Notre Dame Journal of Formal Logic
Volume 43, Number 4, 2002

On Bounded Type-Definable Equivalence Relations

Krzysztof Krupiński and Ludomir Newelski

Abstract We investigate some topological properties of the spaces of classes of bounded type-definable equivalence relations.

1 Introduction

In this paper T is a complete theory in a countable language $\mathcal{L}$ and $\mathfrak{C}$ is a monster model of T. We will consider type-definable (over $\varnothing$) equivalence relations on some $\mathfrak{C}^k$ ($k \in \omega$), that is, relations defined by some type over $\varnothing$. Namely, each such relation E on $\mathfrak{C}^k$ is defined by

$$E(\overline{x}, \overline{y}) \Leftrightarrow \bigwedge_{n<\omega} \theta_n(\overline{x}, \overline{y})$$

for some type $\{\theta_n(\overline{x}, \overline{y}) : n < \omega\}$. By compactness we can assume that each $\theta_n(\overline{x}, \overline{y})$ is symmetric and $\models \theta_{n+1}(\overline{x}, \overline{y}) \wedge \theta_{n+1}(\overline{y}, \overline{z}) \rightarrow \theta_n(\overline{x}, \overline{z})$. We say that E is bounded, if E has boundedly many classes. When we consider two such relations E_1 and E_2, then $E_1 \subseteq E_2$ means that E_1 is finer than E_2.

There is a topology τ on $\mathfrak{C}^k/E$ with a basis of open sets

$$\mathcal{B} = \{[\varphi(\overline{x})] : \varphi(\overline{x}) \in \mathcal{L}(\mathfrak{C})\},$$

where $[\varphi(\overline{x})] = \{\overline{a}/E : \overline{a}/E \subseteq \varphi(\mathfrak{C})\}$. Then a basis of closed sets is of the form $\{\langle\varphi(\overline{x})\rangle : \varphi(\overline{x}) \in \mathcal{L}(\mathfrak{C})\}$, where $\langle\varphi(\overline{x})\rangle = \{\overline{a}/E : \overline{a}/E \cap \varphi(\mathfrak{C}) \neq \varnothing\}$. By compactness (and boundedness of E) we have that $(\mathfrak{C}^k/E, \tau)$ is a compact Hausdorff topological space. It is easy to see that $\{[\theta_n(\overline{x}, \overline{a})] : n \in \omega\}$ is a basis of open neighborhoods of the point $\overline{a}/E$ in $\mathfrak{C}^k/E$. This topology was defined in Hrushovski [1] and Lascar and Pillay [4].

Throughout, E will denote a bounded 0-type-definable equivalence relation on some $\mathfrak{C}^k$ (we will write $\mathfrak{C}$ instead of $\mathfrak{C}^k$). $S(\varnothing)$ denotes $S_k(\varnothing)$ unless stated otherwise. There are three important examples of such equivalence relations which will be denoted in a special way:

Received April 17, 2002; accepted February 13, 2003; printed December 19, 2003
2001 Mathematics Subject Classification: Primary, 03C45
Keywords: type-definable equivalence relations, balanced relations

1. the relation of having the same type over $\varnothing$, denoted by $\equiv$,
2. the relation of having the same strong type over $\varnothing$, denoted by $\overset{s}{\equiv}$,
3. the finest bounded equivalence relation, denoted by $\overset{bd}{\equiv}$.

Remark 1.1 When T is simple then the $\overset{bd}{\equiv}$-classes are Lascar strong types. Similarly, the $\equiv$-classes correspond to complete types over $\varnothing$ and the $\overset{s}{\equiv}$-classes to types in $S(\mathrm{acl}^{\mathrm{eq}}(\varnothing))$. These correspondences are homeomorphisms between $\mathfrak{C}/\equiv$ and $S(\varnothing)$ and between $\mathfrak{C}/\overset{s}{\equiv}$ and $S(\mathrm{acl}^{\mathrm{eq}}(\varnothing))$.

The aim of this paper is to understand what topologies can appear as $\mathfrak{C}/E$ for some bounded type-definable equivalence relation E. Such relations were investigated, for example, in [4], but this elementary aspect seemed neglected.

We discern three cases. In Section 2, we describe fully which topologies are of the form $\mathfrak{C}/E$, where E is coarser than $\equiv$. It turns out that if T is not small, then any compact metric space occurs in this way. Hence in this respect all theories that are not small look alike. Then we investigate relations E finer than $\equiv$ but coarser than $\overset{s}{\equiv}$. In this case $\mathfrak{C}/E$ is 0-dimensional, hence E is an intersection of 0-definable finite equivalence relations. This is folklore but we give a proof.

In Section 3, we investigate relations E finer than $\overset{s}{\equiv}$ but necessarily coarser than $\overset{bd}{\equiv}$. In this case the connected components of $\mathfrak{C}/E$ correspond to strong types ([1], Lemma 2.1). We give an example where the connected components are not locally connected. Using a Haar measure we define an invariant metric on each connected component of $\mathfrak{C}/E$. This leads us to a new proof of a result of Kim [2] that in a small theory $\overset{bd}{\equiv}$ equals $\overset{s}{\equiv}$.

In Section 4, we focus on a connected component X of the space $\mathfrak{C}/E$ for some E finer than $\overset{s}{\equiv}$ but coarser than $\overset{bd}{\equiv}$. Similarly as in [4], we consider the group G of elementary permutations of X. G is a compact topological group acting continuously on X. Hence G is a projective limit of compact Lie groups (Weil [7]). We characterize those Es for which G itself is a Lie group. It turns out that sometimes the very topological nature of X determines G to be a Lie group. This happens, for example, when X is homeomorphic to the circle S^1.

We will often use the following basic facts.

Fact 1.2 Suppose that $E_1 \subseteq E_2$ are as above.

1. The canonical map $\pi : \mathfrak{C}/E_1 \to \mathfrak{C}/E_2$ is continuous.
2. The topology on $\mathfrak{C}/E_2$ is the quotient topology induced by π from the topology on $\mathfrak{C}/E_1$.

