Notre Dame Journal of Formal Logic
Volume 44, Number 4, 2003

Implicit Definability of Subfields

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Abstract We say that a subset A of M is implicitly definable in M if there exists a sentence $\varphi(P)$ in the language $\mathcal{L}(M) \cup \{P\}$ such that A is the unique set with $(M, A) \models \varphi(P)$. We consider implicit definability of subfields of a given field. Among others, we prove the following: $\overline{\mathbb{Q}}$ is not implicitly $\varnothing$-definable in any of its (proper) elementary extension $K \succ \overline{\mathbb{Q}}$. $\mathbb{Q}$ is implicitly $\varnothing$-definable in any field K (of characteristic 0) with $\mathrm{tr.deg}_{\mathbb{Q}} K < \omega$. In a field extension $\mathbb{Q} < K$ with K algebraically closed, $\mathbb{Q}$ is implicitly definable in K if and only if $\mathrm{tr.deg}_{\mathbb{Q}}(K)$ is finite.

1 Introduction

We begin with the definition of implicit definability.

Definition 1.1

1. Let T be a theory in the language $\mathcal{L}$ and P, P' two new unary relation symbols not in $\mathcal{L}$ and $\varphi(P)$ an $\mathcal{L} \cup \{P\}$-sentence. We say that $\varphi(P)$ defines P implicitly in T if $T \vdash \varphi(P) \wedge \varphi(P') \to \forall x (P(x) \leftrightarrow P'(x))$.
2. Let M be an $\mathcal{L}$-structure. We say that a subset A of M is implicitly definable in M if there exists a sentence $\varphi(P)$ in the language $\mathcal{L}(M) \cup \{P\}$ such that A is the unique set with $(M, A) \models \varphi(P)$. In this case, we also say that $\varphi(P)$ implicitly defines A in M. If $\varphi(P)$ does not have parameters, we say that A is implicitly $\varnothing$-definable in M.

Beth's definability theorem states that if $\varphi(P)$ defines P implicitly in T then $P(x)$ is $(T \cup \{\varphi(P)\})$-equivalent to an $\mathcal{L}$-formula (i.e., $\varphi(P)$ defines explicitly P in T). Hence $\varphi(P)$ defines P explicitly in T if and only if $\varphi(P)$ defines P implicitly in T. However, the situation is different if we consider implicit definability in a given structure M. It is clear that any first-order definable subset of M is implicitly definable, but the converse is not true in general. There is a structure

Received March 17, 2003; accepted June 9, 2003; printed July 7, 2004
2000 Mathematics Subject Classification: Primary, 03C40
Keywords: implicit definability, fields

in which two notions (implicit definability and first-order definability) are different. For example, let $M = (\mathbb{N} \cup \mathbb{Z}, <^M)$, where $\mathbb{N}$ and $\mathbb{Z}$ are disjoint and $<^M = <^{\mathbb{N}} \cup <^{\mathbb{Z}} \cup \{(a, b) : a \in \mathbb{N}, b \in \mathbb{Z}\}$. Then $\mathbb{N}$ is not first-order definable, but it is implicitly definable. It can be shown that for a given structure there is an elementary extension in which any implicitly definable subset is first-order definable (see Section 2). One can show that there are M, an implicitly definable subset A in M, and an elementary extension (N, B) of (M, A) such that B is not implicitly definable in N.

In [8], Shelah and Tsuboi considered nonstandard models of **PA** (Peano Arithmetic) and showed that in some elementary extension of the standard model $\mathbb{N}$, the standard part is implicitly $\varnothing$-definable. From a given model M of **PA**, we can easily construct a field in the same manner as we construct $\mathbb{Q}$ from $\mathbb{N}$. By Robinson's result on first-order definability of $\mathbb{N}$ in $\mathbb{Q}$, the constructed field is bi-interpretable with M. So Shelah and Tsuboi showed that there is an elementary (field) extension of $\mathbb{Q}$ in which $\mathbb{Q}$ is implicitly definable. In this paper we shall consider implicit definablity of subfields in a more general setting.

