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Zermelo, Reductionism, and the Philosophy of Mathematics

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Abstract Whereas Zermelo's foundational program is implicitly reductionist, the precise character of his reductionism is quite unclear. Although Zermelo follows Hilbert methodologically, his philosophical viewpoint in 1908 is broadly at odds with that of Hilbert. Zermelo's interest in the semantic paradoxes permits an intuitive concept of mathematical definability to play an important role in his formulation of axioms for set theory. By implication, definability figures in Zermelo's philosophical concept of set, which is seen to be nonstructural in character. Zermelo's advocacy of universal definability is intended to blunt tensions between platonists and constructivists. Finally, the method of justification of mathematical axioms is taken to be of an empirical and public character, at least in part, and, as a consequence, threatens Zermelo's foundational program.

What foundational role, if any, is set theory to play? One relatively straightforward answer has come to be called reductionism. Let us take reductionism to encompass the following claims:

(R1) All mathematical objects are sets.
(R2) All mathematical concepts are definable in terms of membership.
(R3) All mathematical truths are set-theoretic truths.

Reductionism embraces set theory as the metaphysical foundation of the mathematical sciences: mathematical objects, being sets, have whatever sort of reality sets have. We note that, at least according to one common understanding of what it is for a proposition to be true, (R3) is not independent of (R1) and (R2): it is not clear that it makes any sense to affirm (R3) while denying (R1) and (R2), or vice versa. Also, as it stands, (R2) remains vague in that the nature of the definability involved here is left entirely open.

At the other extreme is the view—now widely if not universally accepted—that set theory, despite its important subject, can be foundational only in that

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mathematical theories are *interpretable* in models of set theory in the usual sense that (1) the objects about which such a theory T purports to speak can be tentatively identified with elements of the domain of the set-theoretic model M and (2) the nonlogical predicates of T can be tentatively understood in terms of the membership relation of M in such a way that the theorems of T come out true in M. Adherents of this more modest view deny (R1) through (R3) but readily grant the truth of analogues:

(R1′) All mathematical objects may be understood as sets.
(R2′) All mathematical concepts may be understood in terms of membership.
(R3′) All mathematical truths may be understood as set-theoretic truths.

One is careful to distinguish definability ((R1) and (R2)) from mere interpretability ((R1′) and (R2′)). (R1′) through (R3′) are uncontroversial in themselves. Moreover, (R1) through (R3) presuppose (R1′) through (R3′) but not vice versa.

It is striking that Zermelo's early papers on set theory contain no clear reductionist statement. Nonetheless, Zermelo's goal is clearly a foundational program, and this program is implicitly reductionist in character. The present paper is an investigation of Zermelo's views on the philosophy of mathematics with emphasis upon his earliest publications. As shall become clear, I shall place considerable weight upon his discussion of the paradoxes and, in particular, of the Richard paradox. I shall begin by considering briefly the nature of Zermelo's reductionism. A second paper will explore the philosophical viewpoint discernable in the papers of the 1930's.

I would claim that Zermelo's axioms for set theory are intended (1) to lend a new rigor to set theory by revealing the assumptions underlying earlier work and (2) to provide a foundation in the sense of Descartes for set theory and, via reductionism, for mathematics (arithmetic and analysis) as a whole. This is to place reductionism at the very center of Zermelo's foundational program; in writing about the *Grundlagen der Mengenlehre*, he would be, by implication, speaking of the *Grundlagen der Mathematik*. If Zermelo never states (R1) through (R3), this is in part because he takes the work of Cantor and Dedekind (together with some set-theoretic definition of the natural numbers) to have already demonstrated reductionism at least with respect to finite numbers (see Gillies [6]). The task that Zermelo then sets for himself, on this interpretation, is the provision of a secure foundation for set theory so that the reduction of his predecessors, regarded as a fait accompli, can be seen to have *merit*.

What has just been presented is probably the accepted view of Zermelo: questing for rigor and certainty while assuming an inherited reduction of *finite number*. Clearest support for a reductionist interpretation of Zermelo's thought is to be found in Zermelo's objection to what we would now view as metatheoretic definitions by induction.[1] The Accepted View, as I shall refer to it, nonetheless requires some qualification in the face of apparently disconfirming evidence. This evidence involves (1) the central concepts of order and function, (2) the indeterminate role of number objects in Zermelo's system, and (3) anomalous features of Zermelo's axiomatization. I consider each in turn.

The concepts of order and function present apparent obstacles to attributing to Zermelo either objects- or concepts-reductionism, since, in the period before the Hausdorff-Wiener-Kuratowski definition of the ordered pair, both concepts would present prominent counterexamples to (R1′) and (R2′) and hence to (R1) and (R2). Zermelo's position would at best be analogous to that of the modern physicalist who feels confident that future advances in neurobiology will ultimately reveal the physical basis of all mentalist concepts despite the present unavailability of the required reduction.

Further, the precise role, if any, of number objects in Zermelo's theory of sets is less than clear; this unclarity in turn obscures the character of his reductionism. It is true that certain of Zermelo's remarks around 1908 indicate an overall intention to eschew number objects: thus versions of the principle of mathematical induction are presented starting from the definition of finite set rather than finite number (see Zermelo [18]). But other remarks from the same period point in another direction. For example, Zermelo's discussion of the paradoxes in his [16] assumes real number objects, at least on the face of it.[2] Also, the conclusion of Zermelo's [17] has him characterizing reals, in the usual way, in terms of sets of rationals:

> In practice, every irrational is determined by a "cut," that is, by an infinite set of rational numbers. Similarly, the limit of a function can always be defined by an infinite set of arguments and values.

One has no choice but to conclude that the role of number objects in Zermelo's set theory is largely an unsettled matter.[3] But this almost certainly means that Zermelo's attitude toward reductionism—and in particular (R1)—is unsettled as well, since the inherited reduction of the Accepted View consists largely of set-theoretic definitions of these very number objects. This point concerns not whether Zermelo is a reductionist but, rather, the nature of any reductionism which may be attributed to him. In any case, the Accepted View, which may be an adequate description of Zermelo's ultimate position, does not reflect the ambiguous character of his early thought. This is not to suggest that Zermelo doubts the possibility of a successful reduction of *finite number*. Rather, it may be only that such a reduction appears to him to be quite useless in the absence of any corresponding reduction of *ordinal* and *function*.[4]

Finally, Zermelo's own definite properties pose a problem for (R1′) and hence for (R1). His own manner of proceeding indicates that they form part of the subject matter of set theory and yet they are sets only on pain of contradiction. Of course, one can deny that they are mathematical *objects*, and this is perhaps Zermelo's point of view. But if they then offer no challenge to (R1), they do present a counterexample to (R2′) and (R2), since the *definiteness* concept itself represents the limiting case of an informal mathematical concept that, from Zermelo's point of view, would, in all probability, not itself be characterizable set theoretically. Another issue for (R1′) and (R1) would be the status of the domain of sets itself. Zermelo discusses this domain as if it were indeed a mathematical object. But once again it cannot be a set.

One must conclude that Zermelo's reductionism is programmatic in character, consisting of the view that the needed reduction, while not yet in place, is yet feasible (see Hallett [7]). The analogy to physicalism, previously mentioned, is useful here. Zermelo's inclusion of urelements within the set-theoretic domain is not really hard to square with anything like (R1). Although it is not clear why Zermelo feels that he needs urelements, ultimately no mathematical object will be reduced to an urelement. At worst, mathematical objects will be reduced to sets with urelements in their transitive closures. So there seems to be no conflict with (R1). Finally, Zermelo explicitly asserts:

> (N1′) The set . . . contain[ing] the elements { }, {{ }}, {{{ }}}, and so forth, . . . may be called the *number sequence*, because its elements can take the place of the numerals. ([16], p. 205)

This is, of course, a considerable understatement if Zermelo in fact subscribes to (R1) or the analogous claim concerning the natural numbers. But given that the remark appears during the same period during which number objects are in fact not clearly a part of Zermelo's conception, there is an obvious, alternative reading of Zermelo's remark that leaves room for (R1). I conclude that, despite the cited problems, Zermelo should be regarded as a programmatic reductionist.

