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General Frames for Relevant Modal Logics

Takahiro Seki

Abstract General frames are often used in classical modal logic. Since they are duals of modal algebras, completeness follows automatically as with algebras but the intuitiveness of Kripke frames is also retained. This paper develops basics of general frames for relevant modal logics by showing that they share many important properties with general frames for classical modal logic.

1 Introduction

General completeness results for relevant modal logics were proved in Seki [10] where the notion of general frames was also introduced by analogy with classical modal logic. Although relational semantics for relevant logics has existed since the 1970s and algebraic semantics has also long been known (see, e.g., Dunn [7]), general frames for relevant logics have not been much discussed. Thus duality theory for relevant logics is rather underdeveloped with the notable exceptions of Brink [2], Celani [3], and Urquhart [11].

In this paper we intend to fill this gap and discuss duality theory for relevant modal logics in some detail, making use of general frames as introduced in [10]. Some of the results from the present paper have already been announced without their proofs in [10] in order to develop Sahlqvist-like theorems for relevant modal logics. We will give their detailed proofs here.

General frames occupy a prominent place in classical modal logic where they are often said to combine the intuitiveness of Kripke frames with the universal adequacy of algebraic semantics. General frames are duals of modal algebras in a clear intuitive sense, and for a certain subclass of them, namely, so-called descriptive frames, this duality becomes precisely the duality in a category theory sense. In particular, each modal algebra can be represented as a descriptive frame (for a proof, see Došen [6] or Goldblatt [8]). Thus it may be said that descriptive general frames (later on,

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simply descriptive frames) are the most important. For more on various classes of frames see, for example, Blackburn et al. [1] or Chagrov and Zakharyaschev [4].

Whereas in classical modal logic duality theory is very well developed, in the realm of relevant logic, relatively little has been done along these lines. Brink's Stone-style representation of the (nonmodal) $\mathbf{R}^{\neg}$-algebra in [2] should be mentioned here, as well as Urquhart's Priestley-style duality between relevant algebras and relevant spaces developed in [11] and extended by Celani to certain relevant modal algebras in [3]. This paper will follow in Brink's footsteps and provide Stone-style representation. One difference from Brink's approach is that whereas he makes essential use of subalgebras and subframes, we work directly with algebras and general frames without recourse to any substructures.

This paper is organized as follows. We give a brief survey of relevant modal logics and their semantics in Section 2 followed by the definition of general frames in Section 3. In Section 4 we prove a representation theorem for relevant modal algebras which yields completeness of relevant modal logics with respect to general frames defined in Section 3. We introduce descriptive frames in Section 5 and discuss their properties in comparison to descriptive frames for classical modal logic. Finally, Section 6 proves duality between the category of relevant modal algebras and relevant descriptive frames.

2 Preliminaries

In this section we present basic notions of relevant modal logics. For more information, see [10].

We use &, $\Rightarrow$, $\Leftrightarrow$, $\forall$, and $\exists$ to denote, respectively, conjunction, implication, equivalence, and universal and existential quantifiers in the metalanguage. We omit some parentheses by assuming that $\forall$, $\exists$ bind more strongly than &, and that & binds more strongly than $\Rightarrow$, $\Leftrightarrow$.

The language of relevant modal logics consists of

1. propositional variables,
2. the propositional constant **t**,
3. logical connectives $\rightarrow$, $\wedge$, $\vee$, $\circ$, and $\sim$,
4. modal operators $\Box$ and $\Diamond$.

Formulas are defined in the usual way and are denoted by capital letters A, B, C. Prop and Wff will denote the set of all propositional variables and of formulas, respectively. When necessary, we use $'$ or subscripts. Further, we introduce the following abbreviations:

$$A \leftrightarrow B \stackrel{\text{def}}{=} (A \rightarrow B) \wedge (B \rightarrow A), \qquad \boxdot A \stackrel{\text{def}}{=} \sim \Diamond \sim A, \qquad \Diamond\!\!\!\cdot\, A \stackrel{\text{def}}{=} \sim \Box \sim A.$$

The relevant modal logic $\mathbf{B.C}_{\Box\Diamond}$ is defined as follows.

Axioms

(B1) $A \rightarrow A$
(B2) $A \wedge B \rightarrow A$
(B3) $A \wedge B \rightarrow B$
(B4) $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$
(B5) $A \rightarrow A \vee B$
(B6) $B \rightarrow A \vee B$
(B7) $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$

(B8) $A \wedge (B \vee C) \to (A \wedge B) \vee C$
(B9) $\sim\sim A \to A$
(B10) $\Box A \wedge \Box B \to \Box(A \wedge B)$
(B11) $\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$
(B12) $\mathbf{t}$

Rules of inference

$$\frac{A \to B \quad A}{B} \qquad \frac{A \quad B}{A \wedge B} \qquad \frac{A \to B}{(B \to C) \to (A \to C)} \qquad \frac{A \to B}{(C \to A) \to (C \to B)}$$

$$\frac{A \to \sim B}{B \to \sim A} \qquad \frac{A \circ B \to C}{A \to (B \to C)} \qquad \frac{A \to (B \to C)}{A \circ B \to C}$$

$$\frac{A \to B}{\Box A \to \Box B} \qquad \frac{A \to B}{\Diamond A \to \Diamond B} \qquad \frac{A}{\mathbf{t} \to A}$$

A $\mathbf{B.C}_{\Box\Diamond}$-*frame* is a 7-tuple $\langle O, W, R, S_\Box, S_\Diamond, ^*, e\rangle$ where

(a) W is a set of all worlds,
(b) O is a nonempty subset of W,
(c) R is a ternary relation on W,
(d) $S_\Box$ and $S_\Diamond$ are binary relations on W,
(e) * is an unary operation on W,
(f) e is an element of W called the *null* world.

To simplify the notation, we define a binary relation $\leq$ on W and an element u of O as follows. For all $a, b \in W$,

1. $a \leq b \overset{\text{def}}{\iff} \exists c(c \in O \;\&\; Rcab)$,
2. $u \overset{\text{def}}{=} e^*$.

A $\mathbf{B.C}_{\Box\Diamond}$-frame $\langle O, W, R, S_\Box, S_\Diamond, ^*, e\rangle$ satisfies the following postulates. For all $a, b, c, d \in W$,

(p1) $a \leq a$
(p2) $a \leq b \;\&\; Rbcd \Rightarrow Racd$
(p3) $a \leq c \;\&\; Rbcd \Rightarrow Rbad$
(p4) $d \leq a \;\&\; Rbcd \Rightarrow Rbca$
(p5) $Ruab \Rightarrow a = e$ or $b = u$
(p6) $Reue$
(p7) $a \leq b \Rightarrow b^* \leq a^*$
(p8) $a^{**} = a$
(p9) $a \leq b \;\&\; S_\Box bc \Rightarrow S_\Box ac$
(p10) $S_\Box ee$
(p11) $S_\Box ua \Rightarrow a = u$
(p12) $a \leq b \;\&\; S_\Diamond ac \Rightarrow S_\Diamond bc$
(p13) $S_\Diamond ea \Rightarrow a = e$
(p14) $S_\Diamond uu$
(p15) $a \leq b \;\&\; a \in O \Rightarrow b \in O$
(p16) $e \neq u$.

Furthermore, for a given $\mathbf{B.C}_{\Box\Diamond}$-frame $\langle O, W, R, S_{\Box}, S_{\Diamond}, ^{*}, e\rangle$, we define binary relations $S_{\boxdot}$ and $S_{\dot{\Diamond}}$ on W as follows. For all $a, b \in W$,

1. $S_{\boxdot}ab$ iff $S_{\Diamond}a^{*}b^{*}$,
2. $S_{\dot{\Diamond}}ab$ iff $S_{\Box}a^{*}b^{*}$.

We call an 8-tuple $\langle O, W, R, S_{\Box}, S_{\Diamond}, ^{*}, e, V\rangle$ a $\mathbf{B.C}_{\Box\Diamond}$*-model* on a $\mathbf{B.C}_{\Box\Diamond}$-frame $\mathfrak{F}$ (we simply say a $\mathbf{B.C}_{\Box\Diamond}$-model) if $\mathfrak{F} = \langle O, W, R, S_{\Box}, S_{\Diamond}, ^{*}, e\rangle$ is a $\mathbf{B.C}_{\Box\Diamond}$-frame and V is a mapping from Prop to 2^{W} called a *valuation* on $\mathfrak{F}$, which satisfies the following *hereditary condition* (1), *E-condition* (2), and *U-condition* (3). For all $a, b \in W$ and all $p \in$ Prop,

1. $a \leq b$ & $a \in V(p) \Rightarrow b \in V(p)$,
2. $e \notin V(p)$,
3. $u \in V(p)$.

