A Closer Look at Some Subintuitionistic Logics

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Abstract   In the present paper we study systematically several consequence relations on the usual language of propositional intuitionistic logic that can be defined semantically by using Kripke frames and the same defining truth conditions for the connectives as in intuitionistic logic but without imposing some of the conditions on the Kripke frames that are required in the intuitionistic case. The logics so obtained are called subintuitionistic logics in the literature. We depart from the perspective of considering a logic just as a set of theorems and also depart from the perspective taken by Restall in that we consider standard Kripke models instead of models with a base point. We study the relations between subintuitionistic logics and modal logics given by the translation considered by Došen. Moreover, we classify the logics obtained according to the hierarchy considered in Abstract Algebraic Logic.

1   Introduction

The expression "subintuitionistic logic" is used in the literature (Restall [10], Wansing [18]) to refer to propositional logics in the propositional language with connectives $\wedge, \vee, \rightarrow$ and propositional constant $\perp$ that are defined semantically by using Kripke frames and the same defining truth conditions for the connectives as in intuitionistic logic but without imposing some of the conditions on the Kripke frames that are required in the intuitionistic case. For instance, one can dispense with the reflexivity of the relation R of the Kripke frames or with the transitivity or with the persistence (heredity) condition that requires that the valuations assign to the variables sets of points which are closed under R.

In the first section of [18], Wansing argues very convincingly in favor of taking seriously these weakenings of the usual conditions imposed on Kripke frames for intuitionistic logic. His proposal is to consider the relation of a frame as describing

a possible development of information states instead of an expansion of information states. This last interpretation provides the usual Kripke semantics for intuitionistic logic. We address the reader to this paper.

To our knowledge, the first logic studied in the literature that is a subintuitionistic logic in the sense described above is the consequence relation Basic Propositional Logic introduced in 1981 in Visser [17] and Visser [16]. It is the local consequence relation of the Kripke models that are transitive and whose valuations are persistent. This logic was rediscovered in 1991 by Ruitenburg [11]. Recently it has aroused new interest. In [15], Suzuki, Wolter, and Zakharyaschev prove, among several other facts, that the logic is not protoalgebraic, and in Ardeshir and Ruitenburg [1] and [2], Ruitenburg [12], Suzuki [14], and Sazaki [13], several results for it of proof-theoretic and of model-theoretic nature are obtained. Earlier, in 1976, using only algebraic means, Epstein and Horn [7] studied the $(\wedge, \vee, \top, \bot, \supset)$-fragment of Lewis's systems **S4** and **S5** and several related systems, where $\supset$ is interpreted as strict implication. All can be seen as subintuitionistic logics. But that paper did not consider Kripke semantics. Hacking [8] already studied the mentioned fragments, axiomatizing them using Gentzen systems. A systematic study of several possible subintuitionistic logics is given in Corsi [4] where a logic is defined to be a set of theorems. The logics she deals with are defined by means of the notion of validity in a given class of frames. Moreover, Restall also studies subintuitionistic logics in [10] but now from the perspective of consequence relations defined by means of Kripke frames with a base point related to every point. Given a class of these frames the associated consequence relation is defined by saying that for every model on a frame of the class, if the premises are forced by its base point, then the conclusion is forced too. Several of Restall's logics coincide with several of Corsi's, as far as their theorems are concerned.

In [6], Došen also studies some subintuitionistic logics treated as sets of theorems, mainly the system $K(\sigma)$ whose elements are the formulas valid in every Kripke model. This logic has the same relation to the modal system **K** as intuitionistic logic has to **S4**, namely, there is a translation σ of formulas from the intuitionistic language into the modal language such that an intuitionistic formula is a theorem of $K(\sigma)$ if and only if its modal translation is a theorem of **K**. One of Došen's main aims in [6] was to find a logic with this property. If the Hilbert-style calculus introduced by Došen to axiomatize $K(\sigma)$ is used to define a consequence relation in the standard way, the consequence relation obtained, which we will also denote by $K(\sigma)$, has some shortcomings. Intuitionistic logic has, via Gödel's translation, a stronger relation with **S4** than $K(\sigma)$ has with **K**. If τ denotes Gödel's translation, the relation between intuitionistic logic and **S4** is as follows. For any set of intuitionistic formulas $\Gamma \cup \{\varphi\}$, φ follows from Γ in intuitionistic logic if and only if $\tau(\varphi)$ follows from $\tau[\Gamma]$ in the local consequence relation associated with **S4** if and only if $\tau(\varphi)$ follows from $\tau[\Gamma]$ in the global consequence relation associated with **S4**. The analogous results do not hold for the translation σ considered by Došen and the systems $K(\sigma)$ and **K**.