As my discussion turns now to other issues that arise within the philosophy of mathematics, it must be noted that Zermelo is hardly a philosopher of mathematics. In fact, Zermelo is usually content to produce technical results and usually shuns philosophical discussion. When he does engage in philosophical discussion, this is often in order to bolster support for such a result, as in his philosophical defense of the use of impredicatively defined objects in his proofs of the well-ordering principle. But there are other instances in which it is *philosophical* issues that motivate Zermelo's *mathematical* program. Here I am thinking in particular of Zermelo's late theory of mathematical systems based on a generalized notion of well-foundedness, where the motivation is a philosophically based rejection of finitary first-order systems of the sort to which Gödel's incompleteness and undecidability results apply. I shall take Zermelo's starting point in his earliest period to be not reductionism per se but rather a technical quest for rigor and certainty—in particular, a quest for results that could lend support to a feasible philosophical reductionism. This seems to be the most plausible reading of Zermelo's various remarks to the effect that what he is doing is providing a foundation for mathematics. Although Zermelo in his early period surely cannot hold reductionism to have been *demonstrated*, at the same time he is no doubt inclined toward reductionism: clearly his own work on the foundations of set theory would gain in importance if (R1) through (R3) were true. Moreover, reductionism sometimes provides a plausible explanation for Zermelo's point of view on a given issue.

1 Hilbert and Zermelo: A Shared Methodology The methodological context of Zermelo's [16] is the axiomatic method described by Hilbert, according to which the principal foundational task of the mathematician is the introduction of axioms for given conceptual spheres. Mainstream mathematical activity then consists of the investigation of the consequences of the adopted axioms. The axiomatic method, which forms the methodological component of Hilbert's early program, falls naturally into four parts according to his conception (see Hilbert [11]). First, one determines the intuitive nature of the intended objects of study, and this involves fixing upon some (typically small) network of concepts and operations applicable to and constitutive of these intended objects. One next goes about the presentation of central propositions (axioms) that collectively "define" these concepts and operations. In the typical case, a certain adjustment is then required whereby the initially chosen axioms, having turned out to be derivable in fact from certain deeper propositions, are replaced by the latter. Finally, one must demonstrate that the axioms yield neither too much (consistency) nor too little (completeness). One also establishes that each axiom is independent of the others as well as that there are but finitely many axioms (schemata). Thus, putting a conceptual sphere in order takes the form: (1) intuitive concept identification, (2) initial axiomatization, (3) modification of initial axiomatization, and (4) demonstration of adequacy of modified axiomatization. In general, step (1) may serve later as justification for the axioms of (2) and (3), but in Hilbert's own case this role for (1) must not be exaggerated. In the particular case of sets, where there is potential for paradox, (1) will presumably speak both to the *nature* of sets as well as to the issue of *which sets exist*. The contemporary philosopher of mathematics is likely to feel the absence of a step between (1) and (2) here, whereby one settles upon a certain language or symbolic notation in which to present the axioms. However, Hilbert during this period places relatively little importance on symbolic notation, when compared with the Peano school, and Zermelo follows Hilbert in this regard.

In his [9], Hilbert completes this task for Euclidean geometry. During the same period, he offers a set of axioms for the real number system in Hilbert [10]. Zermelo in [16] sets out to accomplish for Cantorian set theory what Hilbert has accomplished for geometry and the reals. The paper's very title suggests this. As for content, it consists of the presentation of a collection of seven axioms and the derivation of several consequences significant for Cantor's theory of equivalence. Zermelo's remarks in [16] do indeed reflect adherence to the Hilbert methodology. Thus, he expresses concern that he has been unable as yet to prove the consistency of his postulates, and he raises the issue of completeness as well.[5] There is little doubt that Zermelo is following Hilbert *methodologically*. It is equally clear, however, that he takes nothing from Hilbert *philosophically*.

2 Hilbert and Zermelo: Diverging Philosophical Viewpoints To complement the axiomatic method, Hilbert espouses a philosophical conception of axiom systems that is an early prototype of our own *model-theoretic viewpoint*. According to Hilbert's conception, axiom systems have *models*, and this in turn leads him to certain important new ideas concerning mathematical existence and truth. Thus, given a plurality of models of the axioms, it no longer makes any sense to speak of a mathematical object existing independently of the domains of those models: so an existence theorem is now interpreted as meaning only that an object of some description exists in every model of the axioms. Similarly, it now makes no sense to speak of a proposition being *true* independent of those models. In this manner, questions of mathematical existence and truth cease to be central and are replaced by the issues of consistency and completeness of formal systems. The consequences of fully embracing the model-theoretic viewpoint are just such "deductivist" views.

The plurality issue is worth emphasizing. Even if Hilbert's conception is not fully model-theoretic in certain respects, still his correspondence with Frege from the years 1899–1900 as well as his unpublished lectures on geometry from the 1890's indicate that Hilbert is resolutely committed to the view that axiom systems in general admit a plurality of models. In this regard Zermelo is really quite different, since he permits but a very limited plurality: in essence, domains may vary according to Zermelo only with respect to urelements. But more importantly, Zermelo's conception in [16] is not that of any truly model-theoretic viewpoint.[6]

Zermelo plainly parts company with Hilbert when he mentions that "the further, more philosophical, question about . . . the extent to which [these axioms] are valid will not be discussed" ([16], p. 200). Similarly, when Zermelo asserts that axioms are justified by their being shown to be intuitively evident and necessary for the development of mathematics (see Zermelo [15], p. 187), he expresses a view that Hilbert can hardly share. If Zermelo does subscribe to any truly model-theoretic conception, then such remarks are strangely misleading. For such a conception, as described above, entails a certain disengagement with regard to mathematical truth, while Zermelo's remarks suggest no such reserve. Zermelo is hardly dismissing the truth issue. On the contrary, he shuns Hilbert's agnostic construal of mathematical axioms in favor of the ancient doctrine that mathematics is an a priori science resting upon self-evident truths ([18]). The platonism implicit—on one reading—in the quoted remark concerning validity can, of course, be squared with the model-theoretic viewpoint if we assume an intended model whose domain is precisely the *sets* (or so one hopes). With this remark Zermelo would thereby be raising the question whether the domain of this intended model is in fact just the *sets*. This "intended model" reading is out of place, however, since the bulk of his remarks in no way suggests the model-theoretic conception. Note, in this regard, that Zermelo's remark suggests no link between truth and consistency, as urged by Hilbert.