In Section 2, we state basic results on implicit definability. We also consider implicit definability under categoricity assumptions. In Section 3, we give negative results concerning implicit definability. The following statement is a part of Theorem 3.2.

(A) $\overline{\mathbb{Q}}$ is not implicitly $\varnothing$-definable in any of its (proper) elementary extensions $K \succ \overline{\mathbb{Q}}$.

In Section 4, we give positive results concerning implicit definability. Among others, we show that

(B) $\mathbb{Q}$ is implicitly $\varnothing$-definable in any field K (of characteristic 0) with $\mathrm{tr.deg}_{\mathbb{Q}} K < \omega$.

So, in particular, $\mathbb{Q}$ is implicitly definable in its algebraic closure $\overline{\mathbb{Q}}$, while it is not first-order definable there. (Notice that $\overline{\mathbb{Q}}$ is stable, but $\mathbb{Q}$ is not.) Combining this with a result in Section 2, we get

(C) In a field extension $\mathbb{Q} < K$ with K algebraically closed, $\mathbb{Q}$ is implicitly definable in K if and only if $\mathrm{tr.deg}_{\mathbb{Q}}(K)$ is finite.

Throughout, $\mathcal{L}$ is a first-order language. $\mathcal{L}$-structures are denoted by M, M' and so forth. We will simply say that a subset A of M is definable if it is first-order definable in M using parameters. The ring language is the language $\mathcal{L}_{\mathrm{ring}} = \{0, 1, * + *, -*, * \cdot *\}$. A field is regarded as an $\mathcal{L}_{\mathrm{ring}}$-structure. P always denotes a unary predicate symbol not in $\mathcal{L}$. For a formula φ, φ^P is the formula obtained from φ by replacing each quantifier occurrence (Qx) with $(Qx \in P)$. The term "algebraic closure" is used both in the model theoretic sense and in the field theoretic sense. However, after Section 3, it is used only in the field theoretic sense.

2 Basic Facts on Implicit Definability

Let M be an $\mathcal{L}$-structure and A an undefinable subset of M. Then there is an elementary extension (M', A') of (M, A) in which any $\mathcal{L} \cup \{P\}$-sentence does not implicitly define A'. More generally, we can show the following.

Proposition 2.1 *For a given structure, there is an elementary extension of the same cardinality in which every implicitly definable subset is definable.*

This proposition is an easy consequence of Fact 2.2 (stated below) which asserts the existence of resplendent models. So let us recall the definition of resplendent models (see Poizat [5] or Kaye [4], for details): Let M be an $\mathcal{L}$-structure, and $P_1, P_2, \ldots$ are new relation symbols not in $\mathcal{L}$. M is *resplendent* if for every sentence $\varphi(P_{i_1}, \ldots, P_{i_k})$ consistent with $\mathrm{Th}(M, a)_{a \in M}$, there are interpretations of $P_{i_1}, \ldots, P_{i_k}$ on the domain of M.

Fact 2.2 Let M be an $\mathcal{L}$-structure with $|\mathcal{L}| \leq |M|$. Then M has a resplendent elementary extension of the same cardinality.

Proof of Proposition 2.1 Let N be its resplendent elementary extension of the same cardinality. Let A be an undefinable subset of N, and $\varphi(P)$ any $\mathcal{L}(N) \cup \{P\}$-sentence such that $(N, A) \models \varphi(P)$. By Beth's definability theorem, since A is undefinable, $\varphi(P) \wedge \varphi(P') \wedge \{\neg \forall x (P(x) \leftrightarrow P'(x))\}$ is consistent with $\mathrm{Th}(N, a)_{a \in N}$. Hence, by the resplendency of N, there are interpretations of P, P' on N such that the expansion of N satisfies $\varphi(P) \wedge \varphi(P') \wedge P \neq P'$. $\square$

If we apply Proposition 2.1 to structures with categorical theories, we have the following.