Proof (1) Let E_1 and E_2 be defined by $\{\theta^1_n(\overline{x},\overline{y}) : n \in \omega\}$ and $\{\theta^2_n(\overline{x},\overline{y}) : n \in \omega\}$, respectively. We can assume that $\models \theta^1_n(\overline{x},\overline{y}) \to \theta^2_n(\overline{x},\overline{y})$. Let $[\varphi]$ be a basic open set in $\mathfrak{C}/E_2$. We want to show that $\pi^{-1}([\varphi])$ is open in $\mathfrak{C}/E_1$. Take $\overline{a}/E_1 \in \pi^{-1}([\varphi])$. Then $\overline{a}/E_2 \in [\theta^2_n(\overline{x},\overline{a})] \subseteq [\varphi]$ for some $n \in \omega$. It suffices to show that $[\theta^1_{n+1}(\overline{x},\overline{a})] \subseteq \pi^{-1}([\varphi])$. So let $\overline{b}/E_1 \in [\theta^1_{n+1}(\overline{x},\overline{a})]$. Take any $\overline{b'} \in \overline{b}/E_2$. Then $\models \theta^2_{n+1}(\overline{b},\overline{b'})$. We also have $\models \theta^1_{n+1}(\overline{b},\overline{a})$, which implies that $\models \theta^2_{n+1}(\overline{b},\overline{a})$. Therefore $\models \theta^2_n(\overline{b'},\overline{a})$, so $\overline{b}/E_2 \in [\theta^2_n(\overline{x},\overline{a})]$, which means that $\overline{b}/E_1 \in \pi^{-1}([\varphi])$.

(2) This follows from (1) and the compactness of both topologies. $\square$

Fact 1.3 The space $\mathfrak{C}/E$ is metrizable.

Proof $\mathfrak{C}/E$ is compact, so it is sufficient to show that $\mathfrak{C}/E$ has a countable basis. By compactness, for every $n \in \omega$, there are $\overline{a}^n_1, \ldots, \overline{a}^n_{m_n} \in \mathfrak{C}$ (for some $m_n \in \omega$) such that

$$[\theta_n(\overline{x}, \overline{a}^n_1)] \cup \cdots \cup [\theta_n(\overline{x}, \overline{a}^n_{m_n})] = \mathfrak{C}/E.$$

We will prove that the sets $[\theta_n(\overline{x}, \overline{a}^n_i)]$, $n \in \omega$, $i \leq m_n$, form a basis of the topology on $\mathfrak{C}/E$. So let $\overline{a} \in \mathfrak{C}$ and $n \in \omega$. It is enough to show that for some $k \in \omega$ and $i \leq k$ we have $\overline{a}/E \in [\theta_k(\overline{x}, \overline{a}^k_i)] \subseteq [\theta_n(\overline{x}, \overline{a})]$. Let $k = n + 1$. Choose $i \leq m_k$ such that $\overline{a}/E \in [\theta_k(\overline{x}, \overline{a}^k_i)]$. Since $\models \theta_{n+1}(\overline{x}, \overline{y}) \wedge \theta_{n+1}(\overline{y}, \overline{z}) \to \theta_n(\overline{x}, \overline{z})$, we get that $[\theta_k(\overline{x}, \overline{a}^k_i)] \subseteq [\theta_n(\overline{x}, \overline{a})]$. □

2 Equivalence Relations Coarser Than ≡

First we recall an example of Pillay and Poizat [6] of a bounded type-definable equivalence relation (in a stable theory T) which is not an intersection of definable equivalence relations.

Example 2.1 Let $T = \mathrm{Th}(M)$, where M consists of the universe Q and unary predicates $U_a = \{x \in Q : x \leq a\}$ for $a \in Q$. We have $S(\varnothing) = \{t^+_a : a \in Q\} \cup \{t^-_a : a \in Q\} \cup \{t_a : a \notin Q\}$, where

1. t^+_a is determined by $\{U_b(x) : a < b \in Q\} \cup \{\neg U_b(x) : a \geq b \in Q\}$,
2. t^-_a is determined by $\{U_b(x) : a \leq b \in Q\} \cup \{\neg U_b(x) : a > b \in Q\}$,
3. t_a is determined by $\{U_b(x) : a < b \in Q\} \cup \{\neg U_b(x) : a > b \in Q\}$.

Let E be the equivalence relation defined by the conjunction of formulas $(U_a(x) \to U_b(y)) \wedge (U_a(y) \to U_b(x))$ for $a < b$. Then $\mathfrak{C}/E = \{t^+_a(\mathfrak{C}) \cup t^-_a(\mathfrak{C}) : a \in Q\} \cup \{t_a(\mathfrak{C}) : a \notin Q\}$. So we see that E is bounded. One can show that T is stable and E is not a conjunction of definable equivalence relations.

For us it is important that in the above example $\mathfrak{C}/E$ is homeomorphic to the unit interval I (symbolically: $\mathfrak{C}/E \approx I$). Generalizing this example we will fully describe the topological spaces occuring as $\mathfrak{C}/E$ for some E coarser than ≡. It turns out that stability of T is quite irrelevant here.

Theorem 2.2 *Let X be a Hausdorff topological space. Then $X \approx \mathfrak{C}/E$ for some E coarser than ≡ if and only if X is a continuous image of $S(\varnothing)$.*

Proof (⇒) The proof is obvious (Fact 1.2).

(⇐) Let $f : S_k(\varnothing) \to X$ be a continuous surjection. Let E_f be the equivalence relation on $S_k(\varnothing)$ defined by

$$E_f(p, q) \Leftrightarrow f(p) = f(q).$$

As X is Hausdorff we see that $E_f \subseteq S_k(\varnothing) \times S_k(\varnothing)$ is closed. Let $h : \mathfrak{C} \times \mathfrak{C} \to S_{2k}(\varnothing)$ and $g : S_{2k}(\varnothing) \to S_k(\varnothing) \times S_k(\varnothing)$ be defined by $h(\overline{x}, \overline{y}) = tp(\overline{x}\overline{y})$ and $g(tp(\overline{x}\overline{y})) = (tp(\overline{x}), tp(\overline{y}))$. Then $g^{-1}(E_f)$ is closed in $S_{2k}(\varnothing)$, so $E := h^{-1}(g^{-1}(E_f))$ is a type-definable equivalence relation on $\mathfrak{C}$ coarser than ≡. It is easy to see that $\mathfrak{C}/E \approx S_k(\varnothing)/E_f \approx X$. □

If $S(\varnothing)$ is uncountable, then the Cantor set is a continuous image of $S(\varnothing)$, and then in turn (by a well-known topological result), every compact metric space is a continuous image of the Cantor set. Hence we get the following corollary.

