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Neat Embeddings, Omitting Types, and Interpolation: An Overview

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Abstract We survey various results on the relationship among neat embeddings (a notion special to cylindric algebras), complete representations, omitting types, and amalgamation. A hitherto unpublished application of algebraic logic to omitting types of first-order logic is given.

1 Introduction

Henkin et al. [15] proves that a cylindric algebra is representable if and only if it embeds *neatly* into another cylindric algebra in ω extra dimensions (cf. Theorem 1.2 below for a precise formulation). Such algebras are said to have the neat embedding property. Thus the class of representable cylindric algebras coincides with the class of algebras having the neat embedding property. In this paper we show that for a class of representable algebras to have the strong amalgamation property or to consist exclusively of completely representable algebras, each algebra in this class should embed neatly into another algebra in ω extra dimensions in a special way. The algebraic notion of the strong amalgamation property and that of complete representations, to be defined below, are the algebraic counterparts of interpolation and omitting types in the corresponding logics (cf. Pigozzi [35] and Sayed Ahmed [47]). We start by fixing some needed notation. We follow the more or less standard terminology of the monograph Henkin et al. [14]. In particular, CA_β stands for the class of cylindric algebras of dimension β and RCA_β stands for the class of representable CA_βs. Cs_β stands for the class of cylindric set algebras of dimension β and Ws_β stands for the class of weak cylindric set algebras of dimension β. Also, it might be useful to recall that RCA_β is a variety that coincides with the class of subdirect products of (weak) set algebras of dimension β. The central notion that prevails in

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what follows is that of *neat reducts*. The notion of neat reducts, which we now recall, is due to Henkin.

Definition 1.1

1. Let $\alpha < \beta$ be ordinals and let $\mathcal{A} \in \mathrm{CA}_\beta$. Then the *neat-$\alpha$ reduct* of $\mathcal{A}$, in symbols $\mathcal{N}r_\alpha\mathcal{A}$, is the CA_α whose domain is the set of all α-*dimensional* elements of $\mathcal{A}$ defined by
$$Nr_\alpha A = \{a \in A : c_i^{\mathcal{A}} a = a \text{ for all } i \in \beta \smallsetminus \alpha\}.$$
The operations of $\mathcal{N}r_\alpha\mathcal{A}$ are those of the α-dimensional reduct
$$\mathcal{R}d_\alpha\mathcal{A} = \langle A, -^{\mathcal{A}}, \cdot^{\mathcal{A}}, c_i^{\mathcal{A}}, d_{ij}^{\mathcal{A}}\rangle_{i,j\in\alpha}$$
of $\mathcal{A}$, restricted to $Nr_\alpha A$. When no confusion is likely, we omit the superscript $\mathcal{A}$. $Nr_\alpha A$, as easily checked, is closed under the indicated operations and indeed is a CA_α. $\mathcal{N}r_\alpha\mathcal{A}$ is thus a special subreduct of $\mathcal{A}$, that is, a special subalgebra of a reduct in the universal algebraic sense.
2. Let $\alpha < \beta$. Let $L \subseteq \mathrm{CA}_\beta$. Then $Nr_\alpha L$ is the class of all α neat reducts of algebras in L, that is,
$$Nr_\alpha L = \{Nr_\alpha\mathcal{A} : \mathcal{A} \in \mathrm{CA}_\beta\}.$$

For $\alpha > 1$ not every CA_α is representable. We now formulate Henkin's celebrated Neat Embedding Theorem which gives a sufficient and necessary condition for representability of CA_αs. For a class K, we let SK denote the class of all algebras embeddable into members of K.

Theorem 1.2 (Henkin) *For any ordinal α and any $\mathcal{A} \in \mathrm{CA}_\alpha$, the following two conditions are equivalent:*

1. $\mathcal{A} \in \mathrm{RCA}_\alpha$.
2. $\mathcal{A} \in SNr_\alpha\mathrm{CA}_{\alpha+\omega}$.

Theorem 1.2 is proved as Theorem 3.2.10 in [15]. Infinity (appearing as ω) manifests itself in (2) above, and it does so essentially in the case when $\alpha > 2$, in the sense that if $\mathcal{A}$ neatly embeds into an algebra in finitely many extra dimensions, then it might not be representable, as shown by Monk [30]. Indeed in [30] Monk proves that the class $Nr_\alpha\mathrm{CA}_{\alpha+k}$ is a proper super class of RCA_α, for all $k \in \omega$. All ω extra dimensions are needed for representability; one could not truncate ω to any finite ordinal. The ω extra dimensions play the role of added constants (or witnesses) in Henkin's classical completeness proof. Therefore it is no coincidence that variations on Theorem 1.2 lead to metalogical results concerning interpolation and omitting types for the corresponding logic; for such results can be proved by Henkin's methods of constructing models out of constants (Chang and Keisler [11]). Indeed, the main purpose of this article is to show how far variations on the Neat Embedding Theorem for cylindric algebras can lead as far as the algebraic counterparts of interpolation and omitting types are concerned for the corresponding (algebraizable in the sense of Blok and Pigozzi [9]) variants of first-order logic.