Corollary 2.3

1. *Let M be a model of a totally categorical theory. Then any implicitly definable subset of M is definable in M.*
2. *Let M be a model of an $\aleph_1$-categorical countable theory and $|M| \geq \aleph_1$. Then any implicitly definable subset of M is definable in M.*
3. *Let T be a countable strongly minimal theory, $M \models T$, and $\dim_M(M) \geq \aleph_0$. Then any implicitly definable subset of M is definable in M.*

As an algebraically closed field is strongly minimal, from the above we get the following.

Corollary 2.4 *Let M be an algebraically closed field with infinite transcendence degree. Then no undefinable subset of M is implicitly definable in M. In particular, no infinite proper subfield is implicitly definable in M.*

Remark 2.5 In general, if there is an automorphism σ of M with $\sigma(A) \neq A$ and $\sigma | B = \mathrm{id}_B$, then A is not implicitly definable in M using parameters from B. So, if M is a model of a strongly minimal theory T with $|\mathrm{acl}(\varnothing)| < \omega$, then any undefinable (over $\varnothing$) subset A of M is moved by some automorphism, hence A is not $\varnothing$-implicitly definable. Of course, a subset fixed by the automorphism group need not be implicitly definable.

Finally in this section, we consider relative implicit definability of two subsets in a given structure. Let $A, B \subset M$. Let us say that A and B are *interdefinable* in M if A is a definable subset of the structure (M, B), and B is a definable subset of the structure (M, A). Let us state a trivial example: Let M be an infinite set without structure and A an infinite coinfinite subset of M. Let $B = M$. Then A and B are not interdefinable in M. Also we see that B is implicitly definable in M whereas A is not implicitly definable in M.

Lemma 2.6 *Let $A, B \subset M$. If A and B are interdefinable in M, then A is implicitly definable in M if and only if B is implicitly definable in M.*

Proof First prepare two predicates P for A and Q for B. Suppose that $\varphi(P)$ defines A implicitly in M. We show that B is implicitly definable in M. Let $\theta(x, Q)$ be an $\mathcal{L} \cup \{Q\}$-formula that defines A in (M, B), and $\theta'(x, P)$ an $\mathcal{L} \cup \{P\}$-formula that defines B in (M, A). Let $\psi(Q)$ be a formula expressing

1. the set $\{x : \theta(x, Q)\}$ is a (unique) solution of $\varphi(P)$ (i.e., $\varphi(\{x : \theta(x, Q)\})$),
2. $Q(y)$ if and only if $\theta'(y, \{x : \theta(x, Q)\})$.

Then it is easy to see that B is the unique set satisfying $\psi(Q)$ in M. □

Remark 2.7 By Robinson's result, $\mathbb{Q}$ and $\mathbb{N}$ are interdefinable in their extension field. So for $\mathbb{Q} < K$, $\mathbb{N}$ is implicitly definable in K if and only if $\mathbb{Q}$ is implicitly definable in K.

3 Undefinability

The theory ACF_p of algebraically closed fields of a fixed characteristic admits quantifier elimination. So for any field F, the theory $\mathrm{ACF} \cup \mathrm{Diag}(F)$ is a complete theory. However $\mathrm{ACF} \cup \{\varphi^P : F \models \varphi\}$ is usually incomplete in the language $\mathcal{L}_{\mathrm{ring}} \cup \{P\}$. In order to make it complete, we need to add one sentence. The following lemma is due to Chang and Keisler [1], Theorem 5.4.6.

Lemma 3.1 *Let F be any field of characteristic p. Let T be the set*

$$\mathrm{ACF}_p \cup \{\varphi^P : F \models \varphi\} \cup \{\delta\},$$

where δ is an $(\mathcal{L}_{\mathrm{ring}} \cup \{P\})$-sentence which asserts that there is an element in the overfield that is not a zero of a polynomial of degree 2 with coefficients in P. Then T is a complete $(\mathcal{L}_{\mathrm{ring}} \cup \{P\})$-theory.