Zermelo tells us that "set theory is concerned with a domain (*Bereich*) *B* of individuals, which we shall simply call *objects* and among which are the *sets*"; that "certain *fundamental relations* of the form $a\varepsilon b$ obtain between the objects of the domain *B*"; and that "[these] fundamental relations of our domain *B* . . . are subject to [certain] *axioms* or *postulates*" ([16], p. 201). It is tempting to construe Zermelo's intentions as model-theoretic in Hilbert's sense, whereby the seven axioms would define a certain class of Cantorian structures. Again, however, this would be a mistake. For one thing, Zermelo in [16] allows nothing more than the limited domain variability mentioned earlier, speaking always of *a* domain and *the* domain. Given the period in which Zermelo is writing, this manner of speaking is likely to mislead readers: if he indeed intends any real plurality, he might be expected to emphasize this. After all, the idea is relatively new.[7]

In the end, Zermelo's conception in 1908 is anything but model-theoretic, despite his use of the term "domain," since the model-theoretic conception surely presupposes plurality. Zermelo's conception is rather *algebraic* in the sense that the typical axiom expresses a closure condition on a domain of initial objects. Thus we start with some collection of urelements. Axiom I (Extensionality) establishes identity conditions for sets. Axioms II (Elementary Sets) and VII (Infinity) postulate the presence of the null set and ω, respectively. In addition, Axiom II closes under the taking of singletons and pairs. Axioms IV and V close under the taking of power sets and sumsets, respectively. Axiom VI, which postulates the existence of *a single* choice set for any given set of nonempty, disjoint sets, expresses something like a closure condition. But since we do not assume *all pos-*

sible choice sets for a given set, Axiom VI is not a closure condition in the fullest sense. We shall see below that Axiom III (*Aussonderung*) is also anomalous in this regard.[8] It is worth noting at this point that in presenting the individual axioms, Zermelo never engages in anything like the description of a hierarchy or structure that might serve as an intended picture or diagram of the domain he is describing. In this regard, there is a marked contrast to Russell and Whitehead's way of proceeding in presenting the system of *Principia Mathematica* – a point to which I shall return later when discussing Zermelo's philosophical concept of set.

Zermelo's algebraic conception suggests an unstated "terminal" axiom asserting that nothing is a set except what is obtained from urelements and "base" axioms by closure under the set-forming operations of the others. His discussion of the way in which his Axiom III (*Aussonderung*) prevents paradox presumes the minimality of the domain in just this sense. It is then unclear how one is to square this with his explicit allowance for non-well-founded sets: no axiom bans them, but how might they find their way into the domain in the first place? Zermelo's algebraic conception appears to break down at just this point. A more model-theoretic conception would help and is perhaps suggested. Nonetheless the overall conception is definitely not model-theoretic.

Zermelo's not committing himself to the new model-theoretic conception is merely a consequence of the fact that it would in no way facilitate his reductionist program. If in defining the natural numbers as certain sets we are saying what the natural numbers are, then what sense is to be made of the claim that set theory can have multiple models? Is 0 the empty set of model M_1 or is it rather the empty set of alternative M_2? Making out a case for the metaphysical status of the definition of 0 as { } is most natural if some single model is our reference. The alternative – to assume that (pure) sets can be identified *across* models – is philosophically problematic if not altogether incoherent. Alternatively, one might choose to regard "ε" as something like a logical constant that always denotes the membership relation on the domain (assumed to be a class or set). In that case, there is no obvious harm in assuming that the null set of M_1 and the null set of M_2 are identical. So there are ways to make out the metaphysical claim, but they are all highly contrived as an interpretation of Zermelo.

To sum up, we have seen that Hilbert's early program consists of a methodological part and a philosophical part. The methodological part is just steps (1) through (4) above. The philosophical part that complements this methodology is a model-theoretic viewpoint. In fact, what Zermelo takes from Hilbert is a methodology and nothing more. This methodology is reflected in his decision to proceed axiomatically in a quest for rigor while avoiding the paradoxes. On the other hand, Hilbert and Zermelo share nothing *philosophically* just because it is unclear how the model-theoretic conception of the one is to be squared with the reductionism of the other. In general, Hilbert in his thinking about foundational issues is much in advance of Zermelo, who cleaves to a traditional conception of mathematics as a priori science (see also Breger [2]).

I shall turn now to Zermelo's Axiom III and the issues that it raises. It is at the heart of his solution to the set-theoretic paradoxes. It is also the locus of the definability concept that, together with the size-limitation idea, forms the core of his thinking about sets.

3 Avoiding the Semantic Paradoxes: The Bounds of Definability For set theory, unlike other areas of mathematics such as number theory or abstract algebra, the concept of mathematical definability has been central. Comprehension principles, which state which properties or predicates define sets, serve as a starting point in the early investigations. Such principles might be fully unrestricted (Frege), or they might be in some measure restricted (Cantor). Zermelo's *Aussonderungsaxiom* (Axiom of Separation) is a quasi-comprehension principle:

> Whenever the propositional function $F(x)$ is definite for all elements of a set M, M possesses a subset M_F containing as elements precisely those elements x of M for which $F(x)$ is true. ([16], p. 201)

Two very different sorts of restriction are introduced here. First, there is a restriction upon the size of the set defined. Thus the axiom requires that one start with a collection M assumed to be a set. The result, M_F, of applying the axiom to M will be no greater in size than M obviously. Consequently, since M is not "too big" to be a set, neither is M_F. The second restriction introduced in *Aussonderung* seeks to limit the conceptual resources made available for "separating out" M_F. Here Zermelo's source is the semantic paradoxes—in particular the Richard paradox. Since it is well-known how size limitation is useful in eliminating the various set-theoretic paradoxes, my discussion will focus upon this second sort of restriction.

Zermelo's concept of *definiteness* is the locus of his efforts to restrict conceptual means. The philosophical source of the concept of definiteness is an intuitive concept of *logical* definability relative to the new set-theoretic context:

> A question or assertion F is said to be *definite* if the fundamental relations of the domain, by means of the axioms and the universally valid laws of logic, determine without arbitrariness whether it holds or not. Likewise a "propositional function" [*Klassenaussage*] $F(x)$, in which the variable x ranges over all individuals of a class K, is said to be definite if it is definite for *each single* individual x of the class K. Thus the question whether $a \varepsilon b$ or not is always definite, as is the question whether $M \subseteq N$ or not. [Italics in original] ([16], p. 201)

Zermelo's concept of logical definability is of a nonlinguistic character. Thus, rather than looking to syntax to decide definiteness, Zermelo instead makes an appeal to relations holding within the domain. Moreover, application of *Aussonderung* always involves a demonstration of definiteness in which one seeks to show that the given assertion is true or false solely on the basis of the membership relation, the definitions of certain set-forming operations, and logic. (Zermelo in the quoted passage speaks of using the *axioms*. However, the axioms of [16] are extended affairs that incorporate definitions of sumset, power set, and so forth.)

Definiteness in [16] has precious little to do with language.[9] Zermelo's idea is rather that of concepts being definable in terms of other concepts. We start with a certain "conceptual sphere" (*Denkbereich, Begriffssphäre*), to introduce a very Cantorian terminology, which in this case is just the set-theoretic sphere.[10] Now Zermelo wishes to proscribe foreign elements by marking off those properties that are "germane" to this sphere. The appeal to the fundamental relations reflects Zermelo's desire that we restrict our attention to properties characterizable or definable by means of the conceptual resources of the given conceptual sphere. In the case of set theory with urelements, this means membership and equality. Again, the restriction is not to any particular *vocabulary*, for the idea is that of concepts being definable in terms of other concepts.

The envisioned *application* of logic is clear enough. More complex concepts are constructed from the fundamental ones by way of familiar Boolean operations on concepts. Suppose that concepts Φ and Ψ are given. Then a new concept χ can be composed of these as their union: an object x will fall under concept χ if x falls either under Φ or under Ψ or under both. To see how generality can be handled, suppose that concept Φ and two-place relation Λ are given. Now define concept Ψ with the stipulation that an object x will fall under Ψ provided every object y falling under Φ stands to x in the relation Λ. Here we have generalized objects. It is also possible, of course, to introduce generality with respect to (first-order) concepts themselves so as to obtain more complex higher-order concepts. This is all familiar to Zermelo from the work of Frege. Unfortunately, Zermelo says nothing at all to indicate how far we are permitted to go in constructing concepts. His own practice in demonstrating definiteness in [16] never takes him beyond first-order concepts. So it is possible, but not likely, that this is the intended limit. More probably, given his views on definability, Zermelo envisions no restriction whatever on logic. So it is no accident that he speaks of "universally valid laws of logic" without characterizing logic more closely. Thus, whereas the intended *application* of logic is clear enough, the *extent* of logic is not. This, in turn, renders the definiteness concept somewhat murky so that the role of *Aussonderung* is itself less than clear ultimately. Setting that issue to one side, let us see how Zermelo wishes to use definiteness to resolve the Richard paradox. We shall see that Zermelo's solution may presuppose reductionism in the sense of (R2), i.e., concepts-reductionism.

