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On Sequentially Compact Subspaces of $\mathbb{R}$ without the Axiom of Choice

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Abstract We show that the property of sequential compactness for subspaces of $\mathbb{R}$ is countably productive in ZF. Also, in the language of weak choice principles, we give a list of characterizations of the topological statement 'sequentially compact subspaces of $\mathbb{R}$ are compact'. Furthermore, we show that forms 152 (= every non-well-orderable set is the union of a pairwise disjoint well-orderable family of denumerable sets) and 214 (= for every family A of infinite sets there is a function f such that for all $y \in A$, $f(y)$ is a nonempty subset of y and $|f(y)| = \aleph_0$) of Howard and Rubin are equivalent.

1 Introduction

Zermelo's Axiom of Choice AC has been proven to be an indispensable tool in mathematics. One of the most typical examples is Tychonoff's compactness theorem which was proven to be equivalent to AC by Kelley [11] in 1950. Adapting Kelley's proof one can similarly show that Tychonoff's theorem for countable families of nonempty compact spaces implies choice for countable families. For an extensive study on versions of the countable Tychonoff theorem the reader is referred to Howard et al. [7]. Naturally, one may ask what happens when the compact spaces involved in the countable version of Tychonoff's theorem are subsets of the real line. Loeb [16] using the fact that every family of nonempty closed subsets of $\mathbb{R}$ has a constructive choice function (for a generalization of this fact to conditionally complete linearly ordered spaces, see Keremedis and Tachtsis [14]) established that the countable Tychonoff theorem for compact subsets of $\mathbb{R}$ is true in ZF. The interested reader is also referred to De la Cruz et al. [2] for a generalization of this version of Tychonoff's theorem to well-ordered families of compact subsets of $\mathbb{R}$. In the realm

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of sequentially compact subsets of $\mathbb{R}$ we show in Theorem 4.1 that this notion of compactness is also countably productive in the settings of ZF.

Fundamental properties of the topology of the real line such as 'every subspace of $\mathbb{R}$ is separable' or '$x \in \overline{A}$ if and only if there exists a sequence $(x_n)_{n\in\omega} \subseteq A$ converging to x' cannot be proved without using some form of the axiom of choice. In fact, Herrlich and Strecker [6] have shown that the latter statements are topological characterizations of CC($\mathbb{R}$), the axiom of countable choice restricted to nonempty subsets of $\mathbb{R}$. Complete definitions will be given in Section 2. It is a well-known theorem of ZF (the Zermelo-Fraenkel set theory) that "A subspace X of $\mathbb{R}$ is compact if and only if X is closed and bounded." But the well-known ZFC (ZF plus AC) theorem and Form 74 in Howard and Rubin [9], "sequentially compact (= every sequence has a convergent subsequence) subspaces of $\mathbb{R}$ are compact" is not a theorem of ZF. Indeed, in the original Cohen model ($\mathcal{M}1$ in [9]), the set A of the countably many added generic reals is sequentially compact, since it has no countably infinite subsets in the model, which fails to be compact or even Lindelöf. This example implicitly suggests that there is some connection between Form 74 and the principle "every infinite subset of $\mathbb{R}$ has a countably infinite subset" labeled as Form 13 in [9]. In Keremedis [12] it is shown that the statement "every metric space having the property that each of its sequences has a cluster point is countably compact" implies Form 13. Since $\mathbb{R}$ is hereditarily second countable, sequential compactness for subsets of $\mathbb{R}$ coincides with the above-mentioned property and the argument in [12] readily adapts to show that 74 implies 13.

In Theorem 3.1 we give a list of characterizations of 74. For further study on nonconstructive properties of the real line the reader is also referred to Howard and Rubin's long term project *Consequences of the Axiom of Choice* [9], to Jech's book *The Axiom of Choice* [10], and to the papers Feferman and Lévy [3], Gutierres [4], Herrlich [5], [6], Howard et al. [8], Keremedis and Tachtsis [15], and Truss [17].

2 Notation and Some Preliminary Results

Let (X, T) be a topological space.

1. X is called *compact* if every open cover of X has a finite subcover.
2. X is called *countably compact* if every countable open cover of X has a finite subcover.
3. X is called *Lindelöf* if every open cover has a countable subcover.
4. X is called *separable* if X has a countable dense subset.
5. X is called *sequentially compact* if every sequence in X has a convergent subsequence.