Corollary 2.3 *Assume that $S(\varnothing)$ is uncountable. For every metric compact space X there is E coarser than $\equiv$ such that $\mathfrak{C}/E \approx X$.*

In the example of Pillay and Poizat, E is not an intersection of 0-definable equivalence relations. This may be seen directly. However, it may be also deduced from the following proposition and the fact that in this example $\mathfrak{C}/E \approx I$ is not 0-dimensional. This proposition is folklore; it appears in Pillay [5] (without proof).

Proposition 2.4 *$\mathfrak{C}/E$ is 0-dimensional if and only if E is an intersection of definable equivalence relations.*

Proof ($\Leftarrow$) Let $E(\overline{x},\overline{y}) \Leftrightarrow \bigwedge_{i\in\omega} E_i(\overline{x},\overline{y})$ where E_i is a definable equivalence relation. We can assume that $\models E_{i+1}(\overline{x},\overline{y}) \Rightarrow E_i(\overline{x},\overline{y})$. We see that $\{[E_i(\overline{x},\overline{a})] : i \in \omega, \overline{a} \in \mathfrak{C}\}$ is a basis of the topology on $\mathfrak{C}/E$ consisting of clopen sets.

($\Rightarrow$) We will show in Section 3 (Corollary 3.4) that if $\mathfrak{C}/E$ is 0-dimensional, then E is coarser than $\stackrel{s}{\equiv}$, that is, $\stackrel{s}{\equiv} \subseteq E$. Let $\pi : \mathfrak{C}/\stackrel{s}{\equiv} \to \mathfrak{C}/E$ be the canonical mapping and let $\{U_\alpha\}_{\alpha\in I}$ be a basis of clopen sets in $\mathfrak{C}/E$. Then $V_\alpha := \pi^{-1}(U_\alpha)$ is a clopen set in $\mathfrak{C}/\stackrel{s}{\equiv}$ for every $\alpha \in I$. So there is a definable equivalence relation E_α with two classes X_α and Y_α such that $\{\overline{a}/\stackrel{s}{\equiv} : \overline{a} \in X_\alpha\} = V_\alpha$ and $\{\overline{a}/\stackrel{s}{\equiv} : \overline{a} \in Y_\alpha\} = V_\alpha^c$. Obviously E_α is coarser than E and E_α is almost over $\varnothing$. Let E'_α be the conjunction of the conjugates of E_α. We see that E'_α is 0-definable and

$$E(\overline{x},\overline{y}) \Leftrightarrow \bigwedge_{\alpha\in I} E'_\alpha(\overline{x},\overline{y}).$$

□

The above results show that the example of Pillay and Poizat is not exceptional and any compact metric space can be interpreted as $\mathfrak{C}/E$ for some E coarser than $\equiv$.

Now we turn to relations E finer than $\equiv$ but coarser than $\stackrel{s}{\equiv}$.

Fact 2.5 If E is finer than $\equiv$ and coarser than $\stackrel{s}{\equiv}$, then $\mathfrak{C}/E$ is 0-dimensional, so E is a conjunction of definable relations.

Proof Let $G = \mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(\varnothing))$ be the group of elementary permutations of $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$. G is a topological group with the topology of pointwise convergence. One can show that G is a profinite group, so it is a compact 0-dimensional group. The action of G on $\mathfrak{C}/\stackrel{s}{\equiv}$ is continuous, because the basic open sets in $\mathfrak{C}/\stackrel{s}{\equiv}$ are of the form $\{\overline{a}/\stackrel{s}{\equiv} : \models \varphi(\overline{a},\overline{b})\}$, where $\overline{b}$ is a finite sequence of elements from $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$. Via the canonical map $\pi : \mathfrak{C}/\stackrel{s}{\equiv} \to \mathfrak{C}/E$ we get an induced action of G on $\mathfrak{C}/E$, which is also continuous. Denote this action by $\odot$.

Let $p/E = \{\overline{a}/E : \overline{a} \models p\}$ for $p \in S(\varnothing)$. p/E is a closed subspace of $\mathfrak{C}/E$ and G acts transitively upon it. Fix some $a^* = \overline{a}/E \in p/E$ and take a closed subgroup G_{a^*} of G defined by $G_{a^*} = \{g \in G : ga^* = a^*\}$. We get a function $f : G/G_{a^*} \to p/E$ defined by $f(gG_{a^*}) = ga^*$.

We claim that f is a homeomorphism from G/G_{a^*} onto p/E, where G/G_{a^*} is considered with the quotient topology.

To see this it is sufficient to show that f is continuous (because f is a bijection and, moreover, G/G_{a^*} and p/E are compact and Hausdorff). Let $\tau : G \to G/G_{a^*}$ be a canonical map and $\pi_1 : G \times \mathfrak{C}/E \to G$ be a projection on the first coordinate. Fix some open set $U \subseteq p/E$. We have that $\tau^{-1}(f^{-1}(U)) = \pi_1(\odot^{-1}(U) \cap G \times \{a^*\})$ is open, so $f^{-1}(U)$ is open, too, and f is continuous.

Since $G/G_{a^*} \approx p/E$, we get that p/E is 0-dimensional. We also have that $\pi : \mathfrak{C}/E \to \mathfrak{C}/\equiv$ is continuous, $\mathfrak{C}/\equiv \approx S(\varnothing)$ is 0-dimensional, and $\pi^{-1}(\overline{a}/\equiv) = tp(\overline{a})/E$. We conclude that $\mathfrak{C}/E$ is 0-dimensional. □

3 Bounded Equivalence Relations Finer Than $\overset{s}{\equiv}$

In this section we investigate bounded 0-type-definable equivalence relations E finer than $\overset{s}{\equiv}$. Such an E is necessarily coarser than $\overset{bd}{\equiv}$ (which is the finest bounded 0-type-definable equivalence relation). First we describe the connected components of $\mathfrak{C}/E$. For $p \in S(\mathrm{acl}^{\mathrm{eq}}(\varnothing))$ let $p/E = \{\overline{a}/E : \overline{a} \models p\}$. The following proposition appears in [1].

