formula. Furthermore, the relation R remains possibly reflexive, symmetric or transitive, and also euclidean (in this case, recalling that such worlds are the terminal leaves of a $\rightarrow$-tree, that R contains $\rightarrow$ and that R can also only be a symmetric, reflexive, or transitive extension of $\rightarrow$).

Now, let S be the valuation defined as $S(w,A) = \mathbf{t}$ iff A occurs in $\text{left}(w)$, $S(w,A) = \mathbf{f}$ iff A occurs in $\text{right}(w)$, $S(w,A) = \mathbf{u}$ otherwise. S is well-defined: on the contrary, let us suppose a formula A both true and false in a world w; since formulas are never deleted from a part of a world (i.e., $\text{left}_n(v) \subseteq \text{left}_{n+1}(v)$ and $\text{right}_n(v) \subseteq \text{right}_{n+1}(v)$), from an index s on, A occurs both in the left and in the right part of w; so, we had to stop the construction of that branch at such a level, against the hypothesis that the branch is infinite.

Furthermore, S is a Schütte valuation: we prove only that S respects the last modal condition on Schütte valuations, since we can prove that S satisfies the other modal condition in a dual way, and that satisfies the propositional conditions as usual (see [7]). Let us assume $S(w,\Box A) = \mathbf{f}$: by definition of S, $\Box A \in \text{right}(w)$; so, by definition of W, from an index r on, $\Box A$ must appear on the right of w; by construction of the tree, when $\Box A$ is to be examined after a finite number m of steps, A must appear on the right of some w' with $w \rightarrow_{r+m} w'$; so $S(w',A) = \mathbf{f}$ with wRw' (by the definitions of W, $\rightarrow$ and R): by contraposition, we have proved that S agrees with the last modal condition on Schütte valuations.

Finally, by construction, $S(v_0,\bigwedge\Gamma) = \mathbf{t}$ and $S(v_0,\bigvee\Delta) = \mathbf{f}$. This completes Step 2.

3. Let $\lhd$ be an ordering on $\{\mathbf{t},\mathbf{f},\mathbf{u}\}$ (in [7] the symbol $\rhd$ was used) defined as:



$\lhd$ induces an ordering on the Schütte valuations:

$$S \lhd T \text{ iff } S(A) \lhd T(A) \text{ for every } A \in \mathrm{wff}(\mathbf{L}).$$

It is easy to see that any $\lhd$-chain has an upper bound, so that for any Schütte valuation S there is a $\lhd$-maximal valuation V such that $S \lhd V$; but a maximal valuation must be a binary one, and, by the conditions on Schütte valuations, must be a Kripke valuation. Since $\lhd$ maintains the values $\mathbf{t}$ and $\mathbf{f}$, $V(v_0,\bigwedge\Gamma) = \mathbf{t}$ and $V(v_0,\bigvee\Delta) = \mathbf{f}$, so that $V(v_0,\bigwedge\Gamma \to \bigvee\Delta) = \mathbf{f}$; as a particular case, when the sequent is $\vdash A$ we have $V(v_0,A) = \mathbf{f}$, proving completeness.

4. A three-valued model is a triple $\langle W,R,m\rangle$ where W is a non-empty set, R is a binary relation on W, in accord with the properties of accessibility for any system Λ, $m: W \times \mathrm{wff}(\mathbf{L}) \to \{\mathbf{t},\mathbf{f},\mathbf{u}\}$ is a function defined by an induction on the complexity of formulas, as follows (see [7]): given $w \in W$ and $R(w) = \{w' : wRw'\}$,