Let $A \subseteq X$. A point $x \in X$ is called a *cluster point* of A if every neighborhood of x meets A in at least one element other than x. Let $(x_n)_{n\in\omega}$ be a sequence in X. A point $x \in X$ is called a cluster point of $(x_n)_{n\in\omega}$ if every neighborhood of x contains infinitely many terms of $(x_n)_{n\in\omega}$. A subset A of a partially ordered set $(P, \leq)$ is said to be *cofinal* in P if for every $p \in P$, there exists $a \in A$ such that $p \leq a$.

A metric space (X, d) is called *complete* if every Cauchy sequence in X converges.

CC (Form 8 in [9]) For every countable family $\mathcal{A}$ of nonempty sets there exists a function f such that for all $x \in \mathcal{A}$, $f(x) \in x$.

CC($\mathbb{R}$) (Form 94 in [9]) CC restricted to countable families of nonempty subsets of $\mathbb{R}$.

ω-AC($\mathbb{R}$) For every family $\mathcal{A}$ of nonempty subsets of $\mathbb{R}$ there exists a function f such that for all $x \in \mathcal{A}$, $f(x)$ is a nonempty countable subset of x.

ω-CC($\mathbb{R}$) ω-AC($\mathbb{R}$) restricted to countable families.

Pω-CC($\mathbb{R}$) For every countable family $\mathcal{A}$ of nonempty subsets of $\mathbb{R}$ there is an infinite subfamily $\mathcal{B}$ of $\mathcal{A}$ and a function f such that for all $B \in \mathcal{B}$, $f(B)$ is a nonempty countable subset of B.

WO-AC($\mathbb{R}$) For every family $\mathcal{A}$ of nonempty subsets of $\mathbb{R}$ there exists a function f such that for all $x \in \mathcal{A}$, $f(x)$ is a nonempty well-orderable subset of x.

WO-CC($\mathbb{R}$) WO-AC($\mathbb{R}$) restricted to countable families.

CC(c-$\mathbb{R}$) CC($\mathbb{R}$) for families of nonempty complete subsets of $\mathbb{R}$.

CC(sc-$\mathbb{R}$) CC($\mathbb{R}$) for families of nonempty sequentially compact subsets of $\mathbb{R}$.

Form 13 in [9] Every infinite subset of $\mathbb{R}$ has a denumerable (= countably infinite) subset.

Form 74 in [9] Every sequentially compact subspace of $\mathbb{R}$ is compact.

Form 152 in [9] Every non-well-orderable set is the union of a pairwise disjoint well-orderable family of denumerable sets.

Form 214 in [9] For every family $\mathcal{A}$ of infinite sets there is a function f such that for all $y \in \mathcal{A}$, $f(y)$ is a nonempty subset of y and $|f(y)| = \aleph_0$.

The axiom ω-AC($\mathbb{R}$) was introduced in [15] and its weaker forms were first considered in [4].

Theorem 2.1 (**[4]**)

1. ω-CC*($\mathbb{R}$) implies* WO-CC*($\mathbb{R}$).*
2. WO-CC*($\mathbb{R}$) implies every sequentially compact subset of $\mathbb{R}$ is compact.*
3. *If every sequentially compact subset of $\mathbb{R}$ is compact, then Form 13 holds.*

Theorem 2.2 (ZF) *Every sequentially compact, closed subspace of $\mathbb{R}$ is compact.*

Proof Fix A a sequentially compact and closed subspace of $\mathbb{R}$. A is separable follows from the observation that the family $\{A \cap [p, q] : p, q \in \mathbb{Q}\} \backslash \{\varnothing\}$ is a countable family of nonempty closed subsets of $\mathbb{R}$ and the fact that $\mathcal{G} = \{G \subset \mathbb{R} : G \neq \varnothing$ and closed$\}$ has a choice function in ZF (see [2], [14], [16]). The assertion that A is bounded is straightforward. □

Proposition 2.3 *Form 13 if and only if every infinite sequentially compact subset of $\mathbb{R}$ has a denumerable subset.*

Proof ($\Rightarrow$) Straightforward.