Given a $\mathbf{B.C}_{\Box\Diamond}$-model $\langle O, W, R, S_{\Box}, S_{\Diamond}, ^{*}, e, V\rangle$ for $a \in W$ and $A \in$ Wff, a relation $\models$ between W and Wff is defined inductively as follows:

(i) for any $p \in$ Prop, $a \models p$ iff $a \in V(p)$,
(ii) $a \models A \wedge B$ iff $a \models A$ & $a \models B$,
(iii) $a \models A \vee B$ iff $a \models A$ *or* $a \models B$,
(iv) $a \models A \to B$ iff $\forall b \in W \forall c \in W(Rabc$ & $b \models A \Rightarrow c \models B)$,
(v) $a \models A \circ B$ iff $\exists b \in W \exists c \in W(Rbca$ & $b \models A$ & $c \models B)$,
(vi) $a \models \sim A$ iff $a^{*} \not\models A$,
(vii) $a \models \Box A$ iff $\forall b \in W(S_{\Box}ab \Rightarrow b \models A)$,
(viii) $a \models \Diamond A$ iff $\exists b \in W(S_{\Diamond}ab$ & $b \models A)$,
(ix) $a \models \mathbf{t}$ iff $a \in O$,

where $a \not\models A$ means that $a \models A$ does not hold. It is easy to see that

1. $a \models \boxdot A$ iff $\forall b \in W(S_{\boxdot}ab \Rightarrow b \models A)$,
2. $a \models \dot{\Diamond} A$ iff $\exists b \in W(S_{\dot{\Diamond}}ab$ & $b \models A)$.

We put $V(A) = \{a \in W \mid a \models A\}$, for all $A \in$ Wff.

Let $\mathfrak{M} = \langle O, W, R, S_{\Box}, S_{\Diamond}, ^{*}, e, V\rangle$ be a $\mathbf{B.C}_{\Box\Diamond}$-model on a $\mathbf{B.C}_{\Box\Diamond}$-frame $\mathfrak{F} = \langle O, W, R, S_{\Box}, S_{\Diamond}, ^{*}, e\rangle$ and $A \in$ Wff. Then we say

1. A *holds in* $\mathfrak{M}$ if and only if $a \models A$ for every world $a \in O$,
2. A is *valid* in a $\mathbf{B.C}_{\Box\Diamond}$-frame $\mathfrak{F}$ (write $\mathfrak{F} \models A$) if and only if A holds in every $\mathbf{B.C}_{\Box\Diamond}$-model $\mathfrak{M}$ on $\mathfrak{F}$.

Let $\mathbf{L}$ be any extension of $\mathbf{B.C}_{\Box\Diamond}$. Any $\mathbf{B.C}_{\Box\Diamond}$-frame in which all theorems of $\mathbf{L}$ are valid is called an $\mathbf{L}$-*frame*. $\mathbf{L}$-*models* (on $\mathbf{L}$-frames) are defined similarly to $\mathbf{B.C}_{\Box\Diamond}$-models.

In proving completeness of $\mathbf{B.C}_{\Box\Diamond}$, we use the canonical model method. Below we present the definition of the canonical model and refer the reader to [10] for the detailed completeness proof.

1. Σ is an $\mathbf{L}$-*theory* iff Σ satisfies the following:
   (a) $A \in \Sigma$ and $B \in \Sigma$, then $A \wedge B \in \Sigma$;
   (b) $A \to B$ is a theorem of $\mathbf{L}$ and $A \in \Sigma$, then $B \in \Sigma$.
2. For an $\mathbf{L}$-theory Σ,
   (a) Σ is *regular* iff Σ contains all theorems of $\mathbf{L}$;
   (b) Σ is *prime* iff $A \vee B \in \Sigma$ implies either $A \in \Sigma$ or $B \in \Sigma$.

3. Let $\mathsf{Th}(\mathbf{L})$ be the set of all $\mathbf{L}$-theories. Then a ternary relation R on $\mathsf{Th}(\mathbf{L})$, and binary relations on $S_{\Box}$ and $S_{\Diamond}$ on $\mathsf{Th}(\mathbf{L})$, are defined by

$$R\Sigma\Gamma\Delta \quad \text{iff} \quad \text{for any } A, B \in \mathsf{Wff}, \text{ if } A \to B \in \Sigma \text{ and } A \in \Gamma \text{ then } B \in \Delta;$$

$$S_{\Box}\Sigma\Gamma \quad \text{iff} \quad \text{for any } A \in \mathsf{Wff}, \text{ if } \Box A \in \Sigma \text{ then } A \in \Gamma;$$

$$S_{\Diamond}\Sigma\Gamma \quad \text{iff} \quad \text{for any } A \in \mathsf{Wff}, \text{ if } A \in \Gamma \text{ then } \Diamond A \in \Sigma.$$

The *canonical* $\mathbf{L}$-*model* $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c, V_c\rangle$ is defined as follows:

(a) W_c is the set of all prime $\mathbf{L}$-theories;
(b) O_c is the set of all regular prime $\mathbf{L}$-theories;
(c) R_c is the ternary relation R restricted to W_c;
(d) $S_{\Box c}$ is the binary relation $S_{\Box}$ restricted to W_c;
(e) $S_{\Diamond c}$ is the binary relation $S_{\Diamond}$ restricted to W_c;
(f) g_c is the unary operation on W_c defined by $g_c(\Sigma) = \{A \mid \sim A \notin \Sigma\}$;
(g) $e_c = \varnothing$;
(h) V_c is defined by $\Sigma \in V_c(p)$ iff $p \in \Sigma$, for all $p \in \mathsf{Prop}$ and $\Sigma \in W_c$.

Note that $\leq_c$ is the set-theoretic inclusion $\subseteq$ and that $u_c = \mathsf{Wff}$. The following proposition will be used in later sections. For the proof, see [10].

Proposition 2.1

1. *If* A *is not a theorem of* $\mathbf{L}$, *then there exists a regular prime* $\mathbf{L}$-*theory* Π *such that* $A \notin \Pi$.
2. *Let* $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c, e_c, V_c\rangle$ *be the canonical* $\mathbf{L}$-*model. For all* $A \in \mathsf{Wff}$ *and* $\Sigma \in W_c$,

$$\Sigma \models_c A \quad \textit{iff} \quad A \in \Sigma.$$

An algebra $\mathbf{M} = \langle M, \cap, \cup, \to, \cdot, -, \Box, \Diamond, 1\rangle$ is called a $\mathbf{B.C}_{\Box\Diamond}$-*algebra* if it satisfies the following postulates. For all $x, y, z \in M$,

(A1) $\langle M, \cap, \cup\rangle$ is a distributive lattice,
(A2) $x \leq y \Rightarrow x \cdot z \leq y \cdot z$,
(A3) $x \leq y \Rightarrow z \cdot x \leq z \cdot y$,
(A4) $x \cdot y \leq z$ iff $x \leq y \to z$,
(A5) $x \cup y = -(-x \cap -y)$,
(A6) $\Box(x \cap y) = \Box x \cap \Box y$,
(A7) $\Diamond(x \cup y) = \Diamond x \cup \Diamond y$,
(A8) $1 \cdot x = x$,

where $\leq$ denotes the lattice-order, that is, $x \leq y$ is defined by $x \cap y = x$.

For any $\mathbf{B.C}_{\Box\Diamond}$-algebra $\mathbf{M} = \langle M, \cap, \cup, \to, \cdot, -, \Box, \Diamond, 1\rangle$, a mapping v from Prop to M is called a *valuation* on $\mathbf{M}$. Further, given a valuation v on $\mathbf{M}$, a mapping I from Wff to M, called the *interpretation associated with* v, is defined as follows:

(i) for $p \in \mathsf{Prop}$, $I(p) = v(p)$,
(ii) $I(A \wedge B) = I(A) \cap I(B)$,
(iii) $I(A \vee B) = I(A) \cup I(B)$,
(iv) $I(A \to B) = I(A) \to I(B)$,
(v) $I(A \circ B) = I(A) \cdot I(B)$,
(vi) $I(\sim A) = -I(A)$,
(vii) $I(\Box A) = \Box I(A)$,
(viii) $I(\Diamond A) = \Diamond I(A)$,
(ix) $I(\mathbf{t}) = 1$.

Let $\mathbf{M}$ be a $\mathbf{B.C}_{\Box\Diamond}$-algebra, v be a valuation on $\mathbf{M}$, and I be the interpretation associated with v. Then we say

1. A is *valid* in v if and only if $1 \leq I(A)$,
2. A is *valid* in $\mathbf{M}$ if and only if A is valid in any v.

A $\mathbf{B.C}_{\Box\Diamond}$-algebra $\mathbf{M}$ is called an $\mathbf{L}$-*algebra* if all theorems of $\mathbf{L}$ are valid in $\mathbf{M}$.

The Lindenbaum algebra for $\mathbf{L}$, defined as usual, can be used to show the following.

Theorem 2.2 *Any extension $\mathbf{L}$ of $\mathbf{B.C}_{\Box\Diamond}$ is characterized by a class of $\mathbf{L}$-algebras.*

Filters, *prime filters*, and *ideals* in a lattice $\langle M, \cap, \cup \rangle$ are defined as usual, except that, for simplicity, we assume that both $\varnothing$ and M are prime filters and ideals. We state two properties required in later sections. For proofs, see Davey and Priestley [5], for example.