Theorem 3.2 *$\overline{\mathbb{Q}}$ is not implicitly $\varnothing$-definable in any of its (proper) elementary extensions $K \succ \overline{\mathbb{Q}}$. Moreover, for each $\mathcal{L}_{\mathrm{ring}} \cup \{P\}$-sentence $\varphi(P)$ satisfied by $\overline{\mathbb{Q}}$ in K, there are F_1 and F_2 both satisfying $\varphi(P)$ such that $F_1 \subsetneq \overline{\mathbb{Q}} \subsetneq F_2 \subset K$.*

Proof It is sufficient to show the "moreover" part. By Lemma 3.1, the complete theory T defined there proves $\varphi(P)$. So by compactness, there is $n \in \omega$ such that

$$\mathrm{ACF}_0 \cup \{P \textit{ is a field}\} \cup \{\varphi_1, \ldots, \varphi_n\} \cup \{\delta\}$$

proves $\varphi(P)$, where φ_i is the sentence asserting that every polynomial equation of degree i with coefficients from P has a solution in P. We can take $n \geq 3$. Let F_1 be the closure of $\mathbb{Q}$ under the operation of adding roots of polynomials of degree $\leq n$. Then F_1 is a nonalgebraically closed subfield of $\overline{\mathbb{Q}}$ such that any polynomial equation of degree $\leq n$ with coefficients from F_1 has a solution in F_1 and $(K, F_1) \models \delta$. Then we have $(K, F_1) \models \varphi(P)$. Similarly, by taking the closure of $\overline{\mathbb{Q}}(t)$ where $t \in K \setminus \overline{\mathbb{Q}}$, there is a field F_2 with $\overline{\mathbb{Q}} \subsetneq F_2 \subsetneq K$ such that $(K, F_2) \models \varphi(P)$. □

Remark 3.3 Assume $\mathrm{tr.deg}(K/\overline{\mathbb{Q}}) \geq 2$. By Lemma 3.1, we have the elementary equivalence of two $\mathcal{L}_{\mathrm{ring}} \cup \{P\}$-structures $(K, \overline{\mathbb{Q}})$ and $(K, \overline{\mathbb{Q}(a)})$, where $a \in K \setminus \overline{\mathbb{Q}}$. So $\overline{\mathbb{Q}}$ is not implicitly definable in K even if we use an infinite set of formulas.

4 Definability

In Section 3, we showed that $\overline{\mathbb{Q}}$ is not implicitly definable in its elementary extension. In a sense, the proof used the nonfinite axiomatizability of $\overline{\mathbb{Q}}$. But this is not exactly true. Recall that $\mathrm{Th}(\mathbb{Q})$ is not finitely axiomatizable. Despite this fact, we show in this section that $\mathbb{Q}$ is implicitly definable in any extension field of finite transcendence degree.

Proposition 4.1 *Let K be an extension field of $\mathbb{Q}$ with $\mathrm{tr.deg}_{\mathbb{Q}} K < \omega$. Then*

1. *$\mathbb{Q}$ is implicitly $\varnothing$-definable in K;*
2. *in general, if $F < K$ is a finite algebraic extension of $\mathbb{Q}$, then F is implicitly definable in K using parameters;*
3. *furthermore, if F above is normal over $\mathbb{Q}$, then the parameters are not necessary.*

For proving this proposition, the following preparations are necessary.

Definition 4.2 Let $\mathcal{L}_{\mathrm{arith}}$ be the language of arithmetic, and M an $\mathcal{L}_{\mathrm{arith}}$-structure.

1. A partial type $p(\overline{x})$ over $\overline{a}$ is recursive if and only if the set
$$\{\ulcorner \varphi(\overline{x}, \overline{y}) \urcorner : \varphi(\overline{x}, \overline{a}) \in p(\overline{x})\} \subseteq \mathbb{N}$$
is recursive, where $\ulcorner \varphi \urcorner$ denotes the code of the formula φ.
2. A partial type $p(\overline{x})$ over $\overline{a}$ is Σ_n if and only if it consists of Σ_n-formulas.
3. An $\mathcal{L}_{\mathrm{arith}}$-structure M is Σ_n-recursively saturated if and only if every recursive Σ_n-type over a finite subset of M is realized in M.

Now recall that $I\Sigma_n$ is the theory obtained from **PA** by restricting induction formulas to Σ_n-formulas, and note that we can formulate it in the ring language.