Suppose that we are given an enumeration E of all the real numbers between 0 and 1 that are definable in finitely many English words. Included in the enumeration will be numbers defined by expressions such as "point zero one" and "one-half the square root of two." Now consider the following definition of the real number N:

> Let N be the real number between 0 and 1 whose nth decimal digit is the *cyclic sequent* of the n^{th} decimal digit in the n^{th} number in enumeration E.

(We let *1* be the cyclic sequent of *0*, *2* be the cyclic sequent of *1*, . . . , and *0* be the cyclic sequent of *9*.) It follows that N must be different from every number in E. Hence N must not be finitely definable. And yet the given definition of N consists of but finitely many words.

Although Zermelo does not say how individual reals are to be construed set theoretically, he assumes in his discussion of the Richard paradox that the reals collectively form a set. However, says Zermelo, definiteness and *Aussonderung* prevent the *finitely definable reals* from forming a set, since the property of finite definability is not definite. So the Richard paradox is eliminated.

What is the source of Zermelo's claim that finite definability is not definite? Zermelo assumes apparently that the concept of finite definability (via natural language) outstrips the conceptual resources of set theory and, hence, by (R2), of mathematics. Expressed bluntly, Zermelo assumes that the concept of finite definability has nothing to do with membership. This is presumably because he assumes that *definability in natural language* has nothing to do with membership. About this he may be right.[11] But he merely *assumes* this.[12] He does nothing to *demonstrate*, even roughly, that the finite definability concept is not definite. This introduces a certain disanalogy, since Zermelo insists upon demonstrations of definiteness in positive cases; it is reasonable to expect a refutation of definiteness in the case of finite definability. Moreover, such a refutation is readily available to Zermelo. One proceeds indirectly by supposing that finite definability *is* definite. Accordingly, a (denumerable) set S of finitely definable reals is separated out. An enumeration E of S may be assumed. Now Richard's N is both in S and in the complement of S. Clearly Zermelo could reason in this way. But, in any case, he does not do so, and his remarks concerning the Richard paradox indicate that he is not assuming such an argument either. Again, he merely takes it to be obvious, requiring no demonstration, that finite definability transcends the conceptual resources of set theory. From this we should conclude two things. First, we see once again that Zermelo's conception of set theory is hardly model-theoretic. The symbol "ε" is just the name for the real membership relation holding within the given domain of sets. If "ε" were capable of different instantiations, Zermelo's assumption that finite definability transcends set theory would make no particular sense. Second, Zermelo appears to take some understanding of the conceptual "stretch" of the membership relation to come with the domain, so to speak. Alternatively, one might say that he assumes an intuitive understanding of the membership relation that makes it manifest that the finite definability concept will not be logically definable in terms of membership.

On Zermelo's algebraic conception, *Aussonderung* expresses a limited sort of closure condition. Zermelo's idea is not closure under the inclusion relation, as has sometimes been claimed (see Drake [5], pp. 12–13). His brief discussion of the Richard paradox makes this apparent. Rather we close under included collections that are logically definable in terms of the conceptual resources at hand. The problem, as noted above, is that "logically definable" here is left largely unspecified.

What is the relation between the definiteness concept and the concept of mathematical definability? Both may be regarded as higher-order concepts in that only concepts fall under them. Are they extensionally identical? By (R2) all mathematically definable concepts are definable in terms of membership and hence definite. (Note that we are being just as vague regarding definability as is Zermelo.) Thus concepts-reductionism has as consequence that the definite concepts subsume the mathematically definable concepts. The other direction is trivial so long as we follow Zermelo in taking axiomatic set theory to be part of mathematics. If this is correct, then Zermelo can be taken to assert that finite definability is not merely not definite but not mathematically definable either.[13] Thus definiteness turns out to be nothing more than a technical term for mathematical definability. And this is important, since it is clear that Zermelo's goal is not merely to show that the Richard paradox does not arise within set theory. The larger goal is to demonstrate that the paradox is eliminated from mathematics altogether. This probably means that (R2) is a key underlying assumption. Zermelo seeks to show that the finite definability concept is illegitimate, not by citing some circularity as had Richard himself and Poincaré, but rather by asserting that it is not definite in his new sense. In the absence of (R2), however, this assertion would be of no interest from the point of view of eliminating the paradox *from mathematics*, since definiteness, on the face of it, is a matter of logical definability starting from the membership relation whereas finite definability concerns the most general (informal) means of specification. Even if one grants that finite definability is not a definite concept, this by itself shows only that the Richard paradox is not a problem for Zermelo's system. In order for this to have any implication for mathematics generally, (R2) is required. For (R2) and the assumed nondefiniteness of finite definability together entail the desired illegitimacy of finite definability as a mathematical concept, and the paradox is blocked.

It is worth commenting on Zermelo's concept of set at this point. His assumption of (R2) has as a consequence that, in the guise of definiteness, an intuitive concept of mathematical definability is at the root of his concept of set. Size limitation is not an adequate explanation of Zermelo's intentions, as has been assumed, although there is no denying it an important role. Thus Zermelo writes:

> By giving us a large measure of freedom in defining new sets, [*Aussonderung*] in a sense furnishes a substitute for the general definition of set that was cited in the introduction and rejected as untenable. It differs from that definition in that it contains the following restrictions. In the first place, sets may never be *independently defined* by means of this axiom but must always be *separated* as subsets from sets already given; thus contradictory notions such as "the set of all sets" or "the set of all ordinal numbers", and with them the "ultrafinite paradoxes", to use Hessenberg's expression, are excluded. [italics in original] ([16], p. 202)

Here the issue is undoubtedly size. However, Zermelo continues:

> In the second place, moreover, the defining criterion must always be definite in the sense of our definition . . . (that is, for each single element x of [set] M the fundamental relations of the domain must determine whether it holds or not), with the result that, from our point of view, all criteria such as "definable by means of a finite number of words", hence the "Richard antinomy" and the "paradox of finite denotation", vanish. ([16], p. 202)

Again, the role of mathematical definability has been underappreciated because Zermelo's interest in the semantic paradoxes has rarely been stressed. In fact, that interest is the very genesis of definiteness. Ultimately, Zermelo's restricted comprehension principle must be explained, at least as far as his own intentions in 1908 are concerned, both in terms of size limitation *and* mathematical definability. Zermelo's exposition in the two passages just quoted is probably intended to suggest this directly. Describing Zermelo's 1908 axioms as a size-limitation theory as does Hallett [7], while correct as far as it goes,[14] omits fully half the story.[15] In particular, *Aussonderung* is not functioning as a size-limitation principle in its application to the Richard paradox. In that case, it is a limitation on *conceptual resources* rather than on size that prevents the paradox from arising. Moreover, it is apparent that this limitation on conceptual resources is every bit as important for Zermelo's theory as is the limitation on size.

4 Propositional Functions and Logic Definiteness may appear to have little to do with any concept of definability and to be rather only a clumsy way of getting at well-formedness in a formal language. To conclude this would be to assume an object-language/metalanguage distinction that is alien to Zermelo's thought in [16]. Reading Zermelo's remarks on definiteness as merely syntactic in spirit may be good mathematical logic but it is very bad history. Skolem's later construal of definiteness as well-formedness, replacing an informal element with a formal element characterizable in the meta-language, was a much-touted advance just because it was *not* obviously contained in Zermelo's remarks concerning definiteness. Indeed Skolem's characterization of definiteness is a landmark along the road to the model-theoretic conception of formal theories. Zermelo's presentation of axioms for set theory without consideration of language ensures that well-formedness *cannot* be his intent already in 1908. It appears likely that the tendency to underemphasize Zermelo's desire to place limits upon conceptual resources finds its source in a disposition to interpret definiteness as well-formedness.