Proposition 2.3 *Let $\langle M, \cap, \cup \rangle$ be a distributive lattice.*

1. *Suppose that ∇ is a filter and Δ is an ideal such that $\nabla \cap \Delta = \varnothing$. Then there exists a prime filter $\nabla' \supseteq \nabla$ such that $\nabla' \cap \Delta = \varnothing$.*
2. *If $x, y \in M$ satisfy $x \not\leq y$, then there exists a prime filter ∇ such that $x \in \nabla$ and $y \notin \nabla$.*

3 General Frames

In this section, we define general frames for relevant modal logics. Also, for a given general frame $\mathfrak{F}$, we define the dual of $\mathfrak{F}$, which is a $\mathbf{B.C}_{\Box\Diamond}$-algebra. In particular, we show that the Lindenbaum algebra for $\mathbf{L}$ is isomorphic to the dual of the canonical $\mathbf{L}$-frame.

For a given $\mathbf{L}$-frame $\langle O, W, R, S_{\Box}, S_{\Diamond}, ^{*}, e \rangle$, let

$$Up(W)^{+} = \{X \subseteq W \mid X \neq \varnothing \ \& \ X \neq W \ \& \ \forall a \forall b (a \in X \ \& \ a \leq b \Rightarrow b \in X)\}.$$

Note that in the definition of $Up(W)^{+}$, conditions $X \neq \varnothing$ and $X \neq W$ are equivalent to conditions $u \in X$ and $e \notin X$, respectively. A *general* $\mathbf{L}$-*frame* is an 8-tuple $\mathfrak{F} = \langle O, W, R, S_{\Box}, S_{\Diamond}, ^{*}, e, P \rangle$ where

1. $\langle O, W, R, S_{\Box}, S_{\Diamond}, ^{*}, e \rangle$ is an $\mathbf{L}$-frame, later denoted by $\kappa\mathfrak{F}$,
2. P, called a *set of possible values* in $\mathfrak{F}$, is a nonempty subset of $Up(W)^{+}$, containing O and closed under $\cap, \cup$ and the operations $\rightarrow, \cdot, -, \Box$, and $\Diamond$, which are defined as follows. For all $X, Y \subseteq W$,

$$\begin{array}{lrcl} \text{(a)} & X \rightarrow Y & = & \{a \in W \mid \forall b \forall c (Rabc \ \& \ b \in X \Rightarrow c \in Y)\}, \\ \text{(b)} & X \cdot Y & = & \{a \in W \mid \exists b \exists c (Rbca \ \& \ b \in X \ \& \ c \in Y)\}, \\ \text{(c)} & -X & = & \{a \in W \mid a^{*} \notin X\}, \\ \text{(d)} & \Box X & = & \{a \in W \mid \forall b (S_{\Box}ab \Rightarrow b \in X)\}, \\ \text{(e)} & \Diamond X & = & \{a \in W \mid \exists b (S_{\Diamond}ab \ \& \ b \in X)\}. \end{array}$$

Note that any set P of possible values is closed under the operations $\boxdot$ and $\Diamonddot$ defined as follows. For all $X \in P$,

1. $\boxdot X = \{a \in W \mid \forall b (S_{\boxdot}ab \Rightarrow b \in X)\}$,
2. $\Diamonddot X = \{a \in W \mid \exists b (S_{\Diamonddot}ab \ \& \ b \in X)\}$.

Let $\mathfrak{F} = \langle O, W, R, S_{\Box}, S_{\Diamond}, ^{*}, e, P \rangle$ be a general $\mathbf{L}$-frame. We call a 9-tuple $\langle O, W, R, S_{\Box}, S_{\Diamond}, ^{*}, e, P, V \rangle$ an $\mathbf{L}$-*model* on $\mathfrak{F}$, where

1. $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, e, P \rangle$,
2. V is a mapping from Prop to P, called a *valuation* on $\mathfrak{F}$, that is, $V(p) \in P$ for all $p \in$ Prop.

Further, a relation $\models$ between W and Wff is defined as in Section 2. Thus a general **L**-frame $\mathfrak{F}$ with $P = Up(W)^+$ is essentially equal to $\kappa\mathfrak{F}$. Then we write $\mathfrak{F} \models A$ if for any valuation V on a frame $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, e, P \rangle$ and for all $a \in O$, $a \models A$.

The algebra $\langle P, \cap, \cup, \to, \cdot, -, \Box, \Diamond, O \rangle$ is called the *dual* of $\mathfrak{F}$ and denoted by $\mathfrak{F}^+$. For the duals of general frames, we easily see the following.

Theorem 3.1

1. *The dual of a general* $\mathbf{B.C}_{\Box\Diamond}$*-frame is a* $\mathbf{B.C}_{\Box\Diamond}$*-algebra.*
2. *Let* $\mathfrak{F}$ *be a general* $\mathbf{B.C}_{\Box\Diamond}$*-frame. Then A is valid in* $\mathfrak{F}$ *if and only if A is valid in* $\mathfrak{F}^+$.

By Theorem 3.1, we have the following.

Theorem 3.2 *The dual of a general* **L***-frame is an* **L***-algebra.*

Given an **L**-model $\mathfrak{M} = \langle O, W, R, S_\Box, S_\Diamond, ^*, e, V \rangle$, the general **L**-frame $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, e, P \rangle$ with $P = \{V(A) \mid A \in \text{Wff}\}$ is called the *general* **L**-*frame associated with* $\mathfrak{M}$.

The general frame associated with the canonical **L**-model

$$\mathfrak{M}_c = \langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c, e_c, V_c \rangle$$

is denoted by $\gamma\mathfrak{F}_c = \langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c, e_c, P_c \rangle$. We will call $\gamma\mathfrak{F}_c$ the *universal* **L**-*frame*. The following is Theorem 11 of [10]. We will give the proof.

Theorem 3.3 *The Lindenbaum algebra* $\mathbf{M}_L$ *for* **L** *is isomorphic to* $(\gamma\mathfrak{F}_c)^+$, *where the mapping f defined by*

$$f([A]) = V_c(A), \quad \textit{for every } A \in \text{Wff}$$

is an isomorphism.

Proof First of all, we will show that f is bijective. It is clear that f is surjective by the definition of P_c. So we show that f is injective. Suppose that $[A] \neq [B]$. Then $A \leftrightarrow B$ is not a theorem of **L**. By (1) of Proposition 2.1, there exists $\Pi \in O_c$ such that $A \leftrightarrow B \notin \Pi$. Then $\Pi \not\models_c A \leftrightarrow B$ by (2) of Proposition 2.1. This implies that $V_c(A) \neq V_c(B)$, that is, $f([A]) \neq f([B])$.

Next we will prove that f preserves operations of $\mathbf{M}_L$. We will only show some cases. Let $\Sigma \in W_c$.

1. First, suppose that $\Sigma \in f([A] \to [B])$. To show that $\Sigma \in f([A]) \to f([B])$, suppose that $R_c\Sigma\Gamma\Delta$ and $\Gamma \in f([A])$. Then $\Sigma \models_c A \to B$ and $\Gamma \models_c A$, so $\Delta \models_c B$. This means that $\Delta \in f([B])$, which is the desired result.

   Next, suppose that $\Sigma \notin f([A] \to [B])$. Then $\Sigma \not\models_c A \to B$, so there exist $\Gamma, \Delta \in W_c$ such that $R_c\Sigma\Gamma\Delta$, $\Gamma \models_c A$ and $\Delta \not\models_c B$. Hence we have $\Gamma \in V_c(A)$ and $\Delta \notin V_c(B)$, which mean that $\Gamma \in f([A])$ and $\Delta \notin f([B])$. Therefore, $\Sigma \notin f([A]) \to f([B])$.

2. $\Sigma \in f(-[A])$ iff $\Sigma \models_c \sim A$ iff $g_c(\Sigma) \not\models_c A$ iff $g_c(\Sigma) \notin V_c(A)$ iff $\Sigma \in -f([A])$.

3. $\Sigma \in f(\Diamond[A])$ iff $\Sigma \models_c \Diamond A$
iff $\exists\Gamma(S_{\Diamond c}\Sigma\Gamma \ \&\ \Gamma \models_c A)$
iff $\exists\Gamma(S_{\Diamond c}\Sigma\Gamma \ \&\ \Gamma \in V_c(A))$
iff $\Sigma \in \Diamond f([A])$. □

4 Duality

In this section, we consider general frames as duals of algebras. For a given algebra, we will construct a general frame by using the set of prime filters. Further, we will show that the duals of Lindenbaum algebras are isomorphic to universal frames.