($\Leftarrow$) Let A be an infinite set. If A has no denumerable subsets, then A is trivially sequentially compact. By our hypothesis, A has a denumerable subset, a contradiction. □

3 Characterizations of the Axiom CC(sc-$\mathbb{R}$)

Theorem 3.1 *The following are pairwise equivalent:*

1. CC(c-$\mathbb{R}$).
2. CC(sc-$\mathbb{R}$).
3. *A countable Tychonoff product of nonempty sequentially compact subsets of $\mathbb{R}$ is nonempty.*
4. PCC(sc-$\mathbb{R}$) *(= for every countable family $\mathcal{A}$ of nonempty sequentially compact subspaces of $\mathbb{R}$ there is an infinite subfamily $\mathcal{B}$ of $\mathcal{A}$ which has a choice function).*
5. ω-CC(sc-$\mathbb{R}$) *(=* ω-CC($\mathbb{R}$) *for families of nonempty sequentially compact subspaces of $\mathbb{R}$).*
6. Pω-CC(sc-$\mathbb{R}$) *(=* Pω-CC($\mathbb{R}$) *for families of nonempty sequentially compact subspaces of $\mathbb{R}$).*
7. *Every sequentially compact subspace of $\mathbb{R}$ is compact.*
8. *Every sequentially compact subset of $\mathbb{R}$ has a cofinal subset which can be expressed as a well-ordered union of well-orderable sets.*
9. *Every unbounded sequentially compact subset of $\mathbb{R}$ has a countable unbounded subset.*

Proof

(1) $\Leftrightarrow$ (7) This has been established in [4].

(7) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (6), (3) $\Rightarrow$ (5) $\Rightarrow$ (6), and (7) $\Leftrightarrow$ (9) are straightforward.

(6) $\Rightarrow$ (7) In view of Theorem 2.2, it suffices to show that Pω-CC(sc-$\mathbb{R}$) implies *sequentially compact subspaces of $\mathbb{R}$ are closed*. The latter implication is, in view of Theorem 3.2 from [4], straightforward.

(7) $\Rightarrow$ (8) Fix $A \subseteq \mathbb{R}$ a sequentially compact space. By (7) it follows that A is closed and bounded. Therefore $\{\sup A\}$ is the required cofinal subset of A.

(8) $\Rightarrow$ (6) Fix $\mathcal{A} = \{A_i : i \in \omega\}$ a family of nonempty sequentially-compact subsets of $\mathbb{R}$. Without loss of generality assume that for each $i \in \omega$, $A_i \subseteq (i, i+1)$. (Let $f : \mathbb{R} \to (0,1)$ be a homeomorphism and let for each $i \in \omega$, $f_i : \mathbb{R} \to (i, i+1)$ defined by $f_i(x) = f(x) + i$. Then $f_i(A_i)$ is a sequentially compact subset of $\mathbb{R}$ such that $f_i(A_i) \subseteq (i, i+1)$.) Assume that P$\omega$-CC(sc-$\mathbb{R}$) cannot be applied to $\mathcal{A}$. Then clearly $\bigcup \mathcal{A}$ is unbounded and sequentially compact (every sequence $(x_n)_{n\in\omega}$ in $\bigcup \mathcal{A}$ is such that $(x_n)_{n\in\omega} \subseteq \bigcup_{i<m} A_i$ for some $m \in \omega$. Thus, $(x_n)_{n\in\omega}$ necessarily meets some A_i in infinite many terms and since A_i is sequentially compact, $(x_n)_{n\in\omega}$ has a convergent subsequence). Let, by our hypothesis, $B = \bigcup\{B_i : i \in \alpha\}$, α an ordinal, and each B_i well-orderable, a cofinal subset of $\bigcup \mathcal{A}$. Via an easy inductive argument construct a strictly increasing sequence of integers $(i_n)_{n\in\omega}$ and a function f such that for each $n \in \omega$, $f(A_{i_n})$ is a nonempty well-orderable subset of A_{i_n}.