Zermelo eventually does come to view definiteness in a manner more in keeping with the model-theoretic viewpoint, i.e., more in terms of some sort of syntactic definability. And no doubt this change is attributable to Skolem's influence. In a paper (Zermelo [19]) published in 1929, Zermelo presents an inductive definition of *proposition definite relative to R*, where parameter R is the "system" of fundamental relations of a given theory. Definiteness has come to mean propositional connectives and first- and second-order quantifiers, which may or may not be an extension of [16] (viewed syntactically). Since that short paper is intended as a response to years of criticism of the very concept of definiteness, *this* notion rather than set theory is the focus. Consequently, no reformulation of *Aussonderung* is explicitly provided.

Only one year later, in Zermelo [20], *Aussonderung* is given an explicitly second-order formulation. Now there is no reference to definiteness at all.

> Every proposition function [Satzfunktion] f(x) separates out of any given set m a subset m_f which contains all elements x for which f(x) is true. Alternatively: to every part of a given set there corresponds another set which contains all of the elements belonging to this part. ([20], p. 30)

Zermelo says in a footnote that $f(x)$ is here a perfectly arbitrary propositional function but again says nothing about language. At this point in the evolution of his thought it is natural to read "arbitrary function" here as "arbitrary function definable in the language of second-order logic." But this reading founders. For the two formulations taken together entail that every part of an infinite set correspond to a function, which means that mere cardinality considerations block the second-order reading. Perhaps Zermelo is assuming only that the functions are expressible in some infinitary language.[16] The alternative formulation of *Aussonderung* would seem to reflect an intention to close under subsets. As a consequence of this, Sumset and *Aussonderung*-Subset in ZF^2 together should now yield any choice set, thus obviating the Axiom of Choice (henceforth AC). But Zermelo's remarks regarding AC, which is not included among the axioms of [20], suggest that he himself sees things differently somehow. This may mean that, despite the alternative formulation, he does *not* see himself as having closed under subsets. However that may be, taking both formulations with equal seriousness raises an interesting question. For, taken together, they imply that every subclass of an arbitary set is the extension of some propositional function. In particular, each choice set then corresponds to some propositional function which might be thought to define it. Of course this by itself does not yet mean that one can construct the choice set, since, in order to do that, one must have its definition in hand, so to speak; it is not enough to know only that a definition exists in principle. For his part, however, Zermelo tends to draw some stronger and unwarranted conclusion. One can see this in his earlier work.

5 Mathematical Existence and Mathematical Definability Objects-platonism, or simply platonism, is that philosophical doctrine according to which mathematical objects, although abstract and nonphysical in character, exist completely independent of human reasoning about them. Zermelo's AC is the paradigm platonist existence principle. The axiom is nonconstructive in that it asserts the existence of particular choice sets even in the absence of any ability to characterize them conceptually. We saw in [20] what looks like a covert attempt to say that such choice sets are constructible after all.

In [15] Zermelo presents the classically platonist defense of a certain sort of impredicative definition—so-called "definitions from above":

> Once such [an objective] criterion is given, . . . nothing can prevent some of the objects subsumed under the definition from having in addition a special relation to the same notion and thus being determined by, or distinguished from, the remaining ones—say, as common component or minimum. After all, an object is not created through such a "determination". ([15], p. 191)

However, whereas this classical platonist defense tells us something important about Zermelo's views, a competing view is present in his published writings. Zermelo's early platonism must be set alongside strong views concerning mathematical definability.

Immediately after the quoted defense of impredicativity, Zermelo goes on to assert that "every object can be determined in a wide variety of ways" ([15], p. 191). To be sure, such "determinations" (*Bestimmungen*) should not be construed as definitions in any linguistic sense. But they surely do signify some extralinguistic accessibility to the mind via concepts. Indeed, the remark may be regarded, on some such reading, as a consequence of Cantor's 1895 definition of set together with reductionism—in particular (R1). Since Zermelo appears to defend impredicative *definition* on the grounds that alternative predicative "determinations" are always available, it is unclear what force his claim can have if such determinations cannot be associated with corresponding (predicative) definitions. Again, however, this will be a matter of concepts being defined in terms of other concepts. Language is not the issue.

The assumption is that absolutely all mathematical objects are capable of (predicative) "determination" and, hence, are more or less definable. It is clear enough that Zermelo sees universal determinability/definability as tempering the debate between platonists and constructivists despite the fallacy mentioned at the end of the last section. Can one speak of a constructivist thread in Zermelo's thought? Of course constructivism, as usually understood, entails some restriction as to *means* of construction. In this regard, it makes no difference that Zermelo's concept of definiteness, which might serve here, remains an informal concept in 1908. Zermelo seems to suggest that the controversy surrounding AC would disappear if everyone could only recognize the truth of universal definability. Of course, mere definability is not going to satisfy certain parties unless it is a matter of the *right sort* of definability—the restriction of means issue again. However, Zermelo's remark about the "wide variety" of possible determinations of any object suggests that this is not really a problem either. First, it must be pointed out that Zermelo never appeals to universal definability in defending AC itself. Rather, the doctrine is used to defend impredicative definitions only; but it is not clear that the two cases are very different. Zermelo counsels constructivists to countenance impredicative definitions because, although possibly circular, they may in principle be replaced by predicative alternatives.[17] One can easily imagine Zermelo defending AC by analogous reasoning, appealing to the availability in principle of definitions for arbitrary choice sets.

Before continuing, we might ask how Zermelo's treatment of the Richard paradox is to be squared with universal definability. For is not the collection of finitely definable reals a mathematical object and hence definable? At this point one might, of course, take the paradox itself to show that this collection is not a genuine mathematical object and/or that "definable in finitely many words" corresponds to no genuine "determination." Zermelo's remarks in [16] certainly suggest the latter. The assumption that "definable in finitely many words" involves extra-mathematical concepts would imply, by *Aussonderung*, that no set exists. If the collection of finitely definable reals is nonetheless a mathematical object, then there is an apparent conflict with universal definability. Appeal to (R1) would eliminate this conflict.

The doctrine of universal definability may not be unique to Zermelo. Others who use the "finite definability" concept are probably drawn to the idea. For why add the adjective "finite" unless there is another sort of "infinite" definability that is being taken seriously? Further, it might be thought a small step from infinite definitions to universal definability. Zermelo's later denial that "every mathematically definable notion is expressible by a 'finite combination of signs'" demonstrates that he has no prejudice against infinite definitions (see Dawson [4]).

Those who responded to the paradoxes in the early years of this century can be divided into two groups. First, there are those such as Poincaré and Russell for whom definability is central to any solution. Since Poincaré takes the Richard paradox as paradigm, it is not surprising that definability is the core of his vicious-circle principle. Influenced by Poincaré, Russell "ramifies" definability through the introduction of the notion of order. Others in this first group include Richard himself and Peano. For a second group, definability plays no role in the solutions proposed. Here we find the set-theorists Jourdain, Bernstein, Hessenberg, and Mirimanoff, all of whom focus on the Burali-Forti paradox. For them, the key issue is not definability but rather *size*.