Proposition 3.1 *The sets p/E, $p \in S(\mathrm{acl}^{\mathrm{eq}}(\varnothing))$, are the connected components of $\mathfrak{C}/E$.*

Proof It is easy to see that every connected component is contained in some p/E. Indeed, suppose $X \subseteq \mathfrak{C}/E$ is a connected component meeting p/E and q/E for some distinct $p, q \in S(\mathrm{acl}^{\mathrm{eq}}(\varnothing))$. Choose $\overline{a} \models p$ and $\overline{b} \models q$ with $\overline{a}/E$, $\overline{b}/E \in X$. Choose a clopen set $U \subseteq S(\mathrm{acl}^{\mathrm{eq}}(\varnothing))$ such that $p \in U$ and $q \notin U$. Let $\pi : \mathfrak{C}/E \to \mathfrak{C}/\overset{s}{\equiv}$ be canonical. Then $\overline{a}/E \in \pi^{-1}(U)$ and $\overline{b}/E \notin \pi^{-1}(U)$, hence $\pi^{-1}(U)$ and $\pi^{-1}(U^c)$ are distinct clopen sets meeting X, a contradiction.

It is harder to show that p/E is connected for every $p \in S(\mathrm{acl}^{\mathrm{eq}}(\varnothing))$. Suppose for a contradiction that p/E is not connected, that is, there are clopen in p/E nonempty disjoint sets $U, V \subseteq p/E$ such that $U \cup V = p/E$. So there are sets of formulas (with parameters) $\{\varphi_i(\overline{x}) : i \in I\}$ and $\{\psi_j(\overline{x}) : j \in J\}$ closed under finite conjunctions for which

$$U = \{\overline{a}/E : \exists \overline{b} E \overline{a} \bigwedge_{i \in I} \models \varphi_i(\overline{b})\}, \quad V = \{\overline{a}/E : \exists \overline{b} E \overline{a} \bigwedge_{j \in J} \models \psi_j(\overline{b})\}.$$

Claim 3.2 *There is $n \in \omega$ such that for all $\overline{a}/E \in U$ and $\overline{b}/E \in V$ we have $\models \neg\theta_n(\overline{a}, \overline{b})$.*

Proof If not, then the following set of formulas is consistent: $\{\varphi_i(\overline{x}) \wedge \psi_j(\overline{y}) \wedge \theta_n(\overline{x}, \overline{y}) : i \in I, j \in J, n \in \omega\}$. By compactness, there is an $\overline{a}/E \in U \cap V$, which is impossible.

Let $n \in \omega$ be as in the claim. E is bounded, so by compactness there are $\overline{a}_1, \dots, \overline{a}_m \in p(\mathfrak{C})$ (for some $m \in \omega$) such that $p(\mathfrak{C}) \subseteq \theta_n(\mathfrak{C}, \overline{a}_1) \cup \dots \cup \theta_n(\mathfrak{C}, \overline{a}_m)$. Hence there is $n' \in \omega$ such that for all $\overline{a} \in p(\mathfrak{C})$ and $\overline{b} \in p(\mathfrak{C})$ if $\models \bigwedge_{0 \le i \le k} \theta_n(\overline{b}_i, \overline{b}_{i+1})$ for some sequence $\overline{b}_0 = \overline{a}, \dots, \overline{b}_{k+1} = \overline{b}$ of elements of $p(\mathfrak{C})$, then there is such a sequence of length at most n'. Define a relation E^* on $p(\mathfrak{C})$ by

$$E^*(\overline{x}_1, \overline{x}_{n'}) \Leftrightarrow \exists \overline{x}_2, \dots, \overline{x}_{n'-1} \bigwedge_{1 \le i \le n'-1} \models \theta_n(\overline{x}_i, \overline{x}_{i+1}) \wedge \bigwedge_{1 \le i \le n'} \overline{x}_i \models p.$$

E^* is an $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$-type-definable equivalence relation on $p(\mathfrak{C})$, which has finitely many classes (in fact $\leq m$-many classes). So E^* is equivalent on $p(\mathfrak{C})$ to some finite equivalence relation E' definable over $\mathrm{acl}^{\mathrm{eq}}(\varnothing)$. Moreover, by the claim, there are $\overline{a}, \overline{b} \models p$ such that $\neg E'(\overline{a}, \overline{b})$, a contradiction. □

The following example shows that in Proposition 3.1 we cannot prove that p/E is locally connected.

Example 3.3 Let S^1 be the unit circle viewed as the multiplicative group of complex numbers of absolute value 1. Let S^∞ be the projective limit of the system $\{X_n, f_{n+1,n}\}_{n<\omega}$, where $X_n = S^1$ and $f_{n+1,n} : X_{n+1} \to X_n$ is given by $f_{n+1,n}(z) = z^2$. So topologically the group S^∞ is a solenoid (i.e., the projective limit of a system of circles) and is not locally connected.

Let $f_{\infty,n} : S^\infty \to X_n$ be the projection map. The sets $f^{-1}_{\infty,n}[U]$, $U \subseteq X_n$ open, $n < \omega$, form a basis of the topology on S^∞, and if $U \subseteq X_n$ is a short open arc, then $f^{-1}_{\infty,n}[U]$ is homeomorphic with $U \times C$, where C is the Cantor set.

Let d be the usual metric on S^1. We define a first-order structure M with universe S^∞ by

$$M = (S^\infty, \{U^n_q(x, y) : q \in \mathbb{Q}^+\}),$$

where $U^n_q(x, y)$ holds if and only if $d(f_{\infty,n}(x), f_{\infty,n}(y)) < q$.