Let $\mathbf{M} = \langle M, \cap, \cup, \to, \cdot, -, \Box, \Diamond, 1\rangle$ be an $\mathbf{L}$-algebra. Let $\mathsf{F}_\mathbf{M}$ be the set of all filters in $\mathbf{M}$. We define a ternary relation R on $\mathsf{F}_\mathbf{M}$ and binary relations $S_\Box$ and $S_\Diamond$ on $\mathsf{F}_\mathbf{M}$ as follows. For all $\nabla_1, \nabla_2, \nabla_3 \in \mathsf{F}_\mathbf{M}$,

$$R\nabla_1\nabla_2\nabla_3 \quad \text{iff} \quad \text{for all } x, y \in M, \text{ if } x \to y \in \nabla_1 \text{ and } x \in \nabla_2 \text{ then } y \in \nabla_3;$$
$$S_\Box\nabla_1\nabla_2 \quad \text{iff} \quad \text{for all } x \in M, \text{ if } \Box x \in \nabla_1 \text{ then } x \in \nabla_2;$$
$$S_\Diamond\nabla_1\nabla_2 \quad \text{iff} \quad \text{for all } x \in M, \text{ if } x \in \nabla_2 \text{ then } \Diamond x \in \nabla_1.$$

Note that $R\nabla_1\nabla_2\nabla_3$ above means that $\nabla_1 \cdot \nabla_2 \subseteq \nabla_3$ in the notation of [11].

The next lemma shows that relations R, $S_\Box$, and $S_\Diamond$ on $\mathsf{F}_\mathbf{M}$ can be restricted to the class of prime filters. The following results on R can be proved in the standard way. See, for example, Routley et al. [9] or Urquhart [11].

Lemma 4.1

1. *Suppose that ∇_1 and ∇_2 are filters and ∇_3 is a prime filter such that $R\nabla_1\nabla_2\nabla_3$. Then there exists a prime filter $\nabla_1' \supseteq \nabla_1$ such that $R\nabla_1'\nabla_2\nabla_3$.*
2. *Suppose that ∇_1 and ∇_2 are filters and ∇_3 is a prime filter such that $R\nabla_1\nabla_2\nabla_3$. Then there exists a prime filter $\nabla_2' \supseteq \nabla_2$ such that $R\nabla_1\nabla_2'\nabla_3$.*
3. *Suppose that ∇ is a prime filter and both ∇_1 and ∇_2 are filters such that $R\nabla\nabla_1\nabla_2$ and $y \notin \nabla_2$. Then there exist prime filters ∇_1' and ∇_2' such that $R\nabla\nabla_1'\nabla_2'$, $\nabla_1 \subseteq \nabla_1'$ and $y \notin \nabla_2'$.*
4. *For a prime filter ∇ such that $x \to y \notin \nabla$, there exist prime filters ∇_1 and ∇_2 such that $R\nabla\nabla_1\nabla_2$, $x \in \nabla_1$ and $y \notin \nabla_2$.*

Lemma 4.2 *For a prime filter ∇ such that $x \cdot y \in \nabla$, there exist prime filters ∇_1 and ∇_2 such that $R\nabla_1\nabla_2\nabla$, $x \in \nabla_1$, and $y \in \nabla_2$.*

Proof Let ∇_1 and ∇_2 be the filters generated by $\{x\}$ and $\{y\}$, respectively. Then we see easily that $R\nabla_1\nabla_2\nabla$. By (1) and (2) of Lemma 4.1, there exist prime filters $\nabla_1' \supseteq \nabla_1$ and $\nabla_2' \supseteq \nabla_2$ such that $R\nabla_1'\nabla_2'\nabla$. Then it is obvious that $x \in \nabla_1'$ and $y \in \nabla_2'$. □

Lemma 4.3

1. *Suppose that ∇ is a prime filter and ∇_1 is a filter such that $S_\Box\nabla\nabla_1$ and $x \notin \nabla_1$. Then there exists a prime filter ∇_1' such that $S_\Box\nabla\nabla_1'$ and $x \notin \nabla_1$.*
2. *For a prime filter ∇ such that $\Box x \notin \nabla$, there exists a prime filter ∇_1 such that $S_\Box\nabla\nabla_1$ and $x \notin \nabla_1$.*

Proof

1. Let Δ be the ideal generated by $\{x\}$. Then we see easily that $\nabla_1 \cap \Delta = \varnothing$. By (1) of Proposition 2.3, there exists a prime filter $\nabla_1' \supseteq \nabla_1$ such that $\nabla_1' \cap \Delta = \varnothing$. It is obvious that $S_\Box \nabla \nabla_1'$. Further, we have $x \notin \nabla_1'$ since $x \in \Delta$.
2. Let $\nabla_1 = \{y \mid \Box y \in \nabla\}$. Then we easily see that ∇_1 is a filter such that $S_\Box \nabla \nabla_1$ and $x \notin \nabla_1$. By 1, there exists a prime filter $\nabla_1' \supseteq \nabla_1$ such that $S_\Box \nabla \nabla_1'$ and $x \notin \nabla_1'$.

□

Lemma 4.4 *For a prime filter ∇ such that $\Diamond x \in \nabla$, there exists a prime filter ∇_1 such that $S_\Diamond \nabla \nabla_1$ and $x \in \nabla_1$.*

Proof Let ∇_1 be the filter generated by $\{x\}$ and let $\Delta = \{y \mid \Diamond y \notin \nabla\}$. Then we see easily that ∇_1 is an ideal such that $\nabla_1 \cap \Delta = \varnothing$. By (1) of Proposition 2.3, there exists a prime filter $\nabla_1' \supseteq \nabla_1$ such that $\nabla_1' \cap \Delta = \varnothing$. Now suppose that $y \in \nabla_1'$. Then $y \notin \Delta$, so $\Diamond y \in \nabla$. Therefore, $S_\Diamond \nabla \nabla_1'$. It is obvious that $x \in \nabla_1'$. □

For an **L**-algebra $\mathbf{M} = \langle M, \cap, \cup, \to, \cdot, -, \Box, \Diamond, 1\rangle$, the structure

$$\mathbf{M}_+ = \langle O_\mathbf{M}, W_\mathbf{M}, R_\mathbf{M}, S_{\Box\mathbf{M}}, S_{\Diamond\mathbf{M}}, g_\mathbf{M}, e_\mathbf{M}, P_\mathbf{M}\rangle,$$

called the *dual* of **M**, is defined as follows:

(a) $W_\mathbf{M}$ is the set of all prime filters in **M**;
(b) $O_\mathbf{M} = \{\nabla \in W_\mathbf{M} \mid 1 \in \nabla\}$;
(c) $R_\mathbf{M}$ is the restriction of R to $W_\mathbf{M}$;
(d) $S_{\Box\mathbf{M}}$ is the restriction of $S_\Box$ to $W_\mathbf{M}$;
(e) $S_{\Diamond\mathbf{M}}$ is the restriction of $S_\Diamond$ to $W_\mathbf{M}$;
(f) $g_\mathbf{M}(\nabla) = \{x \in M \mid -x \notin \nabla\}$, for $\nabla \in W_\mathbf{M}$;
(g) $e_\mathbf{M} = \varnothing$;
(h) $P_\mathbf{M} = \{f_\mathbf{M}(x) \mid x \in M\}$, where $f_\mathbf{M} : M \to Up(W_\mathbf{M})^+$ is defined by $f_\mathbf{M}(x) = \{\nabla \in W_\mathbf{M} \mid x \in \nabla\}$.

Of course, the binary relation $\leq_\mathbf{M}$ on $W_\mathbf{M}$ is defined by

$$\nabla_1 \leq_\mathbf{M} \nabla_2 \quad \text{iff} \quad \exists \nabla(\nabla \in O_\mathbf{M} \;\&\; R_\mathbf{M} \nabla \nabla_1 \nabla_2),$$

and $u_\mathbf{M} = M$.

Lemma 4.5 *For each* $\mathbf{B.C}_{\Box\Diamond}$*-algebra* **M**, $\langle O_\mathbf{M}, W_\mathbf{M}, R_\mathbf{M}, S_{\Box\mathbf{M}}, S_{\Diamond\mathbf{M}}, g_\mathbf{M}, e_\mathbf{M}\rangle$ *is a* $\mathbf{B.C}_{\Box\Diamond}$*-frame.*

Proof Before we check all postulates, we show that $\leq_\mathbf{M}$ is equal to $\subseteq$. First, take any prime filter ∇_1 and ∇_2 such that $\nabla_1 \subseteq \nabla_2$. Put $\nabla = \{x \mid 1 \leq x\}$. Then it is easy to see that ∇ is a filter. Now suppose that $x \to y \in \nabla$ and $x \in \nabla_1$. Then $1 \leq x \to y$, and so $x \leq y$. Since ∇_1 is a filter, we have $y \in \nabla_1$, which implies $y \in \nabla_2$. So we have $R \nabla \nabla_1 \nabla_2$. By (1) of Lemma 4.1, there exists a prime filter $\nabla' \supseteq \nabla$ such that $R \nabla' \nabla_1 \nabla_2$. Therefore, there exists $\nabla' \in O_\mathbf{M}$ such that $R_\mathbf{M} \nabla' \nabla_1 \nabla_2$.

For the converse, suppose that there exists $\nabla \in O_\mathbf{M}$ such that $R_\mathbf{M} \nabla \nabla_1 \nabla_2$. Further, suppose that $x \in \nabla_1$. Then $1 \in \nabla$, so $x \to x \in \nabla$ since $1 \leq x \to x$. Hence $x \in \nabla_2$. Therefore, $\nabla_1 \subseteq \nabla_2$.