Since for all $n \in \omega$, A_{i_n} is sequentially compact and $f(A_{i_n})$ is well-orderable, it follows that $\overline{f(A_{i_n})} \subseteq A_{i_n}$ for all $n \in \omega$. Then any choice function of the family $C = \{\overline{f(A_{i_n})} : n \in \omega\}$ satisfies Pω-CC(sc-$\mathbb{R}$) for the family $\mathcal{A}$. This contradiction completes the proof of the theorem. $\square$

Remark 3.2

(A) Similarly to the proof of Theorem 3.1, one may show that ω-CC($\mathbb{R}$) is equivalent to its partial version Pω-CC($\mathbb{R}$). A similar argument cannot be applied to complete subspaces of $\mathbb{R}$ since completeness is not preserved under homeomorphisms.

(B) The following statements can be added to the list of Theorem 3.1:

(i) Every sequentially compact subset A of $\mathbb{R}$ is countably compact.

(ii) Every sequentially compact subset A of $\mathbb{R}$ is weakly Lindelöf. (A is *weakly Lindelöf* if every open cover of A has a countable subfamily with dense union in A.)

(iii) Every sequentially compact subset A of $\mathbb{R}$ is pre-Lindelöf. (A is *pre-Lindelöf* if for every $\varepsilon > 0$, A can be covered by countably many open discs of radius ε.)

It is evident that the statement "sequentially compact subsets of $\mathbb{R}$ are compact" implies each one of (i), (ii), and (iii). Now, (i) $\Rightarrow$ (ii) is straightforward (every second countable and countably compact space is compact without appealing to any form of choice) and (ii) $\Rightarrow$ (iii) has been proved in Keremedis [13] generally for metric spaces. To see that (iii) implies back that the notions of compactness and sequential compactness coincide, in view of Theorem 3.1, it suffices to show that (iii) implies every unbounded sequentially compact subset of $\mathbb{R}$ has an unbounded sequence. Fix such a subset $A \subseteq \mathbb{R}$. Consider the open cover $\mathfrak{U} = \{D(x, \varepsilon) : x \in A\}$ of A where $D(x, \varepsilon) = \{y \in A : |x - y| < \varepsilon\}$. As A is pre-Lindelöf, $\mathfrak{U}$ has a countable subcover $\mathfrak{V} = \{D(x_n, \varepsilon) : n \in \omega\}$. It is evident that the sequence $(x_n)_{n\in\omega}$ is unbounded.

Clearly, the statement "if $A \subseteq \mathbb{R}$ is sequentially compact, then $\overline{A}$ is sequentially compact" is a theorem of ZF+CC. We show next that it is unprovable in ZF by establishing its equivalence to the weak choice principle CC(sc-$\mathbb{R}$).

Theorem 3.3 *The following statements are equivalent:*

1. CC(sc-$\mathbb{R}$).
2. *If $A \subseteq \mathbb{R}$ is sequentially compact, then $\overline{A}$ is sequentially compact.*
3. *If $A \subseteq \mathbb{R}$ is sequentially compact, then $\overline{A}$ is bounded.*
4. *If $A \subseteq \mathbb{R}$ is sequentially compact, then $\overline{A}$ is compact.*
5. *If $A \subseteq \mathbb{R}$ is sequentially compact, then $\overline{A}$ is Lindelöf.*

Proof

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are straightforward in view of Theorem 2.2 and the equivalence between (2) and (7) of Theorem 3.1.

(5) $\Rightarrow$ (1) Let $\mathcal{A} = \{A_i : i \in \omega\}$ be a family of nonempty sequentially compact subspaces of $\mathbb{R}$. Assume that $\mathcal{A}$ has no choice function. Then the axiom CC($\mathbb{R}$) fails. Herrlich [5] proved that for T_1 spaces, Lindelöf $\equiv$ compact if and only if CC($\mathbb{R}$) fails. By our hypothesis and Herrlich's result we conclude that for each $i \in \omega$, $\overline{A_i}$ is compact, thus bounded. Therefore, each A_i is bounded. For each $i \in \omega$, let $a_i = \sup(A_i)$.