S^∞ acts on M by translation as a group of automorphisms. In fact, $\mathrm{Aut}(M) = S^\infty \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on M by complex conjugation. We have several type-definable equivalence relations on M. For $n < \omega$ let

$$E_n(x, y) \Leftrightarrow \bigwedge_q U^n_q(x, y) \quad \text{and} \quad E(x, y) \Leftrightarrow \bigwedge_n E_n(x, y).$$

Actually, E equals $\overset{bd}{\equiv}$ here. Since S^∞ acts transitively on M, in $\mathrm{Th}(M)$, $S_1(\varnothing)$ consists of a single type p. Moreover, since S^∞ is Abelian and divisible, it has no proper subgroups of finite index. It follows that there is no 0-definable nontrivial equivalence relation on M with finitely many classes. Hence p is a strong type in $\mathrm{Th}(M)$.

Clearly, for each n, $p/E_n \approx X_n \approx S^1$, while $p/E \approx S^\infty$. So p/E is connected but not locally connected. This example may be modified by replacing the connecting functions z^2 by other powers of z.

Using Proposition 3.1 we get a corollary referred to in the proof of Proposition 2.4.

Corollary 3.4 *Assume E is a bounded 0-type-definable equivalence relation. If $\mathfrak{C}/E$ is 0-dimensional, then E is coarser than $\overset{s}{\equiv}$.*

Proof Suppose that $E \not\supseteq \overset{s}{\equiv}$. Let $E' = E \cap \overset{s}{\equiv}$. Then E' is a type-definable equivalence relation finer than $\overset{s}{\equiv}$. Take $\overline{a}, \overline{b} \in \mathfrak{C}$ such that $\overline{a} \overset{s}{\equiv} \overline{b}$ but $\neg E(\overline{a}, \overline{b})$. Let $p = tp(\overline{a}/\mathrm{acl}^{\mathrm{eq}}(\varnothing))$. By Proposition 3.1 the set p/E' is connected in $\mathfrak{C}/E'$. So the image X of p/E' under the canonical mapping $\mathfrak{C}/E' \to \mathfrak{C}/E$ is connected, but $|X| > 1$, because X contains two distinct points $\overline{a}/E$ and $\overline{b}/E$. This contradicts the assumption that $\mathfrak{C}/E$ is 0-dimensional. □

For any bounded 0-type-definable E let G_E denote the group of elementary permutations of $\mathfrak{C}/E$ induced by automorphisms of $\mathfrak{C}$. Every element of G_E is a homeomorphism of $\mathfrak{C}/E$. We can regard G_E as a closed subset of the space $(\mathfrak{C}/E)^{\mathfrak{C}/E}$

with the Tychonov product topology. G_E with the induced subspace topology is a compact Hausdorff topological group, acting continuously on $\mathfrak{C}/E$ (cf. [4]). When E equals $\overset{bd}{\equiv}$, [1] calls G_E the compact Lascar group.

By Fact 1.3, $\mathfrak{C}/E$ is metrizable. Let d_0 be a metric on $\mathfrak{C}/E$ inducing the topology on it. Modifying d_0 we obtain an equivalent metric d on $\mathfrak{C}/E$, which is invariant under Aut($\mathfrak{C}$). Namely, let μ be the Haar measure on G_E (i.e., the probabilistic measure on G_E invariant under translations). Then the metric d on $\mathfrak{C}/E$ defined by

$$d(\overline{x}, \overline{y}) = \int_{G_E} d_0(g\overline{x}, g\overline{y})d\mu$$

satisfies our demands. Using this metric we can give a new proof of the following result of [2].

Theorem 3.5 *In a small theory,* $\overset{bd}{\equiv}$ *equals* $\overset{s}{\equiv}$.

Proof Suppose for a contradiction that $\overset{bd}{\equiv}$ is essentially finer than $\overset{s}{\equiv}$. This means that for some $p \in S(\mathrm{acl}^{\mathrm{eq}}(\varnothing))$ we have $|p/\overset{bd}{\equiv}| > 1$. Choose $\overline{a}/\overset{bd}{\equiv} \neq \overline{b}/\overset{bd}{\equiv} \in p/\overset{bd}{\equiv}$ and let $\rho = d(\overline{a}/\overset{bd}{\equiv}, \overline{b}/\overset{bd}{\equiv})$. So $\rho > 0$. By Proposition 3.1, $p/\overset{bd}{\equiv}$ is connected, hence, for every δ with $0 < \delta < \rho$ the set $X_\delta = \{\overline{c}/\overset{bd}{\equiv} \in p/\overset{bd}{\equiv} : d(\overline{a}/\overset{bd}{\equiv}, \overline{c}/\overset{bd}{\equiv}) = \delta\}$ is nonempty. For each δ with $0 < \delta < \rho$ choose $\overline{c_\delta}/\overset{bd}{\equiv} \in X_\delta$. Since d is Aut($\mathfrak{C}$) invariant, we see that for $\delta_1 \neq \delta_2$, $tp(\overline{ac_{\delta_1}}) \neq tp(\overline{ac_{\delta_2}})$. Hence $S(\varnothing)$ is uncountable, a contradiction. $\square$

4 Balanced Relations

Throughout this section, E^* is a bounded 0-type-definable equivalence relation on $\mathfrak{C}$, finer than $\overset{s}{\equiv}$. By Proposition 3.1, the connected components of $\mathfrak{C}/E^*$ are of the form p/E^*, $p \in S(\mathrm{acl}^{\mathrm{eq}}(\varnothing))$. Fix a type $p \in S(\mathrm{acl}^{\mathrm{eq}}(\varnothing))$. It is interesting to learn what the structure of p/E^* can be. The structure of $\mathfrak{C}$ is to some extent reflected in the structure of Aut($\mathfrak{C}$). So in order to understand the structure of p/E^* it is reasonable to investigate the structure of the group

$$\begin{aligned} G &= \{f : p/E^* \to p/E^* : f \in \mathrm{Aut}(\mathfrak{C}) \text{ preserves } p(\mathfrak{C})\} \\ &= \{f : p/E^* \to p/E^* : f \in \mathrm{Aut}(\mathfrak{C}/Cb(p))\}, \end{aligned}$$

where $Cb(p) = \{a/E : E \in FE(\varnothing),\ a \models p\} \subseteq \mathrm{acl}^{\mathrm{eq}}(\varnothing)$. Similarly as the group G_{E^*}, G is a compact topological group acting continuously on p/E^*. From the theory of Lie groups [7] we know that

1. every compact group is a projective limit of compact Lie groups;
2. a compact group H is a Lie group if and only if H has DCC on closed subgroups.