Other postulates from (p1) to (p16) can be easily checked. □

Since the relation $\leq_{\mathbf{M}}$ is the set-theoretic inclusion $\subseteq$, we easily see the following.

Lemma 4.6 *Every set $X \in P_{\mathbf{M}}$ is upward closed in $\langle O_{\mathbf{M}}, W_{\mathbf{M}}, R_{\mathbf{M}}, S_{\Box\mathbf{M}}, S_{\Diamond\mathbf{M}}, g_{\mathbf{M}}, e_{\mathbf{M}}\rangle$, that is, if $\nabla \in X$ and $\nabla \leq_{\mathbf{M}} \nabla'$, then $\nabla' \in X$.*

By Lemmas 4.1 through 4.4, we obtain the following.

Lemma 4.7 *$P_{\mathbf{M}}$ is closed under $\rightarrow, \cdot, -, \Box$, and $\Diamond$.*

Proof Here we give proofs only for $\rightarrow$ and $\Box$. Let $f_{\mathbf{M}}(x), f_{\mathbf{M}}(y) \in P_{\mathbf{M}}$ for some $x, y \in M$.

($\rightarrow$) First, suppose that $\nabla \in f_{\mathbf{M}}(x \rightarrow y)$. To show that $\nabla \in f_{\mathbf{M}}(x) \rightarrow f_{\mathbf{M}}(y)$, suppose that $R_{\mathbf{M}}\nabla\nabla_1\nabla_2$ and $\nabla_1 \in f_{\mathbf{M}}(x)$. Then $x \rightarrow y \in \nabla$ and $x \in \nabla_1$, so $y \in \nabla_2$. This is just $\nabla_2 \in f_{\mathbf{M}}(y)$, which is the desired result.

Next, suppose that $\nabla \notin f_{\mathbf{M}}(x \rightarrow y)$. Then $x \rightarrow y \notin \nabla$. By (4) of Lemma 4.1, there exist $\nabla_1, \nabla_2 \in W_{\mathbf{M}}$ such that $R_{\mathbf{M}}\nabla\nabla_1\nabla_2$, $x \in \nabla_1$ and $y \notin \nabla_2$. Then $\nabla_1 \in f_{\mathbf{M}}(x)$ and $\nabla_2 \notin f_{\mathbf{M}}(y)$. Hence $\nabla \notin f_{\mathbf{M}}(x) \rightarrow f_{\mathbf{M}}(y)$.

($\Box$) First, suppose that $\nabla \in f_{\mathbf{M}}(\Box x)$. To show that $\nabla \in \Box f_{\mathbf{M}}(x)$, suppose that $S_{\Box\mathbf{M}}\nabla\nabla_1$. Then $\Box x \in \nabla$, so $x \in \nabla_1$. This means that $\nabla_1 \in f_{\mathbf{M}}(x)$, which is the desired result.

Next, suppose that $\nabla \notin f_{\mathbf{M}}(\Box x)$. Then $\Box x \notin \nabla$, so there exists $\nabla_1 \in W_{\mathbf{M}}$ such that $S_{\Box\mathbf{M}}\nabla\nabla_1$ and $x \notin \nabla_1$ by (2) of Lemma 4.3. Hence we have $\nabla_1 \notin f_{\mathbf{M}}(x)$, so $\nabla \notin \Box f_{\mathbf{M}}(x)$.

$\Box$

By Lemmas 4.5 through 4.7, we have the following.

Theorem 4.8 *Let $\mathbf{M}$ be a $\mathbf{B.C}_{\Box\Diamond}$-algebra. Then the dual $\mathbf{M}_+$ of $\mathbf{M}$ is a general $\mathbf{B.C}_{\Box\Diamond}$-frame.*

Then we have the representation theorem.

Theorem 4.9 *Every $\mathbf{B.C}_{\Box\Diamond}$-algebra $\mathbf{M}$ is isomorphic to $(\mathbf{M}_+)^+$ under the isomorphism $f_{\mathbf{M}}$.*

Proof First of all, we will show that $f_{\mathbf{M}}$ is bijective. It is clear that $f_{\mathbf{M}}$ is surjective, so we show that $f_{\mathbf{M}}$ is injective. Suppose that $x \neq y$. Then either $x \not\leq y$ or $y \not\leq x$. Without loss of generality, we may assume $x \not\leq y$. By (2) of Proposition 2.3, there exists a prime filter ∇ such that $x \in \nabla$ and $y \notin \nabla$. So we have $\nabla \in f_{\mathbf{M}}(x)$ and $\nabla \notin f_{\mathbf{M}}(y)$, and hence, $f_{\mathbf{M}}(x) \neq f_{\mathbf{M}}(y)$.

It remains to show that $f_{\mathbf{M}}$ preserves each operation of $\mathbf{M}$. Since any element of $f_{\mathbf{M}}(x)$ must be a prime filter, it is easy to see that $f_{\mathbf{M}}(x \cap y) = f_{\mathbf{M}}(x) \cap f_{\mathbf{M}}(y)$ and $f_{\mathbf{M}}(x \cup y) = f_{\mathbf{M}}(x) \cup f_{\mathbf{M}}(y)$. For other operations, we have already proved those in Lemma 4.7.

Finally, we show that $O_{\mathbf{M}} \in P_{\mathbf{M}}$. This is obtained from the following:

$$\nabla \in O_{\mathbf{M}} \quad \text{iff} \quad 1 \in \nabla \quad \text{iff} \quad \nabla \in f_{\mathbf{M}}(1).$$

$\Box$

From Theorem 4.9 and (2) of Theorem 3.1, we have the following.

Corollary 4.10 *Let $\mathbf{M}$ be a $\mathbf{B.C}_{\Box\Diamond}$-algebra. Then A is valid in $\mathbf{M}$ if and only if A is valid in $\mathbf{M}_+$.*

Proof By Theorem 4.8, $\mathbf{M}_+$ is a general $\mathbf{B.C}_{\Box\Diamond}$-frame. Further, by (2) of Theorem 3.1, A is valid in $\mathbf{M}_+$ if and only if A is valid in $(\mathbf{M}_+)^+$. From Theorem 4.9, we have the desired result. $\Box$

By Theorems 4.8 and 4.9 and Corollary 4.10, we have the following.

Corollary 4.11 *Let* $\mathbf{M}$ *be an* $\mathbf{L}$*-algebra. Then the dual* $\mathbf{M}_+$ *of* $\mathbf{M}$ *is a general* $\mathbf{L}$*-frame.*

Thus, the following property on general frames (Theorem 12 of [10]) holds.

Corollary 4.12 *Any extension* $\mathbf{L}$ *of* $\mathbf{B.C}_{\Box\Diamond}$ *is complete with respect to the class of all general* $\mathbf{L}$*-frames.*

Proof Suppose that A is not a theorem of $\mathbf{L}$. Then by Theorem 2.2 there exists an $\mathbf{L}$-algebra $\mathbf{M}$ in which A is not valid. By Corollary 4.10, A is not valid in $\mathbf{M}_+$, which is a general $\mathbf{L}$-frame by Corollary 4.11. Hence, there exists a general $\mathbf{L}$-frame in which A is not valid. $\Box$

Further, we can easily see the following relationship between duals of Lindenbaum algebras and universal frames (Theorem 13 of [10]).

Theorem 4.13 *The dual* $(\mathbf{M}_L)_+$ *of the Lindenbaum algebra for* $\mathbf{L}$ *is isomorphic to the universal* $\mathbf{L}$*-frame* $\gamma\mathfrak{F}_c$.

5 Descriptive Frames

In preceding sections, we have seen that the dual of a general frame is an algebra and vice versa. It follows that the bidual (i.e., dual of a dual) of a general frame is also a general frame. But general frames are not always isomorphic to their biduals. In this section we introduce descriptive frames, for which such isomorphism holds, as in classical modal logic.

We first introduce some auxiliary notions. Given a general $\mathbf{L}$-frame $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, {}^*, e, P\rangle$, we say that

(a) $\mathfrak{F}$ is *differentiated* if for any $a, b \in W$,

$$a = b \quad \text{iff} \quad \forall X \in P(a \in X \Leftrightarrow b \in X),$$

(b) $\mathfrak{F}$ is *r-tight* if for any $a, b, c \in W$,

$$Rabc \quad \text{iff} \quad \forall X \in P \forall Y \in P(a \in X \to Y \;\&\; b \in X \Rightarrow c \in Y),$$

(c) $\mathfrak{F}$ is $\Box$-*tight* if for any $a, b \in W$, $S_\Box ab$ iff $\forall X \in P(a \in \Box X \Rightarrow b \in X)$,
(d) $\mathfrak{F}$ is $\boxdot$ -*tight* if for any $a, b \in W$, $S_\boxdot ab$ iff $\forall X \in P(a \in \boxdot X \Rightarrow b \in X)$,
(e) $\mathfrak{F}$ is $\Diamond$-*tight* if for any $a, b \in W$, $S_\Diamond ab$ iff $\forall X \in P(b \in X \Rightarrow a \in \Diamond X)$,
(f) $\mathfrak{F}$ is $\diamonddot$ -*tight* if for any $a, b \in W$, $S_\diamonddot ab$ iff $\forall X \in P(b \in X \Rightarrow a \in \diamonddot X)$,
(g) $\mathfrak{F}$ is *compact* if for any families $\mathcal{X} \subseteq P$ and $\mathcal{Y} \subseteq \overline{P} = \{W - X \mid X \in P\}$,

$$\bigcap(\mathcal{X} \cup \mathcal{Y}) = \{a \mid \forall X \in \mathcal{X} \forall Y \in \mathcal{Y}(a \in X \;\&\; a \in Y)\} \neq \varnothing$$

whenever $\bigcap(\mathcal{X}' \cup \mathcal{Y}') \neq \varnothing$ for all finite subfamilies $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{Y}$.