2 History and Background

In a paper written by Monk in 1991 but published in the *Logic Journal of IGPL* in 2000 (Monk [31]), Andréka writes the final survey section on the subject, to update the article. It is clear from Andréka's section that among the important problems that were still open then in algebraic logic are those in Pigozzi's landmark paper on amalgamation [35] and two problems, both on neat reducts. All of Pigozzi's questions are solved in Madarász and Sayed Ahmed [27]. The two problems on neat reducts are the consecutive problems 2.11 and 2.12 posed by Henkin, Monk, and Tarski in [14]. Hirsch, Hodkinson, and Maddux [23] solve problem 2.12. They show that for $2 < n < \omega$ and $k \in \omega$, $SNr_n\mathrm{CA}_{n+k+1}$ is a proper subclass of $SNr_n\mathrm{CA}_{n+k}$. Thus the decreasing sequence $\mathrm{CA}_n \supseteq SNr_n\mathrm{CA}_{n+1} \supseteq SNr_n\mathrm{CA}_{n+2}\cdots$ converging to RCA_n is not only not eventually constant, as proved by Monk [30], but is in fact *strictly* decreasing.[1] The class $Nr_\alpha\mathrm{CA}_\beta$ is proved to be closed under products homomorphic images and ultraproducts in Németi [33]. Problem 2.11 in [14] asks whether the class $Nr_\alpha\mathrm{CA}_\beta$ for $1 < \alpha < \beta$ is closed under forming subalgebras, for if it is, it would be a variety. A closely related problem to 2.11 appears as item (v) in the introduction of [15] among results of Tarski the proofs of which could not be reconstructed by Henkin and Monk. This problem asks whether generating subreducts in the sense of [14] are necessarily neat reducts. In more detail, for infinite ordinals $\alpha < \beta$ and algebras $\mathcal{A} \in \mathrm{CA}_\alpha$ and $\mathcal{B} \in \mathrm{CA}_\beta$, if it so happens that $\mathcal{A} \subseteq Nr_\alpha\mathcal{B}$ and A is a generating set for $\mathcal{B}$, does this imply that A would exhaust the set of *all* α dimensional elements of $\mathcal{B}$, that is, $A = Nr_\alpha B$? This is the case, for example, when $\mathcal{A}$ is locally finite or dimension complemented [14]. But Tarski conjectured in [14] that this is *not* always the case. Now one might inquire at this point how are problem 2.11 (that appeared in [14]) and Tarski's conjecture (that appeared as item (v) in [15]) related? One connection is the following. If Tarski's conjecture were *false*, then the class $Nr_\alpha\mathrm{CA}_\beta$ for any pair of infinite ordinals $\alpha < \beta$ would be closed under forming subalgebras. To see this, let $\omega \le \alpha < \beta$. Let $\mathcal{A} \in \mathrm{CA}_\alpha$ and $\mathcal{B} \in \mathrm{CA}_\beta$ be such that $\mathcal{A} \subseteq \mathcal{N}r_\alpha B$. Now if one takes $\mathcal{B}'$ to be the subalgebra of $\mathcal{B}$ generated by (the set) A, then the algebra $\mathcal{A}$ would be a generating subreduct of $\mathcal{B}'$, thus—remember we are assuming that Tarski's conjecture is wrong—A would exhaust the set of all α-dimensional elements of $\mathcal{B}$, that is, $\mathcal{A} = \mathcal{N}r_\alpha\mathcal{B}'$. This shows that a subalgebra of a neat reduct of an algebra $\mathcal{B}$ is again a neat reduct of a possibly smaller algebra $\mathcal{B}'$ but a neat reduct all the same. It follows thus that for infinite $\alpha < \beta$, the class $Nr_\alpha\mathrm{CA}_\beta$ is closed under forming subalgebras. But Tarski was right. And indeed, Tarski's conjecture is confirmed in Sayed Ahmed [49]. On the other hand, Németi proves in [33] that for $1 < \alpha < \beta$, the class $Nr_\alpha\mathrm{CA}_\beta$ is not closed under forming subalgebras. The question as to whether the class $Nr_\alpha\mathrm{CA}_\beta$ for $1 < \alpha < \beta$ is perhaps closed under *elementary* subalgebras—equivalently by the celebrated Keisler-Shelah ultrapower theorem whether it is closed under ultraroots—appears as problem 4.4 in the monograph [15]. This is equivalent to asking whether this class is elementary because it is closed under ultraproducts. In [33], Németi conjectures that for $1 < \alpha < \beta$, the class $Nr_\alpha\mathrm{CA}_\beta$ is not closed under elementary subalgebras, hence is not elementary, that is, cannot be axiomatized by any set of first-order axioms. In Sayed Ahmed [45] and [43], Németi's conjecture is confirmed, and a different (interesting model-theoretic) proof than that presented in [45] is given in [48]. The analogous result concerning first-order axiomatizability for certain classes that constitute other algebraizations of first-order logic, like quasi-polyadic algebras and Pinter's substitution algebras, is proved in Sayed Ahmed [46] and [40]. The analogue of Tarski's conjecture for such algebraizations is investigated in Sayed Ahmed and Németi [53] and [49]. Now the notion of neat reducts is an old venerable notion in algebraic logic that has been well investigated in connection to the representation theory of cylindric algebras and related structures like polyadic algebras (see, e.g., [46], [40], and [48] and the references therein). But it often happens that an unexpected viewpoint yields dividends and new insights. Indeed, the repercussions of the very seemingly innocent fact that the class of neat reducts is not closed under forming subalgebras turns out to be enormous as we proceed to show next.