In this section we provide a model-theoretic condition equivalent to G being itself a Lie group. To do this we establish a correspondence between bounded $Cb(p)$-type-definable equivalence relations E on $p(\mathfrak{C})$ coarser than E^* and closed normal subgroups H of G. Namely, for such E and H we define a subgroup $H(E)$ of G and two equivalence relations: E'_H on p/E^* and E_H on $p(\mathfrak{C})$ in the following way.

$$E'_H(\overline{a}/E^*, \overline{b}/E^*) \iff \exists h \in H, h(\overline{a}/E^*) = \overline{b}/E^*.$$

$$E_H = (\pi \times \pi)^{-1}(E'_H), \text{ where } \pi : p(\mathfrak{C}) \to p/E^* \text{ is the natural projection.}$$

$$H(E) = \{h \in G : \forall \overline{a} \models p, h(\overline{a}/E) = \overline{a}/E\}.$$

Remark 4.1

1. E'_H is a closed subset of $p/E^* \times p/E^*$.
2. E_H is $Cb(p)$-type-definable, coarser than E^* on $p(\mathfrak{C})$.
3. $H(E)$ is a closed normal subgroup of G.

Proof (1) This follows from compactness of G and p/E^* and continuity of the action of G on p/E^*.

(2) From (1) and the fact that a set $A \subseteq p/E^* \times p/E^*$ is closed if and only if $(\pi \times \pi)^{-1}(A)$ is type-definable we have that E_H is type-definable. To see that E_H is $Cb(p)$-type-definable we use normality of H in G in the following way. Let $E_H(\overline{a}, \overline{b})$ and $f \in \text{Aut}(\mathfrak{C}/Cb(p))$. Then there is $h \in H$ such that $\overline{a}/E^* = h(\overline{b}/E^*)$, so there is $h_1 \in H$ such that $f(\overline{a}/E^*) = h_1(f(\overline{b}/E^*))$. This means that $E_H(f(\overline{a}), f(\overline{b}))$.

(3) $H(E)$ is closed in G, because $h \in H(E) \Leftrightarrow \forall \overline{a} \models p, h(\overline{a}/E^*) \in \pi_0^{-1}(\overline{a}/E)$ and $\pi_0^{-1}(\overline{a}/E)$ is closed in p/E^* (here $\pi_0 : p/E^* \to p/E$ is the canonical map). $H(E)$ is normal in G, because for $h \in H(E), g \in G$ and any $\overline{a} \models p$ we have $g^{-1}hg(\overline{a}/E) = g^{-1}(h(\overline{b}/E)) = g^{-1}(\overline{b}/E) = \overline{a}/E$, where $g(\overline{a}/E) = \overline{b}/E$. □

Proposition 4.2 *For E and H as above we have*

$$E_{H(E)} \subseteq E, H \subseteq H(E_H), E_{H(E_H)} = E_H, \text{ and } H(E_{H(E)}) = H(E).$$

Proof The first two items follow from definitions and imply the last two. □

Not all bounded $Cb(p)$-type-definable equivalence relations E on $p(\mathfrak{C})$ coarser than E^* are of the form E_H for some closed $H \lhd G$. Similarly, not all closed $H \lhd G$ are of the form $H(E)$. This is the motivation for the following definition.

Definition 4.3 Assume E is an $\text{acl}^{\text{eq}}(\emptyset)$-type-definable equivalence relation on $p(\mathfrak{C})$ coarser than E^* and H is closed normal subgroup of G (symbolically, $H \lhd_c G$).

1. We say that E is balanced on p/E^* if $E = E_{H_1}$ for some $H_1 \lhd_c G$ (by Proposition 4.2, this is equivalent to $E = E_{H(E)}$).
2. We say that H is $*$-closed in G if $H = H(E_1)$ for some $Cb(p)$-type-definable equivalence relation E_1 on $p(\mathfrak{C})$ coarser than E^* (by Proposition 4.2, this is equivalent to $H = H(E_H)$).
3. We say that E^* is balanced if $\overline{x} \equiv \overline{y}$ and $E^*(\overline{x}, \overline{y})$ implies $f(\overline{x}) = \overline{y}$ for some $f \in \text{Aut}(\mathfrak{C})$ with $f|_{\mathfrak{C}/E^*} = \text{id}$ (here E^* is not necessarily finer than $\overset{s}{\equiv}$).
4. For $A \subseteq \mathfrak{C}^{\text{eq}}$ we say that E^* is A-balanced if $E^*(\overline{x}, \overline{y})$ implies $f(\overline{x}) = \overline{y}$ for some $f \in \text{Aut}(\mathfrak{C}/A)$ with $f|_{\mathfrak{C}/E^*} = \text{id}$.

Remark 4.4

1. E is balanced on p/E^* if and only if for $\overline{x}, \overline{y} \models p$, $E(\overline{x}, \overline{y})$ implies $f(\overline{x}/E^*) = \overline{y}/E^*$ for some $f \in \text{Aut}(\mathfrak{C}/Cb(p))$ with $f|_{p/E} = \text{id}$.

2. If E^* is $Cb(p)$-balanced, then E is balanced on p/E^* if and only if for $\overline{x}, \overline{y} \models p$, $E(\overline{x}, \overline{y})$ implies $f(\overline{x}) = \overline{y}$ for some $f \in \mathrm{Aut}(\mathfrak{C}/Cb(p))$ with $f|_{p/E} = \mathrm{id}$.
3. Relations $\equiv$, $\overset{s}{\equiv}$, $\overset{bd}{\equiv}$ are balanced.

Proposition 4.2 yields the following corollary.

Corollary 4.5 *The mapping $H \to E_H$ is a Galois correspondence between $*$-closed subgroups of G and equivalence relations balanced on p/E^*.*

Now we will use balanced equivalence relations to express when our group G is a Lie group.

Proposition 4.6 *G is a Lie group if and only if there is no proper infinite chain $E_0 \supseteq E_1 \supseteq \cdots \supseteq E^*$ of equivalence relations balanced on p/E^*.*

Proof ($\Rightarrow$) This follows easily from Corollary 4.5 and the fact that a Lie group has DCC on closed subgroups.