Claim 3.4 *For all $i \in \omega$, $a_i \in A_i$.*

Proof of the claim Assume on the contrary that for some $i \in \omega$, $a_i \notin A_i$. Let $H : (-\infty, a_i) \to \mathbb{R}$ be an increasing homeomorphism. Then $H(A_i)$ is an unbounded sequentially compact subset of $\mathbb{R}$. Since $\overline{H(A_i)}$ is sequentially compact we see that $\overline{H(A_i)}$ is bounded. Thus, $H(A_i)$ is bounded, a contradiction. This completes the proof of the claim. □

By the claim we immediately have that $f = \{(A_i, a_i) : i \in \omega\}$ is a choice function for $\mathcal{A}$. This contradiction completes the proof of the theorem. □

Remark 3.5 It can be readily verified that each of the following propositions is a theorem of ZF.

1. If $A \subseteq \mathbb{R}$ is separable, then $\overline{A}$ is separable.
2. If $A \subseteq \mathbb{R}$ is complete, then $\overline{A}$ is complete.
3. If $A \subseteq \mathbb{R}$ is bounded, then $\overline{A}$ is bounded.
4. If $A \subseteq \mathbb{R}$ is compact, then $\overline{A}$ is compact.

4 Sequential Compactness for Subsets of $\mathbb{R}$ is Countably Productive in ZF

Theorem 4.1 (ZF) *A countable Tychonoff product of sequentially compact subspaces of* $\mathbb{R}$ *is sequentially compact.*

Proof Let $A = \{X_i : i \in \omega\}$ be a family of sequentially compact subspaces of $\mathbb{R}$ and let $X = \prod_{i\in\omega} X_i$ be their Tychonoff product. For each $i \in \omega$, let π_i be the canonical projection of X onto X_i. Let $(x_n)_{n\in\omega}$ be a sequence in X and let f be a choice function on the set of all nonempty closed subsets of $\mathbb{R}$. Via an easy induction we shall construct a convergent subsequence of $(x_n)_{n\in\omega}$.

For $n = 0$, first define $G_0 = \{x \in X_0 : x$ is a cluster point of $(\pi_0(x_n))_{n\in\omega}\}$. Since X_0 is sequentially compact, it follows that $G_0 \neq \varnothing$. We assert that G_0 is closed in $\mathbb{R}$. To see this, fix $g \in \overline{G_0}$. Then for each $n \in \omega$, the set $V_n = (g - \frac{1}{n}, g + \frac{1}{n}) \cap G_0$ is nonempty and contains infinitely many terms of the sequence $s = (\pi_0(x_n))_{n\in\omega}$. Via a straightforward induction construct a subsequence s^* of s converging to the point g. Since $s^* \subseteq X_0$ and X_0 is sequentially compact, it follows that $g \in X_0$. Consequently, $g \in G_0$ and G_0 is closed. Let $g_0 = f(G_0)$. Construct a subsequence h_0 of $(x_n)_{n\in\omega}$ so that $\pi_0(h_0)$ converges to g_0.

Assume that subsequences of $(x_n)_{n\in\omega}$, $h_0, h_1, \ldots, h_n$ have been constructed so that h_j is a subsequence of h_{j-1} and $\pi_j(h_j)$ converges to $g_j \in X_j$ for all $j = 0, 1, \ldots, n$. Now X_{n+1} is a sequentially compact space, therefore, the set $G_{n+1} = \{x \in X_{n+1} : x$ is a cluster point of $\pi_{n+1}(h_n)\} \neq \varnothing$. Moreover, as in the case $n = 0$, it can be shown that G_{n+1} is closed in $\mathbb{R}$. Thus, let $g_{n+1} = f(G_{n+1})$. Construct a subsequence h_{n+1} of h_n such that $\pi_{n+1}(h_{n+1})$ converges to g_{n+1}.

It can be readily verified that the diagonal $(y_n)_{n\in\omega}$, where y_n is the nth term of h_n, is a subsequence of $(x_n)_{n\in\omega}$ converging to $y \in X$ defined by $y(n) = g_n$ for all $n \in \omega$. □

5 Characterizations of ω-AC($\mathbb{R}$), WO-AC($\mathbb{R}$), and the Independence of WO-CC($\mathbb{R}$) from 13

Theorem 5.1 *The following propositions are equivalent:*

1. *Every non-well-orderable set is the union of a pairwise disjoint well-orderable family of infinite well-orderable sets.*
2. *For every family A of infinite sets there is a function f such that for all $y \in A$, $f(y)$ is an infinite well-orderable subset of y.*