2.1 Neat reducts and amalgamation The form of amalgamation we address in what follows is typically of the following form: Which classes frequently studied in algebraic logic—like, for example, the class of representable cylindric or for that matter the class of polyadic algebras—have the amalgamation property? Amalgamation is proved in algebraic logic to be the algebraic counterpart of the metalogical property of interpolation (see, for example, [53], Sayed Ahmed [42], and [27]). Pigozzi [35] is a milestone in this respect; it gives a comprehensive picture of amalgamation for several classes of cylindric algebras. However, several questions concerning the strong amalgamation property were posed as open questions by Pigozzi in [35].[2] It turns out, rather surprisingly in Sayed Ahmed [38], that the closure of the class of neat reducts under forming subalgebras for certain classes of cylindric algebras turns out to be closely related to the property of strong amalgamation for these classes (cf. [53], [44], [49], [35], [27], [39], and [42]). To formulate the connection between neat embeddings and strong amalgamation, we need the following definition.

Definition 2.1 Let α be an ordinal. Let $L \subseteq \mathrm{RCA}_\alpha$. We say that L has the NS property, short for neat reducts commuting with forming subalgebras, if the following condition holds:

$$(\forall \mathcal{A} \in L)(\exists \mathcal{B} \in \mathrm{CA}_{\alpha+\omega})[A \subseteq Nr_\alpha B \wedge (\forall X)(X \subseteq A) \implies Sg^{\mathcal{A}} X = \mathcal{N}r_\alpha Sg^{\mathcal{B}} X].$$

Here for an algebra $\mathcal{C}$ and $X \subseteq C$, $Sg^{\mathcal{C}} X$ denotes the subalgebra of $\mathcal{C}$ generated by X. Definition 2.1 singles out certain classes of algebras whose members embed neatly into algebras in ω extra dimensions in a special way. Let us dwell a little bit more on the property expressed in the NS property. Let $\mathcal{A} \subseteq \mathcal{N}r_\alpha \mathcal{B}$, with $\mathcal{A} \in \mathrm{CA}_\alpha$ and $\mathcal{B} \in \mathrm{CA}_{\alpha+\omega}$. Let $X \subseteq \mathcal{A} \subseteq \mathcal{N}r_\alpha \mathcal{B}$. Then we can form the subalgebra of $\mathcal{N}r_\alpha \mathcal{B}$ generated by X, which is the same as the subalgebra of $\mathcal{A}$ generated by X. Or alternatively, we can form the subalgebra of $\mathcal{B}$ generated by X (here we are using ω extra operations, so that in principle we can get new α-dimensional elements) and then form the neat-α reduct of the resulting algebra. Then the NS says that taking any of these two paths we will arrive at the same algebra (up to isomorphism), that is, we *do not* get new α-dimensional elements. In short, *the operation of taking α-neat reducts commutes with that of forming subalgebras.*[3] Examples of subclasses of RCA_α that have the NS abound. These classes include Mn_α, the class of minimal cylindric algebras (cf. [14], p. 254), more generally, the class of monadic-generated CAs of dimension α ([14], p. 257), and CA_αs of positive characteristic $k < \alpha \cap \omega$ ([15], p. 68). A minimal cylindric algebra is an algebra with no proper subalgebras, equivalently one that is generated by the diagonal element, while a monadic cylindric algebra is one that is generated by a set X such that $|\Delta x| \leq 1$ for all $x \in X$. Proofs of the results quoted above can be found in [53]. Other classes that have the NS include the class Lf_ω and Dc_ω of locally finite and dimension complemented algebras of dimension ω, respectively. While every RCA_1 has the strong NS, this is not the case for higher dimensions. Indeed, the $\mathcal{A} \in Ws_\alpha$ constructed in Theorem 1 of [53] shows that any L such that $Ws_\alpha \subseteq L \subseteq \mathrm{RCA}_\alpha$ fails to have even the NS when $\alpha > 1$. A slight modification of this algebra $\mathcal{A}$ is used in [49] to show that there are generating subreducts in the sense of [14] that are *not* neat reducts. As indicated above, this confirms a conjecture of Tarski formulated as item (v) in the introduction of [15]. Madárasz [27] proves that any L such that $Ws_\alpha \subseteq L \subseteq \mathrm{RCA}_\alpha$, when α is infinite, fails to have the the strong amalgamation property in the sense of the coming definition. The analogous result for the finite dimensional case is proved in Andréka et al. [2]. The link between NS and SUPAP is given in Theorem 2.3(3) below. Before formulating our theorem we recall from [53] the notions of amalgamation, strong amalgamation, and super amalgamation. The latter notion is due to Maksimova [29]. We slightly modify the conventional definition so that the amalgam, be it strong or super, may be found in a possibly bigger class.