Despite his philosophical and mathematical affinities with the second group and despite the fact that the axioms of [16] are describable in part as a size-limitation theory, it has been largely overlooked that Zermelo has strong affinities with the first group as well. Commentators have ignored his stated interest in the semantic paradoxes and have construed definiteness as a prototype of well-formedness. Via definiteness, definability becomes central. For this reason, the concept of set that is embodied in the Zermelo axioms is yet a *logical* set concept—logical in that (definite) concepts play an important role. (This is another way to describe the link between Zermelo and those in the first group.) Moreover, Zermelo wants it both ways with regard to mathematical existence: mathematical objects (sets) exist independent of human reasoning, and yet each object is ever accessible to the human mind through any one of a "wide variety" of "determinations." Thus, the conflict between platonism and constructivism loses some intensity at least. More to the point, constructivist criticisms of Zermelo's methods of proof and of AC, in particular, are blunted. By 1930, as we have seen, Zermelo might appear to have migrated into the second group: definability has vanished from his formulation of *Aussonderung*. However, the change is more apparent than real. Taken together, his 1930 formulation of *Aussonderung* as *Aussonderung*-Subset continues to urge a convenient coincidence of existence and definition.

One might speculate that it is Zermelo's desire to blunt tensions between platonists and constructivists that underlies both his advocacy of universal definability and his abiding interest in infinitary logic. He is probably not so unusual in this regard either. It is possibly the effort to reconcile the two philosophical tendencies that motivates those few who take infinite definability with any seriousness after about 1920. Gradually, of course, the two tendencies come to be viewed as utterly incompatible. Perhaps the ultimate turn to finitary logics as the standard of the mathematical community, although attributable to a variety of other philosophical and technical issues, is also in part just the result of a new philosophical clarity regarding this incompatibility. Infinite definability represents a last-ditch effort to prevent the splintering of the mathematical community into constructivist and nonconstructivist factions.

6 Zermelo and the Concept of Set Zermelo regards *Aussonderung* as a replacement both for the naive concept of set and for Cantor's 1895 definition of set (see [16], p. 202). Since *Aussonderung* speaks of a given set *M*, it obviously cannot function as a definition independent of the other axioms. We saw that the set concept underlying Zermelo's axioms incorporates two essential components. First, there is some idea of limiting the size of sets. Second, even in cases where size is not a problem, a set may fail to exist because we can access it conceptually only by means of concepts that are nonmathematical (not logically definable in terms of membership). Here I have spoken of placing limits on the conceptual resources available for defining sets.

One way in which the size-limitation idea might be realized is the so-called iterative concept of set. One can find in the literature attributions of the iterative concept to Zermelo—even to Zermelo in his earliest period. (See, for example, Kitcher [12], p. 295 and Kreisel [13], pp. 82–83.) However, at least in this early period Zermelo's concept of set is clearly *not* the iterative concept, as shall be shown. At best, Zermelo, who *may* be an iterativist by 1930, is a latecomer. Most likely, Zermelo *never* adopts the iterative concept.

It is useful to distinguish three related ideas: (1) the iterative concept(s) of set (henceforth IC), (2) the notion of well-foundedness as applied to sets, and (3) the cumulative hierarchy of sets (henceforth CHS). IC is by nature philosophical. As a concept of set, it attempts to say what sort of things sets are by showing how they are built up in *stages*, starting from the null set (or some urelements). At the same time, this genetic characterization and the resulting hierarchy give a certain transparent structure to the entire world of sets. (In this respect, IC is quite different from other set concepts.) IC is closely related to and justifies the notion of well-foundedness in the sense that it implies that every set is well-founded. It is justified, in turn, by well-foundedness in the sense that a set is well-founded only if it can be built up in stages from the null set. Finally, CHS is a technical construction within axiomatic set theory. It can be taken to realize IC in any model of the axioms including Regularity.

CHS is presented rigorously for the first time in [20]. This may well be the source of the frequent assumption that Zermelo already in 1908 starts from IC. However, it is obvious that he does not have in mind either IC or CHS in 1908, since he explicitly allows for the possibility of sets that contain themselves. In the end, it is impossible to find any "structural" concept of set in [16]. Zermelo may well conceive of set theory as the theory of a particular *domain*. However, it is equally clear that set theory for Zermelo is not the theory of a determinate structure. In fact, the hierarchy concept is quite alien to Zermelo's early thought. To see this, consider, in the light of Russell's work of the same period, the following defense of impredicativity:

> It is precisely the form of definition said to be predicative that contains something circular; for, unless we already have the notion, we cannot know at all what objects might at some time be determined by it and would therefore have to be excluded. ([15], p. 191)

One might also have expected an appeal to IC, if Zermelo were an iterativist, in his discussions of the paradoxes. For example, Zermelo specifically cites *Aussonderung* and the concept of definiteness as eliminating the Burali-Forti paradox. If Zermelo had indeed intended IC, then he might be expected to cite it at this point (however, see below). Zermelo's discussion of the Richard paradox is also noteworthy in this regard. As discussed earlier, Zermelo uses definiteness to block the Richard paradox. The collection of all reals is a set. On the other hand, Richard's E (the collection of all definable reals) will *not* be a set according to Zermelo because the function "x is a real number definable in finitely many words" is not definite. Obviously, if Zermelo is thinking along the lines of IC, we should expect the set of all reals to appear at some stage of the iterative hierarchy. Richard's E would appear at that stage as well. However, at least according to the maximal iterative concept, *both* sets would then be available at the next stage, contradicting Zermelo's assertion that E will not belong to the universe of sets.

Consider also that Zermelo describes *Aussonderung* as "giving us a large measure of freedom in defining *new* sets" and states that, using *Aussonderung*, sets must always be separated "as subsets from sets *already* given" ([16], p. 202; emphasis added). Since x and any subset of x appear at one and the same stage under IC, the remark is at least vaguely at odds with that conception. There are, of course, ways in which we might interpret Zermelo's remarks so as to square them with IC, but it seems natural to conclude that IC simply does not underlie Zermelo's thinking about sets.

Another point concerns Zermelo's defense of AC in [15]. IC is normally thought to provide a very natural justification of AC. If Zermelo were an iterativist, he could be expected to appeal to IC, which he does not do.[18] By 1930, however, it is most tempting to regard Zermelo as an iterativist, since his [20], in which CHS is first articulated, certainly suggests IC. But, in fact, not all of Zermelo's remarks even in this period point in the direction of IC. Thus, in [20] he motivates Regularity not by an appeal to IC, as one might expect of a believer, but rather pragmatically by noting its consistency with set-theoretic practice to the present. In itself, this fact cannot be decisive, however, since it assumes that in adopting new axioms, Zermelo, if he is an iterativist, has as his paramount goal a certain fidelity to an intuitive concept. However, the fact that Zermelo has by this time incorporated Fraenkel's Replacement Axiom, itself short on iterative justification, indicates that his attitude toward IC can hardly be so straightforward, assuming for the moment that he *is* an iterativist. Another issue would be his 1929 objection, cited previously, to Skolem's inductive characterization of definiteness as presupposing *finite number* ([19]).[19] Would IC not similarly presuppose *number*? One might then speculate that, whatever the degree of his belief in IC, Zermelo is not open to appeals to IC in justifying the axioms—the issue to which we now turn.

7 Justifying the Axioms Zermelo tells us that mathematical axioms are to be justified "by analyzing the modes of inference that in the course of history have come to be recognized as valid and by pointing out that the principles are intuitively evident and necessary for science" ([15], p. 187). Thus two criteria for adopting axioms are proposed:

(SE): An axiom must be intuitively self-evident.
(NEC): An axiom must be necessary for mathematics.

The analysis of historical reasoning is the method whereby one comes to see that both criteria (SE) and (NEC) are satisfied by a given proposition. Moreover, Zermelo emphasizes that establishing (NEC) is a completely objective procedure. Thus Zermelo's methodology for selecting axioms is historical and ultimately pragmatic. His defense of AC consists in showing that AC satisfies both (SE) and (NEC). The example establishes the consistency of the two criteria: they can be simultaneously satisfied by one and the same proposition. It is easily seen that they are independent of one another as well.