Fact 4.3

1. Let $M \models I\Sigma_n$ be nonstandard. ($M \neq \mathbb{N}$.) Then M is Σ_n-recursively saturated.
2. For $n > 0$, $I\Sigma_n$ is finitely axiomatizable.

For a proof of part 1, the reader can consult [4], p. 150. For part 2, see Hájek and Pudlák [2], p. 78.

Lemma 4.4 *Let $M \models I\Sigma_1$ be nonstandard. Then M contains countably many algebraically independent elements over $\mathbb{Q}$ in the field theoretic sense.*

Proof Suppose M contains $b_1, \ldots, b_n$ which are transcendental over $\mathbb{Q}$. We show that there is another transcendental element over $\mathbb{Q}(\overline{b})$ in M.

Let S be the set of all nontrivial polynomials with coefficients from $\mathbb{Q}(\overline{b})$ and $p(x) = \{f(x) \neq 0 : f(x) \in S\}$. By transposition and multiplication by common denominators, we may consider $p(x)$ as the set of $\mathcal{L}_{\mathrm{ring}}$-formulas. Obviously $p(x)$ is a recursive Σ_0-type over $\overline{b}$ in M. Then part 1 of Fact 4.3 above shows that there is an element which realizes $p(x)$, which is transcendental over $\mathbb{Q}(\overline{b})$. □

Proof of Proposition 4.1

(1) Choose a formula $\theta(x)$ in the ring language such that $\theta(x)$ defines $\mathbb{N}$ in $\mathbb{Q}$. Let $\psi(P)$ be a set of sentences expressing

(a) P is a subfield;
(b) the subset of P defined by $\theta^P(x)$ satisfies $I\Sigma_1$;
(c) $\forall x \in P \exists y \in P \exists z \in P[\theta^P(y) \wedge \theta^P(z) \wedge z \neq 0 \wedge (zx = y \vee zx = -y)]$.

By Fact 4.3, $\psi(P)$ is a first-order sentence. We shall show that $\psi(P)$ implicitly defines $\mathbb{Q}$ in K. Clearly $\mathbb{Q}$ satisfies the formula $\psi(P)$. So by way of contradiction, assume that there is a subset $\mathbb{Q}^* \supsetneq \mathbb{Q}$ of K with $K \models \psi(\mathbb{Q}^*)$. By the properties (b) and (c) above, $\theta(x)$ defines in $\mathbb{Q}^*$ a nonstandard model $\mathbb{N}^*$ of $I\Sigma_1$. So by Lemma 4.4, $\mathbb{N}^*$ has infinitely many algebraically independent elements over $\mathbb{Q}$. This contradicts our assumption $\mathrm{tr.deg}_{\mathbb{Q}} K < \omega$.

(2) Robinson showed in [6] that $\mathbb{Q}$ is (first-order) $\varnothing$-definable in finite algebraic number fields. Together with the fact that a finite algebraic number field has a finite basis over $\mathbb{Q}$, it follows that F and $\mathbb{Q}$ are interdefinable in K (using parameters). Then, by Lemma 2.6 and part (1), we see that F is implicitly definable in K using parameters. However, we give a direct proof below.

Using part (1), choose an $(\mathcal{L}_{\mathrm{ring}} \cup \{Q\})$-formula $\psi(Q)$ that implicitly defines $\mathbb{Q}$ in K. Let $\theta(x)$ be an $\mathcal{L}_{\mathrm{ring}}$-formula that defines $\mathbb{Q}$ in F. Choose $a_1, \ldots, a_k \in F$ such that $F = a_1\mathbb{Q} + \cdots + a_k\mathbb{Q}$. Now let $\varphi(P)$ be a formula expressing the following:

(d) The subset of K defined by $\theta^P(x)$ satisfies ψ;
(e) For all $x \in K$, $x \in P$ iff there are $q_1, \ldots, q_n$ with $\theta^P(q_1) \wedge \cdots \wedge \theta^P(q_k)$ such that $x = a_1q_1 + \cdots + a_kq_k$.