Definition 2.2

1. Let V be a class of algebras (usually but not always assumed to be a variety) and $L_1 \subseteq L_2 \subseteq V$. L_2 is said to have the amalgamation property, or AP for short, over L_1, with respect to V, if for all $\mathcal{A}_0 \in L_1$, all $\mathcal{A}_1$ and $\mathcal{A}_2 \in L_2$, and all monomorphisms i_1 and i_2 of $\mathcal{A}_0$ into $\mathcal{A}_1$, $\mathcal{A}_2$, respectively, there exists $\mathcal{A} \in V$, a monomorphism m_1 from $\mathcal{A}_1$ into $\mathcal{A}$, and a monomorphism m_2 from $\mathcal{A}_2$ into $\mathcal{A}$ such that $m_1 \circ i_1 = m_2 \circ i_2$. In this case we say that $\mathcal{A}$ is an amalgam of $\mathcal{A}_1$ and $\mathcal{A}_2$ over $\mathcal{A}_0$ via m_1 and m_2 or even simply an amalgam.
2. Let everything be as in (1). If, in addition, $(m_1 \circ i_1)A_0 = m_1(A_1) \cap m_2(A_2)$, then we say that L_2 has the strong amalgamation property, or SAP for short, over L_1 with respect to V. In this case, we say that $\mathcal{A}$ is a strong amalgam of $\mathcal{A}_1$ and $\mathcal{A}_2$ over $\mathcal{A}_0$ via m_1 and m_2 or even simply a strong amalgam.
3. Let everything be as in (1). We say that L_2 has SUPAP over L_1 with respect to V, if, for all $\mathcal{A}_0 \in L_1$, $\mathcal{A}_1$ and $\mathcal{A}_2 \in L_2$, and all monomorphisms i_0 and i_1 of $\mathcal{A}_0$ into $\mathcal{A}_1$, $\mathcal{A}_2$, respectively, there exists $\mathcal{A} \in V$, a monomorphism m_1 from $\mathcal{A}_1$ into $\mathcal{A}$, and a monomorphism m_2 from $\mathcal{A}_2$ into $\mathcal{A}$ such that $m_1 \circ i_1 = m_2 \circ i_2$, and
$$(\forall x \in A_j)(\forall y \in A_k)(m_j(x) \leq m_k(y) \implies (\exists z \in A_0)(x \leq i_j(z) \wedge i_k(z) \leq y)),$$
where $\{j, k\} = \{1, 2\}$. In this case we say that $\mathcal{A}$ is a super amalgam of $\mathcal{A}_1$ and $\mathcal{A}_2$ over $\mathcal{A}_0$, via m_1 and m_2, or even simply a super amalgam.
4. When $L_1 = L_2 = L$ in (1), (2), and (3) above we say that L has AP, SAP, and SUPAP, respectively, with respect to V. If, furthermore, $L = V$, we say that V simply has AP, SAP, and SUPAP, respectively.

It is easy to see that for cylindric algebras SUPAP implies SAP. Now the following theorem relating NS to SUPAP is proved in [38] and [27]. It is a generalization of a result of Pigozzi in [35] (cf. 2.12 therein).

Theorem 2.3 *Let α be an ordinal $\leq \omega$. Let $L_1 \subseteq L_2 \subseteq \mathrm{RCA}_\alpha$. Assume there exists a "lifting" function $\mathbf{F}$ from $L_1 \cup L_2$ to $\mathrm{CA}_{\alpha+\omega}$ such that $\mathcal{A} \subseteq \mathfrak{Nr}_\alpha \mathbf{F}(\mathcal{A})$ (here the neat embedding theorem is used) and such that the following two conditions hold:*

1. *Forming subalgebras of $\mathcal{A}$ is the same as forming subalgebras of $\mathcal{N}r_\alpha\mathbf{F}(\mathcal{A})$; formally,*

$$(\forall\mathcal{A}\in L_1\cup L_2)(\forall X\subseteq A)(Sg^{\mathcal{A}}X=Sg^{\mathcal{N}r_\alpha\mathbf{F}(\mathcal{A})}X);$$

2. *Lifting and forming subalgebras commute with forming subalgebras and lifting; formally,*

$$(\forall\mathcal{A}\in L_1\cup L_2)(\forall X)(X\subseteq A)(\mathbf{F}(Sg^{\mathcal{A}}X)=Sg^{\mathbf{F}(\mathcal{A})}X).$$

Then L_2 has AP *over L_1 with respect to* RCA_α. *Assume in addition to (1) and (2) that the following condition is satisfied.*

3. *L_1 has* NS *(with $\mathbf{F}(\mathcal{A})$ playing the role of $\mathcal{B}$ in Definition 2.1), that is,*

$$(\forall\mathcal{A}\in L_1)(\forall X)(X\subseteq A)(Sg^{\mathcal{A}}X=\mathcal{N}r_\alpha Sg^{\mathbf{F}(A)}X).$$

Then L_2 has SUPAP *over L_1 with respect to* RCA_α.