Zermelo does not think that (SE) can be shown to hold *directly* in the case of AC.[20] It is not through some intuition or perception of sets that (SE) is established.[21] Moreover, if we have some sort of *indirect* acquaintance with the world of sets, no appeal to such acquaintance is made in establishing (SE). Rather, one examines community practice. If many mathematicians appeal to a given proposition, then this is taken to establish (SE). Thus (SE) is established for a given proposition if we have:

(APP): Mathematicians regularly appeal to the proposition in proofs, whether explicitly or only implicitly.

Criterion (APP) requires refinement.

Suppose that a mathematician proves a given proposition by an appeal to the Axiom of Inaccessibles. Suppose others do likewise. Since no one takes the Axiom of Inaccessibles to be intuitively evident, would such proofs not present counterexamples to Zermelo's method for establishing (SE)? Of course not: each mathematician will state his/her theorem so as to include the Axiom of Inaccessibles among its hypotheses. So clearly we must read Zermelo as claiming that the appeal to a proposition must occur within proofs without the proposition figuring among the stated hypotheses.

Another apparent problem is suggested by the practice of recursion theorists who regularly appeal to Church's thesis within their proofs. No one takes this in itself to mean that Church's thesis is even true. One can get around this objection by pointing out that appeals to Church's thesis are in every case dispensable. The recursion theorist, say, who appeals to Church's thesis does so merely to facilitate her demonstration. She does not assume its self-evidence or even its *truth*. Rather she assumes the extreme unlikelihood of counterexamples to Church's thesis. Her appeal to Church's thesis expresses her belief that the appeal is eliminable albeit with considerable effort. So we should take Zermelo to mean that establishing (SE) in the case of a given proposition requires determining that criterion (APP′) is satisfied:

(APP′): Mathematicians regularly appeal to a proposition in proofs, whether explicitly or only implicitly, where (1) the proposition does not figure among stated hypotheses and (2) the proposition is not believed to be eliminable.

If we take Zermelo to claim that (APP′) implies (SE), then this claim would be based apparently upon his belief in

(EXP): Extensive appeal to a proposition on the part of mathematicians can be explained only by its self-evidence (see [15], p. 187).

So the idea would be that (EXP) and (APP′) together imply (SE). Now (EXP) is to be distinguished from

(EXP′): Extensive appeal to a proposition on the part of mathematicians can be explained only by their regarding it as self-evident.

Clearly (EXP′) together with (REG) implies (EXP), where (REG) is the principle:

(REG): If many mathematicians regard a proposition as evident, then that proposition is self-evident.

So ultimately (SE) follows for a given proposition from (EXP′), (REG), and (APP′). Unfortunately, neither (REG) nor (EXP′) is obviously true. As for (REG), the history of mathematics is no doubt rife with examples of propositions that for a time were generally held to be self-evident but that were later found to be false in fact.[22] One can imagine other cases in which the proposition in question is yet held to be true although no longer self-evident.

As for (EXP′), my earlier examples involving the Axiom of Inaccessibles and Church's theorem require refinements analogous to those which led from (APP) to (APP′). Beyond this, there is the obvious objection that mathematicians might fail to include a proposition among the hypotheses of the demonstrandum, not because they regard it as self-evident but, rather, because they believe, perhaps erroneously, that the proposition is itself provable.

Of course, Zermelo himself asserts only (EXP), and so one might seek some argument for (EXP) that involves no appeal to (EXP′). One might, for example, claim that (EXP) is true since all "intuitively evident" means is "enjoying communitywide approval." We might call this the *emotivist* theory of the meaning of "intuitively evident." I will not say any more about this. It is impossible to attribute to Zermelo any such view, given both his view of mathematics as an a priori science as well as his great interest in applied mathematics. The emotivist theory would allow for mathematical truths that are not necessary, and it would leave unexplained the efficacy of mathematics in describing the natural world.

Another way to defend (EXP) would be to claim that communitywide appeal to some proposition just reflects the fact that the proposition holds within a mathematical realm to which each mathematician has some sort of access. In other words, communitywide appeal gives inductive evidence of an indirect nature for the proposition holding in the world of mathematics. So ultimately, despite denials of our having any direct intuition of the mathematical realm in its fullest extent, we would be forced to attribute to Zermelo some belief in the accessibility, albeit unconscious, of the mathematical realm in its entirety after all.

Zermelo's methodology for selecting axioms stands in sharp contrast to Hilbert's view wherein truth plays no role. This difference is attributable to Zermelo's reductionism in the sense of (R3): all mathematical propositions are just set-theoretical propositions. If this reduction is to have any foundational merit, then the axioms of set theory must be intuitively evident. Zermelo's historical method for establishing that axioms are intuitively evident suggests that mathematicians must have access to some platonic domain of sets in its entirety. This access will not be direct or immediate, for otherwise the status of mathematics as an a priori science would be undermined. In this respect, Zermelo is probably not unusual: quite likely all philosophies of mathematics assume that one has some sort of access to the mathematical realm. More novel is Zermelo's idea that evidence for this access—whatever its nature (and there is little point in speculating on what Zermelo takes the nature of this access to be)—is gathered *empirically* by examination of the work of practicing mathematicians. This position regarding justification is ultimately troubling to the extent that it risks undermining Zermelo's foundational program.

8 Zermelo's Foundationalism On the one hand, Zermelo's intentions appear to be traditionally foundationalist. So he engages the Cartesian vocabulary of "justification" and "evidence." It has been seen that Zermelo's goal is the reduction of mathematics to set theory. More precisely, assuming the ontic and conceptual reductions accomplished by his predecessors, Zermelo sets out to provide axioms for the new foundational science. There seems little doubt that his intention is to thereby ground mathematics in an epistemic sense. This would seem to be the impetus for his objection to Skolem's inductive characterization of definiteness: induction presupposes knowledge of the natural numbers, and hence it is misguided to appeal to induction in describing set-theoretic concepts since it is set theory that grounds number theory.

On the other hand, when it comes time to defend the axioms, Zermelo adopts a problematic stance. Nowhere does he claim anything like self-evidence for his axioms. As has been seen, the iterative conception, which might have served in this regard, is entirely absent from his non-hieratic thinking about sets. No doubt Zermelo regards some of his axioms as straightforwardly self-evident. It is clear, however, that he does not regard all of them in this way, since what he emphasizes is indispensability. We can determine objectively through the examination of mathematical argumentation presented in written texts that a proposition has often been appealed to in the past. This demonstrates that the proposition is necessary for mathematics. But it also shows that the proposition has been regarded as self-evident and hence is self-evident.[23] Some problems with this argument have been discussed already. The larger question from the point of view of foundations is this: How are the axioms to ground mathematics if our best evidence for them is that very mathematics? What seems to emerge here is a conception of foundations that is not Cartesian at all really. On the face of it, no Archimedean grounding of mathematics in the epistemic sense has been provided. Rather, the conception of the "edifice" of mathematics is one of holism: the architectural metaphor in fact makes little sense since the foundation supports the superstructure as well as vice versa. If this reading of Zermelo is right, then the provision of mathematical "foundations" seems to have proceeded largely in the interest of rigor.

The indispensability criterion by itself also appears to open the door to the possibility that an accepted axiom turn out to be false, thus exposing the inherent anti-foundationalism of the criterion taken alone. This may only show that Zermelo must be interpreted as holding (REG) to be true: the mathematical community cannot be wrong in its judgments regarding self-evidence. Such a move does not eliminate the problem, however. For in the case of a controversial axiom such as AC, the claim of self-evidence cannot be so direct and must rely upon historical practice—as Zermelo well understands: the axiom's self-evidence is established by constant implicit appeals to it. But this reintroduces both fallibilism as well as circularity.