A general $\mathbf{L}$-frame $\mathfrak{F}$ is called *descriptive* if $\mathfrak{F}$ is differentiated, r-tight, $\Box$-tight, $\boxdot$ -tight, $\Diamond$-tight, $\diamonddot$ -tight, compact, and, moreover, satisfies

$$O = \bigcap\{X \in P \mid O \subseteq X\}.$$

The definition of descriptive frames for relevant modal logics, although analogous to the classical one, differs from it in that it introduces several kinds of tightness and a condition on the set O. Different kinds of tightness are due to differences between relevant connectives and their classical counterparts. The condition on O stems from the fact that Routley-Meyer semantics uses so-called distinguished points (see [4]).

In the following, we will investigate the properties of these notions. For a general **L**-frame $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, {}^*, e, P\rangle$ and $a \in W$, define

$$Pa = \{X \in P \mid a \in X\}.$$

The following proposition is easy to prove.

Proposition 5.1 *For every general* **L**-*frame* $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, {}^*, e, P\rangle$ *and every* $a \in W$, Pa *is a prime filter in* $\mathfrak{F}^+$.

Compact general frames are characterized by the following proposition which corresponds to Proposition 8.48 of [4] (p. 255).

Proposition 5.2 *A general* **L**-*frame* $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, {}^*, e, P\rangle$ *is compact if and only if every prime filter* ∇ *in* $\mathfrak{F}^+$ *is of the form* Pa *for some* $a \in W$.

The following theorem (Theorem 14 of [10]) characterizes descriptive frames.

Theorem 5.3 *A general* **L**-*frame* $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, {}^*, e, P\rangle$ *is descriptive if and only if it is isomorphic to* $(\mathfrak{F}^+)_+$.

Proof The 'if' part is proved as follows. Let $\nabla_1, \nabla_2, \nabla_3 \in W_{\mathfrak{F}^+}$, that is, ∇_1, ∇_2, and ∇_3 be prime filters in P. For proofs that $(\mathfrak{F}^+)_+$ is differentiated, $\Box$-tight, $\boxdot$ -tight, and compact, see Proposition 8.51 of [4] (p. 257). Here we will give proofs of the other clauses.

Clause 1 $(\mathfrak{F}^+)_+$ is r-tight. The 'if' part is proved as follows. Suppose that $R_{\mathfrak{F}^+}\nabla_1\nabla_2\nabla_3$ does not hold. Then there exist $X, Y \in P$ such that $X \to Y \in \nabla_1$, $X \in \nabla_2$, and $Y \notin \nabla_3$. Then we have $f_{\mathfrak{F}^+}(X), f_{\mathfrak{F}^+}(Y) \in P_{\mathfrak{F}^+}$ satisfying $\nabla_1 \in f_{\mathfrak{F}^+}(X) \to f_{\mathfrak{F}^+}(Y)$, $\nabla_2 \in f_{\mathfrak{F}^+}(X)$, and $\nabla_3 \notin f_{\mathfrak{F}^+}(Y)$.

The 'only if' part is proved as follows. Suppose that $R_{\mathfrak{F}^+}\nabla_1\nabla_2\nabla_3$. Further, take any $f_{\mathfrak{F}^+}(X), f_{\mathfrak{F}^+}(Y) \in P_{\mathfrak{F}^+}$ satisfying $\nabla_1 \in f_{\mathfrak{F}^+}(X) \to f_{\mathfrak{F}^+}(Y)$ and $\nabla_2 \in f_{\mathfrak{F}^+}(X)$. Then we have $X \to Y \in \nabla_1$ and $X \in \nabla_2$, so $Y \in \nabla_3$, and hence $\nabla_3 \in f_{\mathfrak{F}^+}(Y)$.

Clause 2 $(\mathfrak{F}^+)_+$ is $\Diamond$-tight. The 'if' part is proved as follows. Suppose that $S_{\Diamond\mathfrak{F}^+}\nabla_1\nabla_2$ does not hold. Then there exists $X \in P$ such that $X \in \nabla_2$ and $\Diamond X \notin \nabla_1$. Then we have $f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+}$ satisfying $\nabla_2 \in f_{\mathfrak{F}^+}(X)$ and $\nabla_1 \notin \Diamond f_{\mathfrak{F}^+}(X)$.

The 'only if' part is proved as follows. Suppose that $S_{\Diamond\mathfrak{F}^+}\nabla_1\nabla_2$. Further, take any $f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+}$ satisfying $\nabla_2 \in f_{\mathfrak{F}^+}(X)$. Then we have $X \in \nabla_2$, so $\Diamond X \in \nabla_1$, and hence $\nabla_1 \in \Diamond f_{\mathfrak{F}^+}(X)$.

Clause 3 $(\mathfrak{F}^+)_+$ is $\diamonddot$ -tight. Similar to clause 2.

Clause 4 $O_{\mathfrak{F}^+} = \bigcap\{f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+} \mid O_{\mathfrak{F}^+} \subseteq f_{\mathfrak{F}^+}(X)\}$. First, suppose that $\nabla \in O_{\mathfrak{F}^+}$. Take any $f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+}$ such that $O_{\mathfrak{F}^+} \subseteq f_{\mathfrak{F}^+}(X)$. Then it is clear that $\nabla \in f_{\mathfrak{F}^+}(X)$. Therefore, $\nabla \in \bigcap\{f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+} \mid O_{\mathfrak{F}^+} \subseteq f_{\mathfrak{F}^+}(X)\}$. Next, suppose that $\nabla \notin O_{\mathfrak{F}^+}$. Then $O \notin \nabla$, so $\nabla \notin f_{\mathfrak{F}^+}(O)$. It is easy to show that $O_{\mathfrak{F}^+} \subseteq f_{\mathfrak{F}^+}(O)$. Since $f_{\mathfrak{F}^+}(O) \in P_{\mathfrak{F}^+}$, $\nabla \notin \bigcap\{f_{\mathfrak{F}^+}(X) \in P_{\mathfrak{F}^+} \mid O_{\mathfrak{F}^+} \subseteq f_{\mathfrak{F}^+}(X)\}$. Thus, we see that $(\mathfrak{F}^+)_+$ is descriptive.

The 'only if' part is proved as follows. By Proposition 5.1, for each $a \in W$, Pa is a prime filter in $\mathfrak{F}^+$. We define a mapping $f_{\mathfrak{F}} : W \to W_{\mathfrak{F}^+}$ by

$$f_{\mathfrak{F}}(a) = Pa, \quad \text{for } a \in W.$$

By Proposition 5.2, we have $W_{\mathfrak{F}^+} = \{Pa \mid a \in W\}$. Then it is clear that $f_{\mathfrak{F}}$ is surjective. Take $a, b \in W$ such that $a \neq b$. Since $\mathfrak{F}$ is differentiated, there exists $X \in P$ such that $a \in X$ and $b \notin X$. So we have $X \in Pa$ and $X \notin Pb$. Hence $f_{\mathfrak{F}}(a) \neq f_{\mathfrak{F}}(b)$. Therefore, $f_{\mathfrak{F}}$ is injective.

Further, as in Proposition 8.51 of [4] (p. 257), we show that

1. $S_{\Box}ab$ iff $S_{\Box_{\mathfrak{F}^+}} f_{\mathfrak{F}}(a) f_{\mathfrak{F}}(b)$,
2. $S_{\boxdot}ab$ iff $S_{\boxdot_{\mathfrak{F}^+}} f_{\mathfrak{F}}(a) f_{\mathfrak{F}}(b)$,
3. $X \in P$ iff $f_{\mathfrak{F}}(X) \in P_{\mathfrak{F}^+}$,

for $a, b \in W$. It remains to show the following clauses. For $a, b, c \in W$:

Clause 1 $Rabc$ iff $R_{\mathfrak{F}^+} f_{\mathfrak{F}}(a) f_{\mathfrak{F}}(b) f_{\mathfrak{F}}(c)$.

$$\begin{array}{lll} Rabc & \text{iff} & \forall X \in P \forall Y \in P(a \in X \to Y \ \&\ b \in X \Rightarrow c \in Y) \quad (\mathfrak{F} \text{ is r-tight}) \\ & \text{iff} & \forall X \in P \forall Y \in P(X \to Y \in Pa \ \&\ X \in Pb \Rightarrow Y \in Pc) \\ & \text{iff} & R_{\mathfrak{F}^+} PaPbPc \\ & \text{iff} & R_{\mathfrak{F}^+} f_{\mathfrak{F}}(a) f_{\mathfrak{F}}(b) f_{\mathfrak{F}}(c). \end{array}$$

Clause 2 $S_{\Diamond}ab$ iff $S_{\Diamond_{\mathfrak{F}^+}} f_{\mathfrak{F}}(a) f_{\mathfrak{F}}(b)$.

$$\begin{array}{lll} S_{\Diamond}ab & \text{iff} & \forall X \in P(b \in X \Rightarrow a \in \Diamond X) \quad (\mathfrak{F} \text{ is } \Diamond\text{-tight}) \\ & \text{iff} & \forall X \in P(X \in Pb \Rightarrow \Diamond X \in Pa) \\ & \text{iff} & S_{\Diamond_{\mathfrak{F}^+}} f_{\mathfrak{F}}(a) f_{\mathfrak{F}}(b). \end{array}$$

Clause 3 $S_{\diamonddot}ab$ iff $S_{\diamonddot_{\mathfrak{F}^+}} f_{\mathfrak{F}}(a) f_{\mathfrak{F}}(b)$. Similar to clause 2.