Claim 4.5 *$\varphi(P)$ implicitly defines F in K.*

Clearly $\varphi(F)$ holds in K. Let F' be another solution of $\varphi(P)$. By property (d) and the choice of ψ, we have

$$\mathbb{Q} = \{a \in F' : F' \models \theta(a)\}.$$

So, by property (e), we have

$$F = a_1\mathbb{Q} + \cdots + a_k\mathbb{Q} = \{a_1q_1 + \cdots + a_kq_k : q_i \in F', F' \models \theta(q_i)\} = F'.$$

(3) In the above definition of $\varphi(P)$, choose $A = \{a_1, \ldots, a_k\} \subset P^K$ so that A is fixed by any automorphism of K. (This can be done by the normality.) Then A is $\varnothing$-definable, hence the parameters in $\varphi(P)$ can be eliminated. $\square$

Combining Proposition 4.1 with Corollary 2.4, we get the following.

Corollary 4.6 *Let K be an algebraically closed field of characteristic* 0. *Then $\mathbb{Q}$ is implicitly definable in K if and only if* $\mathrm{tr.deg}_{\mathbb{Q}} K$ *is finite.*

Remark 4.7 It is known that any Peano field different from $\mathbb{Q}$ has infinite transcendence degree over $\mathbb{Q}$. (A field elementary equivalent to $\mathbb{Q}$ is called a Peano field. See Jensen and Lenzing [3].) This fact follows easily from Lemma 4.4. Moreover, from Lemma 4.4 we can say that if $K \neq \mathbb{Q}$ is a field constructed from a model of $I\Sigma_1$, then K has infinite transcendence degree over $\mathbb{Q}$.

As mentioned in the proof of Proposition 4.1, $\mathbb{Q}$ is (first-order) $\varnothing$-definable in finite algebraic extensions of $\mathbb{Q}$. Hence trivially $\mathbb{Q}$ is implicitly $\varnothing$-definable in those extensions. However, only a little is known about definability of subfields in fields. We only know some particular cases. $\mathbb{Q}$ is not definable in algebraically closed fields, since algebraically closed fields are stable. $\mathbb{Q}$ is not $\varnothing$-definable in real closed fields, since the theory of real closed fields is decidable.

It is also known that $\mathbb{Q}$ is $\varnothing$-definable in pure transcendental extensions of $\mathbb{Q}$. (See Robinson [7].) We don't know whether or not $\mathbb{Q}$ is definable in finite algebraic extensions of pure transcendental extensions of $\mathbb{Q}$. However, a similar argument as above proves the following proposition.

Proposition 4.8 *Let K be a finite algebraic extension of $\mathbb{Q}(I)$, where I is an algebraically independent set (a set of indeterminants). Then $\mathbb{Q}$ is implicitly definable in K.*

We will consider the following property (†) for an $\mathcal{L}_{\text{ring}}$-structure K.

(†) There is $a \in K$ such that $a^2 - a \neq 0$ and for each $n \in \omega$, there is $b \in K$ with $a = b^n$.

Remark 4.9 Let I be an algebraically independent set over $\mathbb{Q}$. Then $\mathbb{Q}(I)$ does not have the property (†), since $\mathbb{Q}[I]$ is UFD. Let F be an algebraic number field. Then F does not have the property (†) by Dirichlet's unit theorem and the unique factorization into prime ideals of fractional ideals in the ring of algebraic integers of F.

Lemma 4.10 *Let K be a finite extension of $\mathbb{Q}(I)$. Then K does not have the property (†).*

Proof By way of contradiction, choose an element $a \in K$ witnessing the property (†). By the previous remark, we have $a \notin K \cap \overline{\mathbb{Q}}$. We can choose a minimal finite subset $\{x_1, \ldots, x_k\}$ of I with $a \in \overline{\mathbb{Q}(x_1, \ldots, x_k)}$. Let $M = K \cap \overline{\mathbb{Q}(x_1, \ldots, x_k)}$. Then $M \supseteq \{\sqrt[n]{a} : n \in \omega\}$. Since I is an algebraically independent set over $\mathbb{Q}$, we have $[M : \mathbb{Q}(x_1, \ldots, x_k)] \leq [K : \mathbb{Q}(I)]$. Hence $[M : \mathbb{Q}(x_1, \ldots, x_k)]$ is finite. Put $F = \mathbb{Q}(x_1, \ldots, x_{k-1})$. Then $[M : F(a)]$ is also finite. However, for a prime $m > [M : F(a)]$, we have $[M : F(a)] \geq [F(\sqrt[m]{a}) : F(a)] = m$. This is a contradiction. □