In the present paper we study systematically several subintuitionistic consequence relations that can be defined considering classes of Kripke models, and some of their extensions. Therefore we depart from the standard perspective of considering a logic just as a set of theorems and also depart from the perspective taken by Restall in that we consider standard Kripke models instead of his models with a base point and his

consequence relation. One of the goals of the paper is to introduce two subintuition-istic consequence relations wK_σ and sK_σ that have the following relations with **K** via the translation σ considered by Došen: a formula φ follows from a set of formulas Γ in wK_σ if and only if $\sigma(\varphi)$ follows from $\sigma[\Gamma]$ in the local consequence relation associated with **K**, and φ follows from Γ in sK_σ if and only if $\sigma(\varphi)$ follows from $\sigma[\Gamma]$ in the global consequence relation associated with **K**. It must be emphasized that for arbitrary Γ and φ it does not hold that $\sigma(\varphi)$ follows from $\sigma[\Gamma]$ in the local consequence associated with **K** if and only if $\sigma(\varphi)$ follows from $\sigma[\Gamma]$ in the global consequence associated with **K**. The logic wK_σ will be the local consequence relation defined by the class of all Kripke frames and sK_σ will be the global consequence relation defined by this class.

Besides these two subintuitionistic logics we will concentrate mainly on the local and the global consequence relations defined by the class of reflexive Kripke models and the class of transitive Kripke models. Moreover, we will consider the Basic Propositional Logic of Visser [17] which following [13] we call Visser's Propositional Logic; it turns out also to be the local consequence relation defined by the class of models with an *R*-persistent valuation. For all these logics we consider some weaker versions that do not seem to be characterizable by classes of frames.

We will also classify the logics we obtain according to the hierarchy considered in Abstract Algebraic Logic, which is becoming increasingly popular today, namely, as non-protoalgebraic, protoalgebraic, equivalential, and algebraizable. For information on the hierarchy we address the reader to Czelakowski [5].

2 Preliminaries

The language of subintuitionistic logics, the *si-language* for short, is the same language as that of intuitionistic logic. It contains the connectives $\wedge, \vee, \rightarrow$ and the propositional constant $\bot$. Moreover, it contains a denumerable set of propositional variables. The formulas are defined as usual, that is, the set of subintuitionistic formulas is the smallest set X that contains $\bot$, all the propositional variables, and if φ, ψ belong to X then $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi) \in X$. We abbreviate the formula $\bot \rightarrow \bot$ by $\top$. Let us denote the set of all formulas by *Fm*.

A *logic*, or deductive system, in the si-language is a pair $\mathcal{S} = \langle Fm, \vdash_{\mathcal{S}} \rangle$ where $\vdash_{\mathcal{S}}$ is a relation, called the *entailment relation* or the *consequence relation* of $\mathcal{S}$, between sets of formulas and formulas such that

1. if $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathcal{S}} \varphi$;
2. if $\Gamma \vdash_{\mathcal{S}} \varphi$ and for every $\psi \in \Gamma$, $\Delta \vdash_{\mathcal{S}} \psi$, then $\Delta \vdash_{\mathcal{S}} \varphi$;
3. if $\Gamma \vdash_{\mathcal{S}} \varphi$, then for any substitution e, $e[\Gamma] \vdash_{\mathcal{S}} e(\varphi)$, where a *substitution* is a homomorphism from the formula algebra *Fm* into itself—this property is called *substitution invariance*.

From (1) and (2) it follows that

4. if $\Gamma \vdash_{\mathcal{S}} \varphi$, then for any formula ψ, $\Gamma \cup \{\psi\} \vdash_{\mathcal{S}} \varphi$.

Logics can be defined in many ways using either syntactic or semantic means. A logic $\mathcal{S}$ is said to be *finitary* if for every set of formulas $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{\mathcal{S}} \varphi$ implies that $\Gamma' \vdash_{\mathcal{S}} \varphi$ for some finite $\Gamma' \subseteq \Gamma$. All the logics we deal with in the paper are finitary. We will usually identify a logic with its consequence relation. Given a logic $\mathcal{S}$, an *extension* of $\mathcal{S}$ is a logic $\mathcal{S}'$ in the same set of formulas such that $\vdash_{\mathcal{S}} \subseteq \vdash_{S'}$.