Proof See [27]. □

As an application of Theorem 2.3, it is proved in [27] that the classes of monadic generated CAs and CAs of positive characteristic have SUPAP. In [53] it is proved that the classes of semi-simple ω-dimensional cylindric algebras, or Sc_ω for short, and the class of ω-dimensional diagonal cylindric algebras, or Di_ω for short, in the sense of [35], do not have NS. Emphasizing the link between NS and SUPAP, Madárasz [27] proves that Sc_ω and Di_ω do not have SAP. In fact, these classes do not have SAP even with respect to the strictly bigger class RCA_ω. In [38] and [53] it is proved that the so-called strong amalgam base of RCA_α in the sense of Andréka et al. [4] coincides with the class of algebras having NS. In particular Lf_ω and Dc_ω lie in the strong amalgam base of RCA_ω. Determining the strong amalgam base of RCA_α settles a question raised by Monk (cf. [4], p. 739). In contrast, it is shown in [39] that Di_ω, Sc_ω, and RCA_ω have the strong embedding property in the sense of Pigozzi [35]. Roughly the strong embedding property is the restriction of the strong amalgamation property when the base algebra is a minimal one, that is, has no proper subalgebras. In [39], Theorem 4 is generalized to the very general context of systems of varieties definable by schemes in the sense of [15]. The notion of systems of varieties definable by schemes is a very broad one that covers almost all algebraic logics in the literature (cf. [15] and Andréka and Németi [5]). It is known that if V is a quasi variety that has the amalgamation property, then V has SAP if and only if in V epimorphisms (in the categorial sense) are surjective or, V has ES for short. When dealing with systems of varieties definable by schemes, ES can be, and indeed is, replaced by the NS property as illustrated in Theorem 4(iii). Loosely speaking, the NS can be paraphrased as follows: "Term functions that are definable with extra variables are already term definable without extra variables."[4] Also in [39], Theorem 4 is compared and likened to Németi's techniques on amalgamation adopted in Németi [32]. Presenting Theorem 4 and Németi's technique formulated as Lemma 3 in [32] in a functorial context as adjoint situations, it is shown in [39] that both can be seen as instances of the use of the Keisler-Shelah ultrapower theorem in proving Robinson's Joint Consistency Theorem. Another aspect of this unification consists of presenting Theorem 4 and Németi's Lemma 3 in [32] as transforming a diagram of algebras to be strongly amalgamated into certain saturated representations of these algebras that can be strongly amalgamated and then returning to the original diagram using an inverse operator. The categorial formulation makes the notion of inverse operator precise. In the case of Theorem 4 it is the neat reduct functor[5] (an inverse to a neat embedding functor—denoted by **F** taking an algebra into ω extra dimensions, that is, to a classical representation), whereas in Németi's Lemma 3 it is basically the operation of forming atom structures that is an inverse of taking an algebra to its canonical extension (which can be seen as a modal representation). This can be seen as an instance of the triple duality existing between Boolean algebras with operators Kripke frames and modal logic (Goldblatt [12]). A purely modal approach to proving Németi's results in [32] was found independently by Weaver and Welaish in [56]. In [56] the argument used is a back and forth construction to establish a modal analogue of Robinson's joint consistency theorem. Also it is shown in [39] and Sayed Ahmed [41] that both techniques take the representation problem expressed by a two-sorted defining theory, as shown by Hirsch and Hodkinson in [21], a step further, asking that the second sort be a saturated representation.[6] This is not surprising since (in extensions of first-order logic and multimodal logics) interpolation is stronger than completeness (see, e.g., Henkin's proof of Craig interpolation in [11] and its algebraization to wider contexts in [35] and Sayed Ahmed [50]). Indeed this is illustrated in Theorem 4 because of the following reasoning. Completeness corresponds to representability. A class of algebras consists of representable algebras, if each of its members embed neatly into ω extra dimensions. On the other hand, for this class to have SUPAP, its members should embed in ω extra dimensions in a special way; this class should have the NS. From a model-theoretic point of view, this is expected. Let us explain why. Henkin's neat embedding theorem is an algebraization of his celebrated completeness proof, which is an instance of Robinson's finite forcing in model theory. As Hodges says,"if the forcing conditions are chosen cleverly, this yields interpolation results" [24]. Pigozzi did exactly that in his proof that the dimension-restricted free algebras that happen to be dimension complemented have the interpolation property ([35], Theorem 2.2.6). Theorem 2.2.6 is the central theorem in [35], §2.2. Indeed, from this single theorem (together with one or two others of a nontrivial but still auxiliary character) there follows via the general results summarized in [35], §2, all positive results in Pigozzi's paper, including the SAP version of our Theorem 2.3 (cf. Corollary 2.2.12 in [35]). In [42] a similar argument is used to show that the class of algebras studied in Sain [36] has the strong amalgamation property. Another instance of Robinson's finite forcing is that of omitting types as we proceed to show. But before that we make a short detour into the (very much related) notion of complete representation.

2.2 Neat embeddings, complete representations, and omitting types. Using neat embeddings the following new characterization of countable algebras in the class of the so-called completely representable algebras finite dimensional cylindric algebras is given in [47]. Such algebras are investigated in, for example, Lyndon [26], Hirsch and Hodkinson [19], and Hirsch and Hodkinson [18]. We first define the notion of complete representations. For this we need to fix some notation. For an algebra $\mathcal{A}$ with a Boolean reduct and $X \subseteq A$, $\sum X$ and $\prod X$ denote the supremum and infimum of X whenever these exist, respectively. For algebras $\mathcal{A}$ and $\mathcal{B}$, $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ denotes the set of all homomorphisms from $\mathcal{A}$ to $\mathcal{B}$. We recall that Cs_n denotes the class of cylindric set algebras of dimension n. A complete representation of a CA_n is one that preserves infinite meets (and joins) carrying it to set theoretic intersections (and unions), more precisely, as in the following definition.

Definition 2.4 Let $n < \omega$. Let $\mathcal{A} \in \mathrm{CA}_n$. $\mathcal{A}$ is completely representable if for all nonzero $a \in A$, there exists $\mathcal{B} \in Cs_n$ and $f \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ such that $f(a) \neq 0$ and such that

$$f(\sum X) = \cup_{x \in X} f(x)$$

for all $X \subseteq A$ whenever $\sum X$ exists.

Before formulating the connection between neat embedding and complete representations, we also need the following definition.

Definition 2.5 For K a class with a Boolean reduct we define

$$S_c K = \{\mathcal{A} : \exists \mathcal{B} \in K : A \subseteq B, \text{ and whenever } \sum X = 1 \text{ in } \mathcal{A}, \\ \text{then } \sum X = 1 \text{ in } \mathcal{B} \text{ for all } X \subseteq A\}.$$

Roughly $S_c K$ denotes the operation of forming *complete* subalgebras of algebras in K. Now the following theorem is proved in [47]. It says that for finite n and for a countable atomic n-dimensional algebra to be completely representable, it has to embed neatly and *completely* into another algebra in ω extra dimensions, that is, an algebra in $S_c Nr_n \mathrm{CA}_\omega$. We refer to algebras in the latter class as algebras having the *strong neat embedding property*. (In [47] it is shown that the strong neat embedding property is indeed stronger than the neat embedding property. An example of a representable algebra that does not have the strong neat embedding property is given below in the proof of Theorem 2.10. However, the two notions coincide in other contexts, for example, with the algebras studied in [36]). We recall that an atom is a minimal nonzero element and that an algebra is atomic if every nonzero element contains an atom.