Proof of Proposition 4.8 Suppose otherwise. By a similar argument as in the proof of Proposition 4.1, we can show the existence of a nonstandard model $\mathbb{N}^* \subset K$ of $I\Sigma_1$. Consider the following Σ_1-recursive type

$$p(x) = \{x^2 - x \neq 0\} \cup \{\exists y(x = y^n) : n \in \omega, n \geq 1\}.$$

Then by Fact 4.3, $p(x)$ is realized in $\mathbb{N}^*$. Hence there is $a \in K$ which witnesses the property (†). However, by Lemma 4.10, we know that this is impossible. □

5 Examples

We state some examples.

Example 5.1 Clearly $\mathbb{Q}$ is not $\varnothing$-definable in $\mathbb{R}$. But $\mathbb{Q}$ is implicitly definable in $\mathbb{R}$. Let τ be the sentence

$$\forall x, y \in (P \wedge \theta^P)[\exists z \in (P \wedge \theta^P)(x + z = y) \iff \exists z(x + z^2 = y)]$$

and put $\psi'(P) = \psi(P) \wedge \tau$, where $\theta(x)$, $\psi(P)$ is defined as in the proof of Proposition 4.1. Clearly $\mathbb{Q}$ satisfies the formula $\psi'(P)$. We shall show that $\psi'(P)$ implicitly defines $\mathbb{Q}$ in $\mathbb{R}$. Suppose otherwise. Then there is a subset $\mathbb{Q}^* \neq \mathbb{Q}$ satisfying $\psi'(P)$. So the subset $\mathbb{N}^*$ of $\mathbb{Q}^*$ defined by θ is a nonstandard model of $I\Sigma_1$. The order structure $\leq$ of $\mathbb{N}^*$ (defined by $\exists z(x + z = y)$) coincides with the original order in $\mathbb{R}$. (This is guaranteed by the sentence τ.) This shows that $\mathbb{R}$ is a non-Archimedean field. A contradiction.

Example 5.2 In $\overline{\mathbb{Q}}$, the real closed field $R = \mathbb{R} \cap \overline{\mathbb{Q}}$ is not implicitly definable. Let A be an arbitrary finite subset of $\overline{\mathbb{Q}}$. Let $n = [\mathbb{Q}(A) : \mathbb{Q}]$. Let us consider the polynomial $f(x) = x^p - 2$, where $p > n$ is a prime. By Eisenstein's criterion, $f(x)$ is irreducible. R has the unique solution, say α_1, of $f(x) = 0$. Clearly α_1 does not belong to $\mathbb{Q}(A)$. So the minimal polynomial $g(x)$ of α_1 over $\mathbb{Q}(A)$ has a root, say α_2, other than α_1. Let σ be an isomorphism of $\overline{\mathbb{Q}}$ such that $\sigma(\alpha_1) = \alpha_2$. Then $\sigma(R) \neq R$. This shows that R is not implicitly A-definable in $\overline{\mathbb{Q}}$.

Finally, we end this paper by stating some open questions.

Question 5.3 *By [8], there is an elementary extension M of $\mathbb{Q}$ with $|M| = \aleph_1$ in which $\mathbb{Q}$ is implicitly definable. Can we require M to be countable?*

Question 5.4 *Let K be an algebraically closed field of finite transcendence degree. Then by Proposition 4.1, any field $\mathbb{Q}(a) < K$ with a algebraic is implicitly definable in K. What happens if we require that a is transcendental?*

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Acknowledgments

Second author was partially supported by Japanese Grant-in-Aid for Scientific Research (C) 15540104.

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