A *Hilbert-style rule* will be a pair $\langle \Gamma, \varphi \rangle$ where Γ is a finite set of formulas and φ is a formula. Given a Hilbert-style rule $\langle \Gamma, \varphi \rangle$ and a substitution e, we say that the pair $\langle e[\Gamma], e(\varphi) \rangle$ is a substitution instance of the rule $\langle \Gamma, \varphi \rangle$. A *Hilbert-style calculus* consists of a set of rules closed under substitution instances. The rules of the form $\langle \varnothing, \varphi \rangle$ are called *axioms*; thus axioms can be identified with formulas. When we say that we add a rule $\langle \Gamma, \varphi \rangle$ to a Hilbert-style calculus we are implicitly saying that we add all its substitution instances, that is, we treat the rules as schemata. A Hilbert-style calculus $\mathcal{H}$ defines in the standard way a logic $\mathcal{S}_{\mathcal{H}}$, namely, $\Gamma \vdash_{\mathcal{S}_{\mathcal{H}}} \varphi$ if and only if there is a proof of φ in $\mathcal{H}$ using as premises formulas in Γ. A Hilbert-style calculus $\mathcal{H}$ is said to be a Hilbert calculus for a logic $\mathcal{S}$ if $\vdash_{\mathcal{S}} = \vdash_{\mathcal{S}_{\mathcal{H}}}$.

In this paper a *sequent* is any pair $\langle \Gamma, \varphi \rangle$ where Γ is a (possibly empty) finite set of formulas and φ is a formula; we will use the standard notation $\Gamma \vdash \varphi$ for sequents. The Gentzen calculi we will consider are calculi for sequents of that form. When we add a sequent as an axiom to a given Gentzen-style calculus we will also treat it schematically and assume that we add all its substitution instances as well.

When we say that a logic $\mathcal{S}$ has $\langle \Gamma, \varphi \rangle$ as a rule, or that $\langle \Gamma, \varphi \rangle$ is a rule of $\mathcal{S}$, we mean that $\Gamma \vdash_{\mathcal{S}} \varphi$. Given a logic S and several (Hilbert) rules $R_1, \ldots, R_n$, the least extension of $\mathcal{S}$ that also has $R_1, \ldots, R_n$ as rules is denoted by

$$\mathcal{S} + R_1 + \cdots + R_n.$$

Given a Hilbert-style calculus $\mathcal{H}$ for $\mathcal{S}$ the extension $\mathcal{S} + R_1 + \cdots + R_n$ can be axiomatized by the Hilbert calculus whose rules are the rules of $\mathcal{H}$ plus the rules $R_1, \ldots, R_n$.

We will now introduce the basic notions considered in Abstract Algebraic Logic for the classification of logics that will be needed in the paper. For each one we take as definition the one that best suits the purposes of the paper but specialized to the si-language. For an extensive and detailed exposition of the classification, we address the reader to [5].

A *matrix* for a logic $\mathcal{S}$ is a pair $\langle A, F \rangle$ where A is an algebra of the similarity type of the si-language and F is a subset of its universe A. A matrix $\langle A, F \rangle$ for $\mathcal{S}$ is said to be a *model* of $\mathcal{S}$, and F is said to be an $\mathcal{S}$-filter of A, if for every set of formulas Γ and every formula φ, if $\Gamma \vdash_{\mathcal{S}} \varphi$, then for every valuation v from the set of propositional variables into A such that $v[\Gamma] \subseteq F$, it holds that $v(\varphi) \in F$; a valuation is simply a homomorphism from the formula algebra into the algebra A.

The Leibniz congruence of a matrix $\langle A, F \rangle$ is the greatest congruence of A which is compatible with F, that is, one that does not relate elements of F with elements not in F. As customary we denote the Leibniz congruence of $\langle A, F \rangle$ by $\mathbf{\Omega}_A(F)$.

A logic $\mathcal{S}$ is *protoalgebraic* when for every algebra A the operator $\mathbf{\Omega}_A$ is monotonic on the set of all $\mathcal{S}$-filters of A, in the sense that if F and G are $\mathcal{S}$-filters and $F \subseteq G$ then $\mathbf{\Omega}_A(F) \subseteq \mathbf{\Omega}_A(G)$. It can be proved that a logic is protoalgebraic if and only if there is a set of formulas $\Delta(p, q)$ in at most two variables such that

1. for every formula $\delta(p, q) \in \Delta, \vdash_{\mathcal{S}} \delta(p, p)$,
2. $p, \Delta(p, q) \vdash_{\mathcal{S}} q$.