Proof (1) $\Rightarrow$ (2) Fix a family $\mathcal{A} = \{A_j : j \in k\}$ of infinite sets and let $\mathcal{B} = \bigcup \mathcal{A}$. If $\mathcal{B}$ is well-ordered, then there is nothing to show. Thus, we may assume that $\mathcal{B}$ is not well-orderable, and by (1), we can express it as $\bigcup\{B_i : i \in \aleph\}$, where $\aleph$ is a well-ordered cardinal and each B_i is an infinite well-orderable set. For every $j \in k$, let i_j be the first $i \in \aleph$ such that $B_i \cap A_j$ is infinite and put $f(A_j) = B_{i_j} \cap A_j$. If no such i exists then $|B_i \cap A_j| < \aleph_0$ for all $i \in \aleph$, and we may pick inductively a sequence $\{i_{j_v} : v \in \omega\} \subset \aleph$ such that $0 < |B_{i_{j_v}} \cap A_j| < \aleph_0$. In this case put $f(A_j) = \bigcup\{B_{i_{j_v}} \cap A_j : v \in \omega\}$.

Claim 5.2 $|\bigcup\{B_{i_{j_v}} \cap A_j : v \in \omega\}| = \aleph_0$.

Proof of the claim For each $v \in \omega$, let n_v be the unique integer such that $|B_{i_{j_v}} \cap A_j| = n_v$. Put $K_{n_v} = \{f \in (B_{i_{j_v}} \cap A_j)^{n_v} : f \text{ is injective}\}$. Clearly, K_{n_v} is nonempty and (1) implies that $\bigcup\{\prod_{k<v} K_{n_k} : v \in \omega\}$ has a denumerable subset $F = \{f_n : n \in \omega\}$. On the basis of F and via a straightforward induction construct an enumeration for $\bigcup\{B_{i_{j_v}} \cap A_j : v \in \omega\}$. This completes the proof of the claim and the implication. □

(2) $\Rightarrow$ (1) Let X be a non-well-orderable set and let f be a function which satisfies (2) for the set $\mathcal{P}^{\infty}(X) = \{Y \subseteq X : |Y| = \infty\}$. Using f and transfinite induction we construct a well-ordered cover $\{X_i : i \in \alpha\}$, α an ordinal number, of X consisting of infinite well-orderable subsets of X.

For $i = 0$, put $X_0 = f(X)$. For $i = \lambda + 1$ a successor ordinal and having chosen infinite and well-orderable subsets X_j, $j < \lambda + 1$, we put $X_i = f(X \setminus \bigcup\{X_j : j < \lambda + 1\})$ if the latter set difference is infinite. Otherwise the induction terminates. For i a limit ordinal we work as in the nonlimit case.

Since $\mathcal{P}^{\infty}(X)$ is a set, the induction must terminate at some ordinal stage α. This means that $X \setminus \bigcup\{X_i : i \in \alpha\}$ is finite. Consequently, X is expressible as a well-ordered union of infinite well-orderable sets and the proof of the theorem is complete. □

In Note 140 of [9] it is shown that 214 $\Longrightarrow$ 152 and in Table I of [9] the status of the reverse implication is indicated as unknown (see the section notation and terminology for the definitions of the axioms). We fill this gap in the next corollary of Theorem 5.1 obtaining also two equivalent forms of the axioms ω-AC($\mathbb{R}$) and WO-AC($\mathbb{R}$), respectively.

Corollary 5.3

1. *152 if and only if 214.*
2. *The following are equivalent:*
   (a) ω-AC($\mathbb{R}$).
   (b) $\mathbb{R}$ *can be written as a well-ordered union of denumerable subsets.*
3. *The following are equivalent:*
   (a) WO-AC($\mathbb{R}$).
   (b) $\mathbb{R}$ *can be written as a well-ordered union of well-orderable subsets.*

In Sections 1 and 2 we mentioned that the implications WO-CC($\mathbb{R}$) $\Longrightarrow$ Form 74 $\Longrightarrow$ Form 13 are valid. In this section we show that in ZF, Form 13 does not imply WO-CC($\mathbb{R}$). We also give a straight proof of the implication WO-CC($\mathbb{R}$) $\Longrightarrow$ Form 13.