($\Leftarrow$) Suppose that G is not a Lie group. Still G, as a compact group, is a projective limit of Lie groups, so there exists a family $\{H_\alpha\}_{\alpha \in I}$ of closed normal subgroups of G such that

1. $\bigcap_{\alpha \in I} H_\alpha = \{\mathrm{id}\}$,
2. $\forall \alpha, \beta \in I\ \exists \gamma \in I$, $H_\gamma \subseteq H_\alpha \cap H_\beta$,
3. G/H_α is a Lie group for all $\alpha \in I$.

So, $H_\alpha \neq \{\mathrm{id}\}$ for all $\alpha \in I$. Obviously for all $\alpha, \beta \in I$ we have that $E_\alpha := E_{H_\alpha}$ is balanced on p/E and $H_\alpha \supseteq H_\beta$ implies $E_\alpha \supseteq E_\beta$.

It is sufficient to show that for every $\alpha \in I$ there exists a $\beta \in I$ such that $E_\alpha \supseteq E_\beta$ and $E_\alpha \neq E_\beta$. First we prove that $E_\alpha \neq E^*$ on $p(\mathfrak{C})$. Otherwise, by Proposition 4.2, we have $H_\alpha \subseteq H(E_{H_\alpha}) = H(E_\alpha) = H(E^*|p(\mathfrak{C}) \times p(\mathfrak{C})) = \{\mathrm{id}\}$, which is impossible.

So choose $\overline{a}/E^* \neq \overline{b}/E^* \in p/E^*$ with $E'_{H_\alpha}(\overline{a}/E^*, \overline{b}/E^*)$. Then $\mathrm{id} \notin \{g \in G : g(\overline{a}/E^*) = \overline{b}/E^*\}$ and the last set is closed in G. By compactness of G and the choice of the family $\{H_\alpha\}_{\alpha \in I}$ we get some $\beta \in I$ such that $H_\alpha \supseteq H_\beta$ and $H_\beta \cap \{g \in G : g(\overline{a}/E^*) = \overline{b}/E^*\} = \varnothing$. So $\neg E'_{H_\beta}(\overline{a}/E^*, \overline{b}/E^*)$, hence $E_\alpha \neq E_\beta$ and of course $E_\alpha \supseteq E_\beta$. □

Sometimes the very topological structure of p/E^* implies that the condition from Proposition 4.6 is satisfied, whence G is a Lie group. It is so with another example of Poizat (Lascar [3]), where p/E^* is homeomorphic to the circle S^1.

Remark 4.7 If p/E^* is homeomorphic to the circle S^1, then G is a Lie group.

Proof We want to prove that the condition from Proposition 4.6 is satisfied. By Definition 4.3, every equivalence relation balanced on p/E^* is of the form E_H for some $H \lhd_c G$. G acts transitively on p/E^* so all classes of E'_H on p/E^* have the same cardinality. So it suffices to prove that for every $H \lhd_c G$, the orbit O of some $\overline{a}/E^* \in p/E^*$ under H is finite or equal to p/E^*.

Suppose for a contradiction that O is infinite and $O \neq p/E^*$. We identify topologically p/E^* with S^1. So O is a closed homogeneous subset of S^1, hence it is an uncountable perfect set. As $O \neq S_1$, there is an open arc I on S_1 disjoint from O, with endpoints in O. H acts transitively on O, so every point $z \in O$ is an endpoint of an open arc I_z disjoint from O, but with both endpoints in O. By construction, for distinct $z, z' \in O$ the arcs $I_z, I_{z'}$ are equal or disjoint, hence there are countably many of them. However, each such arc has only two endpoints, while O is uncountable, a contradiction. □

We end this paper with some examples of unbalanced equivalence relations.

Example 4.8 (finite unbalanced equivalence relation finer than $\equiv$) Let E be an equivalence relation on a countable set V with n infinite classes, where $n \geq 3$. Let $V/E = \{a_0, \dots, a_{n-1}\}$. For $\sigma \in \mathrm{Sym}(n)$ let $\overline{a}_\sigma = a_{\sigma(0)}, a_{\sigma(1)}, \dots, a_{\sigma(n-1)}$. We choose $U(\overline{a}_\sigma), \sigma \in \mathrm{Sym}(n)$, a family of infinite disjoint subsets of V such that for every $i < n,\ a_i = \bigcup\{U(\overline{a}_\sigma) : \sigma(0) = i\}$.

We define an $(n+1)$-ary relation R on V by

$$R(x_0, \dots, x_n) \Leftrightarrow x_0, \dots, x_{n-1} \text{ are pairwise not } E\text{-equivalent and } x_n \in U(\overline{a}_\sigma),$$

where $\overline{a}_\sigma = \langle x_0/E, \dots, x_{n-1}/E\rangle$.

Finally, let $M = (V; E, R)$. We will show that E is unbalanced in M. First notice that in M^{eq} we can define the sets $U(\overline{a}_\sigma), \sigma \in \mathrm{Sym}(n)$, over $\{a_0, \dots, a_{n-1}\}$. Indeed, $U(\overline{a}_\sigma) = R(b_0, \dots, b_{n-1}, M)$, where $b_0, \dots, b_{n-1}$ are arbitrary elements of M satisfying $b_i/E = a_{\sigma(i)}$.

Secondly we show that $\mathrm{Aut}(M)$ acts transitively on M, hence E is finer than $\equiv$. To see this, for any $\tau \in \mathrm{Sym}(n)$ let $f_\tau : M \to M$ be a bijection mapping each set $U(\overline{a}_\sigma)$ onto $U(\overline{a}_{\tau\sigma})$. Clearly, $f_\tau \in \mathrm{Aut}(M)$, so the orbit of each point in M meets each set $U(\overline{a}_\sigma)$. Also, for a fixed σ all elements of $U(\overline{a}_\sigma)$ are in the same orbit, so we are done.