if $m(w,A) = \mathbf{t}$ and $m(w,B) = \mathbf{t}$	then $m(w,A \wedge B) = \mathbf{t}$
if $m(w,A) = \mathbf{f}$ or $m(w,B) = \mathbf{f}$	then $m(w,A \wedge B) = \mathbf{f}$
if $m(w,A) = \mathbf{u}$ and $m(w,B) \neq \mathbf{f}$, or vice versa	then $m(w,A \wedge B) = \mathbf{u}$
if $m(w,A) = \mathbf{t}$ or $m(w,B) = \mathbf{t}$	then $m(w,A \vee B) = \mathbf{t}$
if $m(w,A) = \mathbf{f}$ and $m(w,B) = \mathbf{f}$	then $m(w,A \vee B) = \mathbf{f}$
if $m(w,A) = \mathbf{u}$ and $m(w,B) \neq \mathbf{t}$, or vice versa	then $m(w,A \vee B) = \mathbf{u}$
if $m(w,A) = \mathbf{t}$	then $m(w,\neg A) = \mathbf{f}$
if $m(w,A) = \mathbf{f}$	then $m(w,\neg A) = \mathbf{t}$
if $m(w,A) = \mathbf{u}$	then $m(w,\neg A) = \mathbf{u}$
if $m(w,A) = \mathbf{f}$ or $m(w,B) = \mathbf{t}$	then $m(w,A \to B) = \mathbf{t}$
if $m(w,A) = \mathbf{t}$ and $m(w,B) = \mathbf{f}$	then $m(w,A \to B) = \mathbf{f}$
if $m(w,A) = \mathbf{u}$ and $m(w,B) \neq \mathbf{t}$	then $m(w,A \to B) = \mathbf{u}$
if $m(w,A) \neq \mathbf{f}$ and $m(w,B) = \mathbf{u}$	then $m(w,A \to B) = \mathbf{u}$
if for every $w' \in R(w)$ $m(w',A) = \mathbf{t}$	then $m(w,\Box A) = \mathbf{t}$
if exists $w' \in R(w)$ with $m(w',A) = \mathbf{f}$	then $m(w,\Box A) = \mathbf{f}$
if for every $w' \in R(w)$ $m(w',A) \neq \mathbf{f}$ and exists $w' \in R(w)$ $m(w',A) = \mathbf{u}$	then $m(w,\Box A) = \mathbf{u}$.

A three-valued valuation is really a Schütte valuation where the values of formulas are strictly determined by the values of their subformulas (see [7]): in fact, when both the value of A and the value of B is $\mathbf{t}$ then the value of $A \wedge B$ could be either **t** or **u** for a Schütte valuation, whereas it must only be **t** for a three-valued valuation.

To stress the link between three-valued valuations and Schütte valuations, we introduce a refinement of the ordering $\lhd$; namely, we define the relation $\leqslant$ on the Schütte valuations as:

$$S \leqslant T \text{ iff } S \lhd T \text{ and } S|\mathcal{P} = T|\mathcal{P}.$$

It is easy to see that any $\leqslant$-chain has an upper bound, so that for any Schütte valuation S there is a $\leqslant$-maximal valuation m such that $S \leqslant m$; and it is also clear that such a maximal valuation must be a three-valued valuation. Since $\leqslant$ (as $\lhd$) maintains the values **t** and **f**, $m(v_0,\bigwedge\Gamma) = \mathbf{t}$, $m(v_0,\bigvee\Delta) = \mathbf{f}$ and $m(v_0,\bigwedge\Gamma \to \bigvee\Delta) = \mathbf{f}$. Furthermore, we can restrict the relation $\lhd$ to the three-valued valuations: reasoning as above, for any three-valued valuation m there is a $\lhd$-maximal valuation V such that $m \lhd V$; such a V must be a Kripke valuation and $V(v_0,\bigwedge\Gamma \to \bigvee\Delta) = \mathbf{f}$; as a particular case, when the sequent is $\vdash A$ we have $V(v_0,A) = \mathbf{f}$, proving, in another way, completeness. This finishes the proof.

Finally, as usual immediate corollaries of cut-elimination, we have both the consistency of those fifteen main normal modal logics and the subformula property for them:

Corollary 7 (Consistency and subformula property) *Every system among* **K**, **KB**, **KD**, **KT**, **K4**, **K5**, **KBD**, **KBT**, **KB4**, **KD4**, **KD5**, **KD45**, **K45**, **KT4**, **KT5** *is consistent; furthermore, for each of them the subformula property holds.*

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