Theorem 5.4

1. WO-CC($\mathbb{R}$) *implies Form 13.*
2. *In* ZF, *Form 13 does not imply* WO-CC($\mathbb{R}$).

Proof (1) Fix A an infinite subset of $\mathbb{R}$. Assume that A has no countably infinite subsets. For each $n \in \omega$, define $A_n = \{f \in A^n : f \text{ is injective}\}$. Clearly, each A_n is nonempty and can be considered as a subspace of $\mathbb{R}^\omega(|A_n| \leq |A^n \times \prod_{i \geq n}\{0\}| \leq |\mathbb{R}^\omega|)$. As $\mathbb{R}^\omega$ with the Tychonoff topology is a separable metrizable space, it follows that $|\mathbb{R}^\omega| = |\mathbb{R}|$ and consequently we may consider each A_n as a subset of $\mathbb{R}$. By WO-CC($\mathbb{R}$), let g be a function such that for all $n \in \omega$, $g(A_n)$ is a nonempty well-orderable subset of A_n. Since A has no countably infinite subsets, it can be readily verified that none of the $g(A_n)$s can be infinite (otherwise, $\bigcup\{f[n] : f \in g(A_n)\}$ would be an infinite well-ordered union of finite (linearly ordered) subsets of $\mathbb{R}$, thus well-ordered and infinite). Therefore, we may pick the least element f_n from each $g(A_n)$. Via a straightforward induction construct a countably infinite subset of A. This contradicts our hypothesis and completes the proof of (1).

(2) For our purpose we shall use Truss's forcing model $\mathfrak{M}_\aleph$ in [17], where $\aleph$ is a singular cardinal. (This model is $\mathcal{M}12(\aleph)$ in [9].) First we recall its definition. Let $\mathcal{M}$ be a countable transitive model of ZF + V = L and let $\aleph$ be a singular cardinal in $\mathcal{M}$. For each ordinal α, the set of conditions Q^α is the set of finite sets p of triples (β, n, γ), where $\gamma < \beta < \alpha$ and $n \in \omega$ such that if $(\beta, n, \gamma_1), (\beta, n, \gamma_2) \in p$, then $\gamma_1 = \gamma_2$. Q_α is the set of finite sets p of pairs of the form (n, γ), where $\gamma < \alpha$ and $n \in \omega$ such that if $(n, \gamma_1), (n, \gamma_2) \in p$, then $\gamma_1 = \gamma_2$. Let G be an $\mathcal{M}$-generic subset of $Q^\aleph$. Then G_α, the projection of G onto Q_α, is an $\mathcal{M}$-generic subset of Q_α for each $\alpha < \aleph$. Let $f_\alpha = \bigcup G_\alpha$. Then $\mathfrak{M}_\aleph$ is the smallest model of ZF containing the same ordinals as $\mathcal{M}$ and each sequence $(f_\beta)_{\beta<\alpha}$ for $\alpha < \aleph$.

Truss shows that in $\mathfrak{M}_\aleph$ the following statements are true:

1. $\aleph_1$ is singular, that is, it can be written as a countable union of countable sets;
2. a well-ordered union of well-orderable subsets of $\mathbb{R}$ is well-orderable; and
3. every uncountable subset of $\mathbb{R}$ has a perfect subset (= closed with no isolated points).

In view of the validity of (1) and (2) in $\mathfrak{M}_\aleph$, we see that WO-CC($\mathbb{R}$) must fail in $\mathfrak{M}_\aleph$. Otherwise, CC($\mathbb{R}$) would hold true in the model (obviously, (2) + WO-CC($\mathbb{R}$) implies CC($\mathbb{R}$)). Now, CC($\mathbb{R}$) implies that $\aleph_1$ is a regular cardinal (see Church [1], [8]). This contradicts the validity of (1) in $\mathfrak{M}_\aleph$.

On the other hand, Truss shows that in ZF every perfect subset of $\mathbb{R}$ has cardinality $2^{\aleph_0}$. By this fact and the validity of (3) in $\mathfrak{M}_\aleph$, we deduce that Form 13 is true in this model and the proof is complete. □

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