Theorem 2.6 *Assume that $n < \omega$. Let $\mathcal{A} \in \mathrm{CA}_n$ be countable. Then $\mathcal{A}$ is completely representable if and only if $\mathcal{A}$ is atomic and $\mathcal{A} \in S_c Nr_n \mathrm{CA}_\omega$.*

Theorem 2.6 follows from the stronger following theorem proved in [47]. But first we recall from [15] that Ws_n denotes the class of weak set algebras of dimension n.

Theorem 2.7 *If $\mathcal{A} \in S_c Nr_n \mathrm{CA}_{n+\omega}$ is countable, $n \leq \omega$ (note that here n is allowed to be infinite) and $\{X_i : i < \omega\}$ is a family of subsets of A such that $\prod X_i = 0$ for all $i < \omega$, then for every nonzero $a \in A$ there exists $\mathcal{B} \in Ws_n$, with countable base, and $f \in Hom(\mathcal{A}, \mathcal{B})$ such that $f(a) \neq 0$ and for all $i \in \omega$ we have $\cap_{x \in X_i} f(x) = \emptyset$.*

The statement expressed in Theorem 2.7 is an algebraic version of an Omitting Types Theorem. Indeed, bearing in mind the correspondence established in [15], §4.3, between theories and algebras on the one hand and models and set algebras on the other, the X_is are nothing more than *nonprincipal types* and $\mathcal{B}$ is nothing more than a representation *omitting* these types. Such analogies are worked out in more detail in [47] and [50]. Note that for $n < \omega$ we have $Ws_n = Cs_n$, so that a unit of a Ws_n is simply of the form ${}^nU$. We show how the "only if" part of Theorem 2.6 follows from Theorem 2.7. The other implication is direct.

Proof of Theorem 8 modulo Theorem 9 Assume Theorem 2.7 and let $n < \omega$. Let $\mathcal{A} \in S_c Nr_n \mathrm{CA}_\omega$ be countable and atomic. We can assume, without loss of generality, that $\mathcal{A}$ is simple, for if not then we could replace it by its simple components. Then taking for all $i \in \omega$, $X_i = Y = \{-b : b \text{ is an atom of } \mathcal{A}\}$, and applying $(*)$ for any nonzero a in A, upon noting that $\prod Y = 0$ since $\mathcal{A}$ is atomic, we get an atomic representation in the sense of [18], hence a complete representation of $\mathcal{A}$. $\square$

Theorem 2.7 was used to prove an Omitting Types Theorem for certain infinitary extensions of first-order logic studied in Andréka et al. [3] (cf. [47]). A variation thereof is used to prove an Omitting Types Theorem for other extensions of first-order logic (without equality) introduced as one possible solution to the finitization problem in Sain and Gyuris [37] and [36] (cf. [50]). The proof of Theorem 9 in [47] is a typical omitting types construction; it is a variation on a theme occurring in Casanovas and Farré [10] and Newelski [34]. In fact a slight modification of this proof gives the results in the latter two papers as shown in [47]. An instance of Robinson's finite forcing in model theory, the proof of Theorem 9 is indeed a Baire Category argument at heart as illustrated in [47]. When one asks for the omission of $< {}^{\omega}2$ types one is led to an instance of the famous independent Martin's axiom, establishing an interesting connection with set theory (cf. [53]). Indeed the resulting statement turns out to be equivalent in Zermelo-Fraenkel set theory (ZF) to Martin's axiom restricted to countable Boolean algebras. Also countability of the algebras involved cannot be omitted from the above characterization.[7] However, in the absence of the continuum hypothesis and for that matter Martin's axiom, ω_1 many types can be omitted. In ZF one can omit $covK$ many types, where $covK$ is the least cardinal κ such that the real line can be covered by κ many closed nowhere dense sets (cf. [47]). Martin's axiom implies that $covK = {}^{\omega}2$ but it is consistent that $covK = \omega_1 < {}^{\omega}2$. For more on these descriptive set-theoretic notions in connection to omitting types the reader is referred to [10], [34], and [47].

2.3 Omitting types for the finite variable fragments of first-order logic We now give yet another novel application of algebraic logic to first-order logic. We show how the construction of certain relation algebras can be used to show that the classical Henkin-Orey Omitting Types Theorem fails for the finite variable fragments of first-order logic, in a rather strong sense (to be made precise shortly). These algebras were originally constructed to serve an entirely different purpose, namely, to show that the classes of representable relation algebras and representable cylindric algebras of finite dimension > 2 are neither *atom-canonical* nor *single-persistent* nor *Sahlqvist* nor closed under *minimal completions*. We refer the reader to Goldblatt [13] for the definition of these notions. The author used cylindric algebras constructed by Hirsch and Hodkinson in [18] and Hodkinson in [25] to prove a result weaker than Theorem 2.10 concerning the failure of Omitting Types (cf. [53], Theorem 4). In Theorem 2.10, the construction of the relation algebra (with n-dimensional cylindric bases) is due to Andréka [1], whereas the connection with omitting types is due to the present author. However we will not give the Andréka construction since it will be published elsewhere. To formulate our joint result with Andréka, we need to fix some notation and recall some terminology.