So which reading of Zermelo's remarks is the correct one? Is he a classical foundationalist? In one sense, yes, since (SE) expresses a necessary property of axioms according to him. The difficulty in reading him in this way is a consequence of the manner in which self-evidence is to be established. It seems impossible to square the empirical procedure justifying AC (and by implication at least some of the other axioms) with the traditional foundational goal. Again, how has mathematics been grounded epistemically if our best evidence for AC, say, consists in its having figured regularly in the history of this very mathematics?

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NOTES

1. In [19] Zermelo claims that such definitions presuppose the concept *finite number*. Hence, he continues, such a way of proceeding is circular in the case of axiomatic set theory, where natural numbers have been defined as certain sets (or where the concept of finite number has been defined in terms of membership). Clearly (R1) *and (R2), rather than (R1′) and (R2′), are the sources of Zermelo's objection: if the* set concept turns out to presuppose the number concept, then the metaphysical status of the reduction is nil.

2. See also the section "Avoiding the Semantic Paradoxes: The Bounds of Definability" of the present paper—particularly the final quotations from Zermelo's [16].

3. For a different view, see Hallett's penetrating analysis of Zermelo's theory in [7]. Hallett argues there that Zermelo follows Hessenberg in shunning number objects. I would claim, on the contrary, that Zermelo's intentions with regard to number objects are just not all that clear, which Hallett himself seems to concede ultimately ([7], p. 248). In any case, by about 1915, as reported in Bernays [1], Zermelo will have developed a theory of ordinals that is independent of the theory of ordered sets. I read this later development as an indication that Zermelo is never without interest in number objects.

4. The issue here is not the availability of a reductionist (i.e., set-theoretic) definition of *ordinal* but, rather, the extent of the ordinals within Zermelo's 1908 system. In the absence of anything like Replacement, the so-called Zermelo ordinals $\{\ \}$, $\{\{\ \}\}$, $\{\{\{\ \}\}\}, \ldots$, say, exist in that system only up to, but not including, $\omega 2$.

5. This indicates that a consistency proof for his system is at least a possibility for Zermelo and further suggests that he has himself tried to obtain such a result, which raises the issue of just what such a demonstration would consist of from Zermelo's perspective in 1908. Poincaré raises this issue against Zermelo (see Moore [14], pp. 162–163).

6. I make this claim based on the bulk of the textual evidence. There is one passage in [15], however, in which Zermelo, following Hilbert, describes his own use of the Axiom of Choice as the free adoption of a "hypothesis" whose consequences he seeks to explore ([15], p. 189). It would be a mistake to assimilate Zermelo's philosophical conception of axiom systems to that of Hilbert based on this single remark. In fact Zermelo's conception is far more traditional than Hilbert's.

7. *It must be pointed out that Hilbert himself adopts a similar "misleading" idiom in* his [9], always speaking in terms of *a* geometry or *the* geometry, and yet his correspondence and lectures demonstrate unambiguously that he nonetheless has plurality in mind. So, by itself, this argument is perhaps rather weak.

   Much later in [19], where not set theory but rather axiom systems generally are the issue, Zermelo does adopt a more explicit mode of exposition. There he speaks of "*Modellen*" in the plural. It is also perfectly obvious that by this point Zermelo does not see set theory as special: *ZF*, too, will have multiple models. Of course there is no reason to suppose that merely because the later conception allows for plurality that Zermelo's earlier conception is similarly pluralistic. On the contrary, the quoted passage indicates that Zermelo can be very careful in expressing himself with regard to the model-theoretic conception despite the fact that there is at this point in time less danger of misunderstanding; by 1929, due to the influence of Löwenheim and Skolem, talk of multiple models for axiomatic theories is common coin. That Zermelo does not express himself in this careful manner in 1908 can hardly mean that he is assuming his reader's thorough understanding of the model-theoretic position. One plausible explanation is a certain uneasiness regarding plurality at least in the case of set theory.

8. Zermelo considers but ultimately rejects inclusion of an axiom asserting that no set is self-membered (see Moore [14], pp. 155–157). Since such an axiom does not express a closure condition, this may or may not support my claim that Zermelo's conception of his axioms is algebraic rather than model-theoretic.

9. It is true that in fn. 11 of [15] (p. 192) Zermelo discusses definability in a way that emphasizes language. Still, if language were really primary, one would expect Zermelo to specify it in some way, which he does not do.

10. In fact, Zermelo's description of definiteness owes a certain amount to Cantor. (See Cantor [3], p. 150.)

11. But if Church's thesis and the computational model of mind are both true, then he may be wrong. For suppose that my linguistic competence with respect to finite English strings purporting to name real numbers is realized by some Turing machine M. (This amounts to assuming both Church's thesis and the computational model of mind.) Now M can be defined as a certain set of tuples. Whatever the problems with my example, it at least shows that it is not obvious that definability in natural language has nothing to do with membership.

12. One possibility which should be mentioned is that Zermelo is presupposing something like Hessenberg's discussion in Section XXIII of [8]. The conclusion there is that the predicate "is finitely definable" has no coherent application in the case of individual numbers at least. So perhaps what Zermelo assumes in [16] is that such *incoherent* predicates cannot be understood in terms of membership.

13. One problem here is that the claim that finite definability is not definable in terms of membership may seem to beg the question: the presumed inability to define the concept of finite definability in terms of the fundamental relations of set theory might be taken to show just that (R2) is false—mathematical concepts are not just set-theoretic concepts. Zermelo might be expected to hold, however, that the reduction achieved previously by Cantor et al. constitutes independent grounds for believing (R2).

14. The term "size-limitation" is due to Russell. Like von Neumann much later, Russell specifically bans sets that can be placed in 1-1 correspondence with some paradoxical collection such as that of all sets. We might speak of "proscriptive" theories. Zermelo, on the other hand, bans nothing. That the paradoxical collections fail to materialize is just a consequence of the nature of the power-set operation. Zermelo's approach is conservative to the extent that possibly non-paradoxical collections may fail to be sets as well. In any case, one may well question the wisdom of grouping both proscriptive theories and iterative theories under the same heading.

15. I do not mean to suggest that Hallett ignores definability. In fact, he devotes the final section of his chapter on Zermelo to the definiteness concept ([7], pp. 266–269). But what is stressed there is, as usual, the confusion surrounding the notion as well as the degree to which Zermelo's use of it appears to be incompatible with reductionism. What one misses in [7], I would claim, is adequate recognition of the fact that, from Zermelo's point of view in 1908, the limitation upon size and the limitation upon conceptual resources are of equal importance.

16. In [22] Zermelo describes an infinitary language with the usual propositional connectives and names for all individuals. Sentences may be of arbitrary infinite length.

17. Zermelo, in fact, says nothing quite this strong. However, it is unclear what force his argument can have if such substitution is not possible.

18. Regarding this point, Zermelo's views concerning justification of mathematical axioms, which will be discussed in the next section, really point in another direction. So one should probably not be overly impressed by the absence of appeals to IC in justifying AC.

19. Zermelo's attitude toward metamathematical discourse in the late period is hard to make out. For in [20] and in later papers Zermelo uses ordinals within metamathematical discourse.

20. This is perhaps the only reasonable reaction to the controversy surrounding AC. Still, this pretty much settles the question whether Zermelo is an iterativist in his early period.

21. Much later, Zermelo explicitly denies intuition of the mathematical infinite (see his [21], p. 85). However, the same passage probably suggests that we do possess some direct intuition or grasp of the finite portions of mathematics.

22. The obvious example here is the naive assumption that every property or predicate determines a class. For an example from mainstream mathematics, consider the proposition that all continuous functions are somewhere differentiable. As a recent example from differentiable geometry, consider the assumption that any manifold possesses but a single differentiable structure—shown to be false by S. Donaldson. I owe the last two examples to Seamus Moran.

23. Zermelo does not quite say this, but it is clear enough from his discussion of AC—especially his citation of cases of implicit appeal on the part of skeptics.

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