Clause 4 $f_{\mathfrak{F}}(a^*) = g_{\mathfrak{F}^+}(f_{\mathfrak{F}}(a))$.

$$\begin{array}{lll} X \in f_{\mathfrak{F}}(a^*) & \text{iff} & a^* \in X \\ & \text{iff} & a \notin -X \\ & \text{iff} & -X \notin f_{\mathfrak{F}}(a) \\ & \text{iff} & X \in g_{\mathfrak{F}^+}(f_{\mathfrak{F}}(a)). \end{array}$$

Clause 5 $f_{\mathfrak{F}}(O) = O_{\mathfrak{F}^+}$. First, suppose that $\nabla \in f_{\mathfrak{F}}(O)$. Then there exists $a \in O$ such that $\nabla = f_{\mathfrak{F}}(a)$. Since $\mathfrak{F}$ is descriptive, $a \in \bigcap\{X \in P \mid O \subseteq X\}$. It means that $\forall X \in P(O \subseteq X \Rightarrow X \in Pa)$. Since $O \in P$, we have $O \in Pa$, that is, $O \in \nabla$. Hence we have $\nabla \in O_{\mathfrak{F}^+}$.

For the reverse inclusion, suppose that $\nabla \in O_{\mathfrak{F}^+}$. Since ∇ is a prime filter in $\mathfrak{F}^+$, $\nabla = Pa$ for some $a \in W$ by Proposition 5.2. Then $O \in Pa$, so $a \in O$. Therefore, $\nabla \in f_{\mathfrak{F}}(O)$. □

Thus we have the following theorem (Theorem 15 of [10]).

Theorem 5.4 *Any extension* **L** *of* $\mathbf{B.C}_{\Box\Diamond}$ *is characterized by the class of descriptive* **L**-*frames.*

Proof Let $\mathfrak{F}$ be any descriptive **L**-frame. By the definition of **L**-frames, if A is a theorem of **L**, then $\mathfrak{F} \models A$. On the other hand, if A is not a theorem of **L**, then A is not valid in the Lindenbaum algebra $\mathbf{M}_L$ for **L**. By Corollary 4.10, A is not valid in $(\mathbf{M}_L)_+$. By Theorem 4.13, A is not valid in the universal **L**-frame $\gamma\mathfrak{F}_c$. Further, we see that $\gamma\mathfrak{F}_c$ is descriptive by Theorems 3.3, 4.13, and 5.3. Therefore, there exists a descriptive **L**-frame in which A is not valid. □

6 The Categories of Descriptive Frames and Algebras

In Section 5, we introduced descriptive frames. Considering duality between descriptive frames and algebras in detail, we have shown that each algebra can be represented by descriptive frames and vice versa. This fact can be stated clearly with the help of categorical notions. In this section, following [6] and [8], we will show that descriptive frames and algebras are duals in the category theory sense.

Let $\mathscr{A}$ be the category of **L**-algebras defined as follows:

1. objects are **L**-algebras;
2. morphisms are homomorphisms.

Now we will introduce frame morphisms, which will be the morphisms in the category of descriptive frames that we are going to define. Let $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, e, P\rangle$ and $\mathfrak{F}' = \langle O', W', R', S'_\Box, S'_\Diamond, ^{*'}, e', P'\rangle$ be general **L**-frames. Then a mapping $q : W \to W'$ is a *frame morphism* from $\mathfrak{F}$ to $\mathfrak{F}'$ if the following conditions hold. For all $a, b, c \in W$ and $a', b', c' \in W'$,

(m1) $Rabc \Rightarrow R'q(a)q(b)q(c)$,
(m2) $R'a'b'q(c) \Rightarrow \exists a \in W\exists b \in W(Rabc \;\&\; a' \leq' q(a) \;\&\; b' \leq' q(b))$,
(m3) $R'q(a)b'c' \Rightarrow \exists b \in W\exists c \in W(Rabc \;\&\; b' \leq' q(b) \;\&\; q(c) \leq' c')$,
(m4) $S_\Box ab \Rightarrow S'_\Box q(a)q(b)$,
(m5) $S'_\Box q(a)b' \Rightarrow \exists b \in W(S_\Box ab \;\&\; q(b) \leq' b')$,
(m6) $S_\Diamond ab \Rightarrow S'_\Diamond q(a)q(b)$,
(m7) $S'_\Diamond q(a)b' \Rightarrow \exists b \in W(S_\Diamond ab \;\&\; b' \leq' q(b))$,
(m8) $q(a^*) = (q(a))^{*'}$,
(m9) $q^{-1}(O') = O$,
(m10) $q(e) = e'$,
(m11) $X \in P' \Rightarrow q^{-1}(X) \in P$.

Then it is clear that the identity map on any frame is a frame morphism and that the composition of frame morphisms is also a frame morphism. So the collection of all descriptive frames and frame morphisms between descriptive frames forms a category. Let $\mathscr{F}$ be the category of descriptive **L**-frames defined as follows:

1. objects are descriptive **L**-frames; and
2. morphisms are frame morphisms.

We proceed to define functors between these categories. We begin by showing that every algebra homomorphism has its corresponding frame morphism. Let **M** and **M**$'$ be **L**-algebras. For a homomorphism h from **M** to **M**$'$, a mapping $h_+ : \mathbf{M}'_+ \to \mathbf{M}_+$ is defined by

$$h_+(\nabla) = h^{-1}(\nabla),$$

for every prime filter ∇ in **M**$'$. Then we have the following.

Lemma 6.1 *If h is a homomorphism from* **M** *to* **M**$'$*, then the mapping h_+ is a frame morphism from $\mathbf{M}'_+$ to $\mathbf{M}_+$.*

Proof It suffices to check the conditions (m1) through (m11). Here we give the proof for the conditions (m2) and (m5).

(m2) Suppose that $R_{\mathbf{M}}\nabla_1\nabla_2 h_+(\nabla'_3)$ for $\nabla_1, \nabla_2 \in W_{\mathbf{M}}$ and $\nabla_3 \in W_{\mathbf{M}'}$. Putting $\nabla'_4 = \{y' \in M' \mid \exists x \in \nabla_1(h(x) \leq y')\}$ and $\nabla'_5 = \{y' \in M' \mid \exists x \in \nabla_2(h(x) \leq y')\}$, it is easy to see that ∇'_4 and ∇'_5 are filters in $\mathbf{M}'$. Further, suppose that $y' \to z' \in \nabla'_4$ and $y' \in \nabla'_5$. Then there exist $x_1 \in \nabla_1$ and $x_2 \in \nabla_2$ such that $h(x_1) \leq y' \to z'$ and $h(x_2) \leq y'$. Since h is a homomorphism, we have $h(x_1 \cdot x_2) = h(x_1) \cdot h(x_2) \leq (y' \to z') \cdot y' \leq z'$, and $x_1 \cdot x_2 \in h_+(\nabla'_3)$, that is, $h(x_1 \cdot x_2) \in \nabla'_3$ by the assumption. Since ∇'_3 is a filter, $z' \in \nabla'_3$. Thus, we have $R\nabla'_4\nabla'_5\nabla'_3$.

Then, by (1) and (2) of Lemma 4.1, there exist prime filters ∇'_1 and ∇'_2 such that $\nabla'_4 \subseteq \nabla'_1$, $\nabla'_5 \subseteq \nabla'_2$, and $R_{\mathbf{M}'}\nabla'_1\nabla'_2\nabla'_3$. Here, taking $x \in \nabla_1$, we have $h(x) \in \nabla'_4 \subseteq \nabla'_1$ since $h(x) \leq h(x)$, and hence $x \in h_+(\nabla'_1)$. So we have $\nabla_1 \subseteq h_+(\nabla'_1)$ and, similarly, $\nabla_2 \subseteq h_+(\nabla'_2)$. Therefore, there exist $\nabla'_1, \nabla'_2 \in W_{\mathbf{M}'}$ such that $R_{\mathbf{M}'}\nabla'_1\nabla'_2\nabla'_3$, $\nabla_1 \leq_{\mathbf{M}'} h_+(\nabla'_1)$, and $\nabla_2 \leq_{\mathbf{M}'} h_+(\nabla'_2)$.