Notation 2.8 Let L_n denote first-order logic restricted to the first n variables. Let T be a countable consistent L_n theory. Let Γ be a countable set of L_n formulas that is consistent over T, that is, no contradiction is derivable from $T \cup \Gamma$. For a formula φ and a model M, we recall that φ^M denotes the set of all assignments that satisfy φ

in M, that is,

$$\varphi^M = \{s \in {}^\omega M : M \models \varphi[s]\}.$$

Definition 2.9 We say that Γ is *implicitly principal* over T, if for all $M \models T$, $\bigcap_{\varphi\in\Gamma}\varphi^M \neq \varnothing$. We say that Γ is *explicitly k-principal* over T, if there exists a formula φ built up of at most k variables and consistent with Γ that isolates Γ, that is, $T \models \varphi \implies \psi$ for all $\psi \in \Gamma$.

The classical Henkin-Orey Omitting Types Theorem ([11], Theorem 2.2.9), or rather the contrapositive thereof, implies that if Γ is implicitly principal (over T) then Γ is explicity k principal (over T) for some $k \in \omega$. The question is, Do we guarantee that $k \leq n$, that is, the formula isolating Γ stays inside L_n, or do we have to "step outside" L_n, resorting to extra variables? The following result was announced in Andréka and Sayed Ahmed [6]. It contrasts positive results on omitting types proved in [47] and [50].

Theorem 2.10 *For all $3 \leq n < k$ there exists a countable L_n theory T, a type Γ consistent over T such that Γ is implicitly principal but not k-explicitly principal over T.*

Sketch of proof For undefined notions the reader is referred to Maddux [28]. Now let $3 \leq n < \omega$. Andréka constructs a countable symmetric integral (hence simple) representable atomic relation algebra R_n such that its completion, that is, the complex algebra of its atom structure, is not representable. This means that R_n is not completely representable, for if it were then this complete representation will give a representation of the complex algebra of its atom structure. Also R_n has the additional property that the (countable) set B_nR_n of all n by n basic matrices over R_n constitutes an *n-dimensional cylindric basis* in the sense of Maddux [28] (Definition 4, p. 953). Thus B_nR_n is a cylindric atom structure. This means that the full complex algebra $Ca(B_nR_n)$ with universe the power set of B_nR_n can be turned into an n-dimensional cylindric basis in a natural way. Since R_n is not completely representable, then $Ca(B_nR_n)$, sure enough a CA_n, is not a representable one. However the *term algebra* over the atom structure B_nR_n, which is the subalgebra of $Ca(B_nR_n)$ generated by the countable set of n by n basic matrices, $Tm(B_nR_n)$ for short, is a countable representable CA_n, and further it is simple so that it is in fact (isomorphic to) a Cs_n. $Tm(B_nR_n)$ is representable, but it does not have a complete representation. If it did, then R_n will also have a complete representation, which is not the case. Now let $3 \leq n < k$. Then $Tm(B_nR_n)$ embeds neatly in $Tm(B_kR_k)$. In fact, it turns out that $Tm(B_nR_n) \cong Nr_n(T_kR_k)$. Thus $Tm(B_nR_n)$ is an example of a countable atomic algebra in $Nr_n\mathrm{RCA}_k$ with no complete representation. For brevity, let $\mathcal{A} = Tm(B_nR_n)$, and let $\mathcal{B} = Tm(B_kR_k)$. Then by [14], Theorem 4.3.28(ii), there is a countable first-order language $\mathcal{L}$, such that $\mathcal{B} \cong (Fm^{\mathcal{L}_k}/T)$ for some (countable) L_k theory $T \subseteq Fm^{\mathcal{L}_k}$. Here $\mathcal{L}_k$ is the restriction of the language $\mathcal{L}$ to the first k variables and $Fm^{\mathcal{L}_k}/T$ is the Lindenbaum-Tarski representable k-dimensional representable cylindric algebra corresponding to T. We can assume that T consists of sentences only, that is, that no free variables occur in formulas in T. It follows that $\mathcal{A} \cong Fm^{\mathcal{L}_n}/T$. Fix θ an isomorphism from $Fm^{\mathcal{L}_n}/T$ to $\mathcal{A}$ and let AtA denote the set of atoms of $\mathcal{A}$. Put

$$\Gamma = \cup\{\neg\varphi/T : \theta(\varphi/T) \in AtA\}.$$

Here φ/T denotes the equivalence class of φ, consisting of all formulas equivalent to φ modulo T. Since $\mathcal{A}$ is atomic we have $\sum At^A A = 1$. Since $A = Nr_n B$, we have $\mathcal{A}$ is a complete subalgebra of $\mathcal{B}$. It then follows that $\sum At^B A = 1$, thus (+)

$$\prod\{\theta(\psi/T) : \psi \in \Gamma\} = 0.$$

Now we check that Γ is implicitly principal but not k-explicitly principal. To see that Γ is not explicitly k-principal, assume to the contrary that there exists $\varphi \in Fm^{\mathcal{L}_k}$ such that φ is consistent with T and φ isolates Γ. We can assume without loss of generality that the free variables occurring in φ are among the first n. But then we get that for all $\psi \in \Gamma$

$$0 < \theta(\varphi/T) \leq \theta(\psi/T).$$

This contradicts (+). To see that Γ is implicitly principal we assume to the contrary that there exists a model M that omits Γ, that is, $M \models T$ is such that $\cap_{\varphi\in\Gamma}\varphi^M = \emptyset$. Then $\cup_{\varphi\in\Gamma}(\neg\varphi^M) = {}^nM$. But then $\{\varphi^M : \varphi \in Fm^{\mathcal{L}_n}\}$ would be (the domain of) an atomic representation of A in the sense of [18], Definition 4, thus by Theorem 5 of [18], it is a complete representation of A, contradiction. □

We note that the algebra constructed by Andréka is binary generated, that is, generated by its two-dimensional elements. It follows from [15] that the diagonal free reduct of this algebra is not completely representable. This gives the following theorem.