Finally E is not balanced, since for $\sigma \neq \tau \in \mathrm{Sym}(n)$ with $\sigma(0) = \tau(0)$, elements of $U(\overline{a}_\sigma)$ and $U(\overline{a}_\tau)$ lie in the same E-class but in different orbits over $\{a_0, \dots, a_{n-1}\}$. In fact, here E is also unbalanced on $p/\overset{s}{\equiv}$, where p is the complete 1-type in $\mathrm{Th}(M)$.

In this example we must have assumed that $n > 2$, since each equivalence relation finer than $\equiv$, which has 2 classes, is balanced.

Example 4.9 (unbalanced relation with infinitely many classes, finer than $\equiv$ but coarser than $\overset{s}{\equiv}$) Let $n > 2$ and, for $k < \omega$, let E_k, R_k be defined on a countable infinite set V as in Example 4.8 and additionally so, that the corresponding partitions $\{U_k(\overline{a}_\sigma) : \sigma \in \mathrm{Sym}(n)\}, k < \omega$, are independent. Let $M = (V, \{E_k, R_k : k < \omega\})$ and let $E = \bigwedge_k E_k$. As in Example 4.8, in $\mathrm{Th}(M)$ there is just one complete 1-type p and E is not balanced. Also, E is not balanced on $p/\overset{s}{\equiv}$.

Example 4.10 (unbalanced relation finer than $\overset{s}{\equiv}$ but coarser than $\overset{bd}{\equiv}$) Consider the group of rotations $SO(3, \mathbb{R})$ acting on S^2, the unit sphere in $\mathbb{R}^3$. Let d be the usual metric on $\mathbb{R}^3$. Let $M = (S^2, \{U_q(x, y)\}_{q \in \mathbb{Q}^+})$, where $U_q(x, y) \Leftrightarrow d(x, y) < q$.

As in the example in Section 3, in $\mathrm{Th}(M)$ there is just one complete 1-type p over $\varnothing$. Also, on M there is no finite 0-definable equivalence relation, hence p is a strong type. This is because $SO(3, \mathbb{R})$, being a connected Lie group, has no proper subgroup of finite index.

On M we have a type-definable equivalence relation E_0 given by

$$E_0(x, y) \Leftrightarrow \bigwedge_q U_q(x, y).$$

We have $p/E_0 \approx S^2$. In fact, E_0 equals $\overset{bd}{\equiv}$ here, hence it is balanced. However, we will define an unbalanced relation on some strong 2-type.

Fix $a \in S^2$. For $\rho \in (0, 2)$ let $S^1_\rho(a) = \{b \in S^2 : d(a, b) = \rho\}$. The circle $S^1_{1.2}(a)$ is a-definable in M by the formula $\neg U_{1.2}(a, x) \wedge \neg U_{1.6}(a^+, x)$, where a^+ is the antipode of a defined by the formula $\neg U_2(a, x)$. Let $b \in S^1_{1.2}(a)$ and $q = tp(ab)$. We see that q is isolated. p is a strong type and the group of rotations of S^2 around the axis going through a acts transitively on $S^1_{1.2}(a)$, hence q is a strong type. On q we define two equivalence relations:

$$E(x, y; x', y') \Leftrightarrow E_0(x, x') \text{ and } E^*(x, y; x'y') \Leftrightarrow E_0(x, x') \wedge E_0(y, y').$$

We see that E and E^* are bounded equivalence relations on q.

$$\begin{aligned} q/E^* &\approx \{(a/E_0, b/E_0) : a \in S^2 \wedge b \in S^1_{1.2}(a)\} \\ &\approx \{(a, b) : a \in S^2 \wedge b \in S^1_{1.2}(a)\} \subseteq S^2 \times S^2. \end{aligned}$$

Notice that q/E^* is not homeomorphic to $S^2 \times S^1$. In the monster model $\mathfrak{C}$ we have

$$ab/E = \{(a', b') : a'E_0a \wedge b' \in S^1_{1.2}(a')\} \subseteq a/E_0 \times \bigcup\{b'/E_0 : b' \in S^1_{1.2}(a)\},$$

hence any $f \in \mathrm{Aut}(\mathfrak{C})$ fixing ab/E setwise fixes a/E_0 setwise.

Now E is not balanced. Indeed, any $f \in \mathrm{Aut}(\mathfrak{C})$ fixing $\mathfrak{C}/E$ fixes also $\mathfrak{C}/E_0$. So if $b' \in S^1_{1.2}(a)$ and $b' \neq b$, then $f(ab) \neq ab'$. Likewise, E is not balanced on q/E^*.

References

[1] Hrushovski, E., "Simplicity and the Lascar group," preprint, 1997. 231, 232, 235, 237

[2] Kim, B., "A note on Lascar strong types in simple theories," *The Journal of Symbolic Logic*, vol. 63 (1998), pp. 926–36. Zbl 0961.03030. MR 2000a:03053. 232, 237

[3] Lascar, D., "The group of automorphisms of a relational saturated structure," pp. 225–36 in *Finite and Infinite Combinatorics in Sets and Logic*, vol. 411 of *NATO Science Series: Mathematical and Physical Sciences*, Kluwer Academic Publishers, Dordrecht, 1993. Zbl 0845.03015. MR 95d:03060. 239

[4] Lascar, D., and A. Pillay, "Hyperimaginaries and automorphism groups," *The Journal of Symbolic Logic*, vol. 66 (2001), pp. 127–43. Zbl 1002.03027. MR 2002f:03067. 231, 232, 237

[5] Pillay, A., "Compact groups and automorphism groups," lecture notes, Barcelona, 1997. 234

[6] Pillay, A., and B. Poizat, "Pas d'imaginaires dans l'infini!" *The Journal of Symbolic Logic*, vol. 52 (1987), pp. 400–403. Zbl 0631.03014. MR 88j:03019. 233

[7] Weil, A., *L'intègration dans les Groupes Topologiques et ses Applications*, Actualités Scientifiques et Industrielles, Hermann, Paris, 1940. Zbl 0063.08195. MR 3,198b. 232, 237

Acknowledgments

We would like to thank Piotr Kowalski for some helpful comments. Research supported by KBN Grant 2 P03A 002 16.

Mathematical Institute
University of Wroclaw
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND
kkrup@math.uni.wroc.pl
newelski@math.uni.wroc.pl