(m5) Suppose that $S_{\Box\mathbf{M}} h_+(\nabla'_1)\nabla_2$ for $\nabla'_1 \in W_{\mathbf{M}'}$ and $\nabla_2 \in W_{\mathbf{M}}$. Putting $\nabla'_3 = \{x' \in M' \mid \Box x' \in \nabla'_1\}$, it is easy to see that ∇'_3 is a filter in $\mathbf{M}'$ satisfying $S_\Box\nabla'_1\nabla'_3$. Let Δ be the ideal generated by $\{h(x) \mid x \notin \nabla_2\}$. Assuming that $x' \in \nabla'_3 \cap \Delta$, we have $\Box x' \in \nabla'_1$, $x' \leq h(x)$, and $x \notin \nabla_2$. Since ∇'_1 is a filter and h is a homomorphism, $h(\Box x) \in \nabla'_1$, that is, $\Box x \in h_+(\nabla'_1)$, and $x \notin \nabla_2$. This is a contradiction, so $\nabla'_3 \cap \Delta = \varnothing$. By (1) of Proposition 2.3, there exists a prime filter $\nabla'_2 \supseteq \nabla'_3$ such that $\nabla'_2 \cap \Delta = \varnothing$. It is obvious that $S_{\Box\mathbf{M}'}\nabla'_1\nabla'_3$. Further, suppose that $x \in h_+(\nabla'_2)$. Then $h(x) \in \nabla'_2$, and hence $h(x) \notin \Delta$. So, we have $x \in \nabla_2$. Thus, $h_+(\nabla'_2) \subseteq \nabla_2$. □

Then we show that every frame morphism has its corresponding algebra homomorphism. Let $\mathfrak{F}$ and $\mathfrak{F}'$ be descriptive **L**-frames. For a frame morphism q from $\mathfrak{F}$ to $\mathfrak{F}'$, a mapping $q^+ : \mathfrak{F}'^+ \to \mathfrak{F}^+$ is defined by

$$q^+(X) = q^{-1}(X),$$

for every $X \in P'$. Then we have the following.

Lemma 6.2 *If q is a frame morphism from $\mathfrak{F}$ to $\mathfrak{F}'$, then the mapping q^+ is a homomorphism from $\mathfrak{F}'^+$ to $\mathfrak{F}^+$.*

Proof Here we only show that (1) $q^+(X \to Y) = q^+(X) \to q^+(Y)$ and (2) $q^+(\Diamond X) = \Diamond q^+(X)$, for $X, Y \in P'$.

(1) First, suppose that $a \in q^+(X \to Y)$. To show that $a \in q^+(X) \to q^+(Y)$, suppose that $Rabc$ and $b \in q^+(X)$. Then we have $q(a) \in X \to Y$ and $q(b) \in X$. By (m1), $R'q(a)q(b)q(c)$, so $q(c) \in Y$. This means that $c \in q^+(Y)$, which is the desired result.

For the converse, suppose that $a \in q^+(X) \to q^+(Y)$. To show that $a \in q^+(X \to Y)$, suppose that $R'q(a)b'c'$ and $b' \in X$. By (m3), there exist $b, c \in W$ such that $Rabc$ and $b' \leq' q(b)$ and $q(c) \leq' c'$. Since X is upward closed, $q(b) \in X$, so we have $b \in q^+(X)$. Hence $c \in q^+(Y)$, so $c' \in Y$. This is the desired result.

(2) First, suppose that $a \in q^+(\Diamond X)$. Then $q(a) \in \Diamond X$, so there exists $b' \in X$ such that $S'_\Diamond q(a)b'$. By (m7), there exists $b \in W$ such that $S_\Diamond ab$ and $b' \leq' q(b)$.

Since X is upward closed, we have $q(b) \in X$, so $b \in q^+(X)$. Therefore, we have $a \in \Diamond q^+(X)$.

For the converse, suppose that $a \in \Diamond q^+(X)$. Then there exists $b \in q^+(X)$ such that $S_\Diamond ab$. Then $q(b) \in X$, and $S'_\Diamond q(a)q(b)$ by (m6). So we have $q(a) \in \Diamond X$, and hence $a \in q^+(\Diamond X)$. □

By Lemmas 6.1 and 6.2, we obtain the following facts:

1. if h is a homomorphism from $\mathbf{M}$ to $\mathbf{M}'$, then $(h_+)^+$ is a homomorphism from $(\mathbf{M}_+)^+$ to $(\mathbf{M}'_+)^+$;
2. if q is a frame morphism from $\mathfrak{F}$ to $\mathfrak{F}'$, then $(q^+)_+$ is a frame morphism from $(\mathfrak{F}^+)_+$ to $(\mathfrak{F}'^+)_+$.

Since $\mathbf{M}$ and $(\mathbf{M}_+)^+$, $\mathbf{M}'$ and $(\mathbf{M}'_+)^+$ are respectively isomorphic, it is natural to consider the relation between h and $(h_+)^+$.

Theorem 6.3 *For an* $\mathbf{L}$*-algebra* $\mathbf{M}$*, let* $f_\mathbf{M}$ *be the isomorphism from* $\mathbf{M}$ *to* $(\mathbf{M}_+)^+$ *of Theorem 4.9. Then for any homomorphism* h *from* $\mathbf{M}$ *to* $\mathbf{M}'$*,* $(h_+)^+ \circ f_\mathbf{M} = f_{\mathbf{M}'} \circ h$.

Proof For any $x \in M$ and $\nabla \in W_{\mathbf{M}'}$, $\nabla \in (h_+)^+(f_\mathbf{M}(x))$ iff $h_+(\nabla) \in f_\mathbf{M}(x)$ iff $x \in h^{-1}(\nabla)$ iff $h(x) \in \nabla$ iff $\nabla \in f_{\mathbf{M}'}(h(x))$. □

We can associate a descriptive $\mathbf{L}$-frame $\mathbf{M}_+$ and a frame morphism h_+ from $\mathbf{M}'_+$ to $\mathbf{M}_+$ with an $\mathbf{L}$-algebra $\mathbf{M}$ and a homomorphism h from $\mathbf{M}$ to $\mathbf{M}'$, respectively. It is easy to check that

1. $(\mathrm{id}_\mathfrak{F})_+ = \mathrm{id}_{\mathfrak{F}^+}$, where 'id' denotes identity maps,
2. $(h_1 \circ h_2)_+ = (h_2)_+ \circ (h_1)_+$.

Thus $(\,\cdot\,)_+$ defines a contravariant functor from $\mathcal{A}$ to $\mathcal{F}$.

The analogous result for frame morphisms is given below.

Theorem 6.4 *For each descriptive* $\mathbf{L}$*-frame* $\mathfrak{F}$*, let* $f_\mathfrak{F}$ *be the isomorphism from* $\mathfrak{F}$ *to* $(\mathfrak{F}^+)_+$ *in the proof of Theorem 5.3. Then for any frame morphism* q *from* $\mathfrak{F}$ *to* $\mathfrak{F}'$*,* $(q^+)_+ \circ f_\mathfrak{F} = f_{\mathfrak{F}'} \circ q$.

Proof For any $a \in W$ and $X \in P'$, $X \in (q^+)_+(f_\mathfrak{F}(a))$ iff $q^+(X) \in f_\mathfrak{F}(a)$ iff $a \in q^{-1}(X)$ iff $q(a) \in X$ iff $X \in f_{\mathfrak{F}'}(q(a))$. □

We can associate an $\mathbf{L}$-algebra $\mathfrak{F}^+$ and a homomorphism q^+ from $\mathfrak{F}'^+$ to $\mathfrak{F}^+$ with a descriptive $\mathbf{L}$-frame $\mathfrak{F}$ and a frame morphism q from $\mathfrak{F}$ to $\mathfrak{F}'$, respectively. It is easy to check that

1. $(\mathrm{id}_\mathbf{M})^+ = \mathrm{id}_{\mathbf{M}_+}$, where 'id' denotes identity maps,
2. $(q_1 \circ q_2)^+ = q_2^+ \circ q_1^+$.

So $(\,\cdot\,)^+$ defines a contravariant functor from $\mathcal{F}$ to $\mathcal{A}$.

Thus Theorem 6.3 shows that the collection of isomorphisms h_+ constructs a natural isomorphism between the composite functor $((\,\cdot\,)_+)^+$ and the identity functor on $\mathcal{A}$, and Theorem 6.4 shows that the collection of isomorphisms q^+ constructs a natural isomorphism between the composite functor $((\,\cdot\,)^+)_+$ and the identity functor on $\mathcal{F}$. Hence we have the following theorem.

Theorem 6.5 *The categories* $\mathcal{A}$ *and* $\mathcal{F}$ *are dual by the functors* $(\,\cdot\,)_+$ *and* $(\,\cdot\,)^+$.

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Computing and Communications Center
Kyushu University
6-10-1 Hakozaki, Higashi-ku
812-8581 Fukuoka
JAPAN
tseki@cc.kyushu-u.ac.jp