Theorem 2.11 *Let $n > 2$ be finite. Then the Omitting Types Theorem fails for the equality free version of L_n, and the multimodal logic* $\mathbf{S5}^n$.

The multimodal logic $\mathbf{S5}^n$ is studied in, for example, Venema [55], Venema [54], and Hirsch et al. [16]. Theorems 14 and 15 say that, though there are always formulas isolating or witnessing realizable types, we cannot control the number of variables occurring in such formulas. They grow without bound. This adds to the list of deep (negative) results known for L_n (and its equality free reduct) concerning the complexity of its proof theory for $n > 2$ (cf. [30], Andréka [7], Hirsch et al. [22], and Hirsch and Hodkinson [20]; (failure of) interpolation for $n > 1$, [2] and [53]; and undecidability of the validities for $n > 2$, [15], Theorem 4.2.18). Also this adds to our knowledge of omitting types for usual first-order logic, since it says that there is no bound on the number of variables needed to isolate nonomissible types. What this entails in general terms is that in isolating types, all ω variables are needed. This is indeed analogous to the situation with provability and interpolation. We point out that from Theorem 2.11, one can easily prove that the Omitting Types Theorem fails for any first-order definable expansion of L_n for $2 < n < \omega$ as defined in Biró [8]. In contrast, we have the following theorem.

Theorem 2.12 *The Omitting Types Theorem holds for L_1, and more generally for countable L_n theories, $n \leq \omega$, with only (countably) many* unary *relation symbols.*

Proof See [47]. □

While every atomic RCA_2 is completely representable (this follows easily from [15], Theorem 3.2.65), we do not know whether the Omitting Types Theorem holds for L_2. We pose this as an open question.

Question 2.13 *Does the Omitting Types Theorem hold for first-order logic restricted to the first two variables?*

As we mentioned earlier, the algebra R_n for $2 < n < \omega$ constructed by Andréka and used above was originally designed to show that RRA and RCA_n are not Sahlqvist varieties, that is, cannot be characterized by Sahlqvist equations. Thus they are not atom-canonical nor closed under completions [13]. That RCA_ω is not Sahlqvist is proved in Sayed Ahmed [51]. Also it is shown therein that the Omitting Types Theorem in the sense of Henkin-Orey fails for the corresponding logic L_ω. L_ω is an algebraizable extension of first-order logic introduced in [15], §4.3. In contrast, it is proved in [50] that other algebraizable extensions of first-order logic enjoy an Omitting Types Theorem. The algebraic counterparts of these logics are studied in [42], [37], and [36]. These turn out to be atom-canonical reducts of polyadic algebras (Sayed Ahmed [52]). We conclude this article by the following theorem addressing the closure of the class of neat reducts under forming subalgebras.

Theorem 2.14 *Let* $2 < n < \omega$. *Then we have*

1. *the class* $S_c Nr_n \mathrm{CA}_\omega$ *is not elementary;*
2. *the following inclusions are proper:*

$$Nr_n\mathrm{CA}_\omega \subset S_c Nr_n \mathrm{CA}_\omega \subset SNr_n\mathrm{CA}_\omega.$$

Proof For a class K we let $\mathrm{El}K$ stand for the least elementary class containing K. Let $2 < n < \omega$. Let C_n be the countable cylindric algebra used in [19], Theorem 34. By [19], Theorem 34, C_n is *elementary equivalent* to a completely representable algebra. It follows from the characterization established in the theorem that $C_n \in \mathrm{El}S_c Nr_n\mathrm{CA}_\omega \smallsetminus S_c Nr_n \mathrm{CA}_\omega$. This proves (1) and takes care of the second inclusion of (2), since C_n is representable. The first inclusion of (2) follows from [48] where two representable CA_ns, $\mathcal{A}$ and $\mathcal{B}$, are constructed such that $\mathcal{A} \in Nr_n\mathrm{CA}_\omega$, $\mathcal{B}$ is a complete elementary subalgebra of $\mathcal{A}$, and $\mathcal{B} \notin Nr_n\mathrm{CA}_{n+1}$. □

Notes

1. This shows that for any Hilbert-style axiomatization of first-order logic and for every $3 \le n < m$ there is a formula built up of n variables that can be proved using $m+1$ variables but cannot be proved using m variables.
2. We recall that a class has the strong amalgamation property if, whenever two algebras in this class have a given common subalgebra, they can be embedded in a third algebra in this class in such a way that the intersection of the images of the two algebras is the image of the given common subalgebra.
3. Below we shall show that the NS is, in essence, a definability condition.
4. Very roughly in quasi varieties we have AP + ES = SAP, whereas for systems of varieties we have AP + NS = SAP.

5. Let $\alpha < \beta$. Then the neat reduct operator, Nr_α for short, can be viewed as a functor from the category CA_β to CA_α. Its action on objects is obvious. Its action on morphisms is their (natural) restriction.

6. It turns out, as discovered by Hirsch and Hodkinson ([21], ch. 9), that many natural kinds of representations of an algebra can be described in a first-order 2-sorted language. The first sort of a model of this *defining theory* is the algebra itself, while the second sort is a representation of it. The defining theory specifies the relation between the two, and its axioms depend on what kind of representation we are considering. Thus the representable algebras are those models of the first sort of the defining theory with the second sort providing the representation.

7. Personal communications, and see Hirsch and Sayed Ahmed [17].

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