A logic is *equivalential* if there is a set of formulas $\Delta(p, q)$ in at most two variables such that

1. for every formula $\delta(p, q) \in \Delta, \vdash_{\mathcal{S}} \delta(p, p)$,
2. for every binary connective $\star$, $\Delta(p, p') \cup \Delta(q, q') \vdash_{\mathcal{S}} \delta(p \star q, p' \star q')$, for every $\delta(p, q) \in \Delta$;

3. $p, \Delta(p, q) \vdash_\delta q$.

A set Δ with the above properties is called a *set of equivalence formulas* for δ.

A logic is *algebraizable* if it is equivalential, say with $\Delta(p, q)$ as a set of equivalence formulas, and there is a set of equations in at most one variable $E(p)$ such that for every equation $\varphi \approx \psi \in E(p)$ and every $\delta(p, q) \in \Delta$, $p \vdash_\delta \delta(\varphi, \psi)$ and $\{\delta(\varphi, \psi) : \delta(p, q) \in \Delta, \varphi \approx \psi \in E(p)\} \vdash_\delta p$. This set of equations is called a *set of defining equations*.

2.1 Kripke semantics   A *Kripke frame* is a pair $\langle W, R \rangle$ where W is a nonempty set and R is a binary relation on W. A *Kripke model* is a triple $\langle W, R, V \rangle$ where $\langle W, R \rangle$ is a Kripke frame and V is a function that assigns to each propositional variable a subset of W. This function can be extended to every formula by means of the following clauses:

1. $V(\bot) = \varnothing$,
2. $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$,
3. $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$,
4. $V(\varphi \to \psi) = \{x \in W : \forall y(xRy \ \& \ y \in V(\varphi) \Rightarrow y \in V(\psi))\}$.

Notice that $V(\top) = W$. Given a model $\langle W, R, V \rangle$ and a set of formulas Γ we define $V(\Gamma)$ by

$$V(\Gamma) = \bigcap_{\varphi \in \Gamma} V(\varphi).$$

Given a frame $\langle W, R \rangle$ and $x \in W$, we denote the set $\{y \in W : xRy\}$ of R-successors of x by $R(x)$. Then condition 4 can be written as follows:

$$V(\varphi \to \psi) = \{x \in W : R(x) \cap V(\varphi) \subseteq V(\psi)\}.$$

Of a model $\langle W, R, V \rangle$ we say that it is a *model based on the frame* $\langle W, R \rangle$. A formula φ is *valid*, or *holds*, in a Kripke model $\langle W, R, V \rangle$, in symbols $\langle W, R, V \rangle \models \varphi$, if $V(\varphi) = W$. If Γ is a set of formulas, $\langle W, R, V \rangle \models \Gamma$ means that for every $\varphi \in \Gamma$, $\langle W, R, V \rangle \models \varphi$. A sequent $\Gamma \vdash \varphi$ is *true at a point x of a model* $\mathcal{M} = \langle W, R, V \rangle$, in symbols $\mathcal{M} \models_x \Gamma \vdash \varphi$, if $x \notin V(\Gamma)$ or $x \in V(\varphi)$. It is *valid in* $\mathcal{M}$, in symbols $\mathcal{M} \models \Gamma \vdash \varphi$, if $V(\Gamma) \subseteq V(\varphi)$. The rule associated with a sequent is valid in a model $\mathcal{M}$ if every substitution instance of the sequent is valid in $\mathcal{M}$. A formula is *valid in a frame* if it is valid in every model on the frame; analogously we speak of a sequent being valid in a frame.

To any class F of frames we can associate two consequence relations in the si-language (and therefore two logics), the local one and the global one. They are defined as follows. Let Γ be a set of formulas and let φ be a formula. We say that φ is a *local consequence* from Γ relative to F, in symbols $\Gamma \models_{l\mathsf{F}} \varphi$, if for every model $\langle W, R, V \rangle$ based on a frame in F, $V(\Gamma) \subseteq V(\varphi)$. We say that φ is a *global consequence* form Γ relative to F, in symbols $\Gamma \models_{g\mathsf{F}} \varphi$, if every model $\langle W, R, V \rangle$ based on a frame in F such that $V(\Gamma) = W$ also verifies that $V(\varphi) = W$. It is not difficult to show that the relations defined are in fact consequence relations. The most difficult point is to show that they are substitution invariant. This is achived by the following lemma.

Lemma 2.1   *Let $\langle W, R \rangle$ be a frame and e a substitution. Then for every valuation V on $\langle W, R \rangle$ there is a valuation V' such that for every formula φ,*

$$V(e(\varphi)) = V'(\varphi).$$

It is not necessary that the consequence relations obtained are finitary; they are if the class of frames is closed under ultraproducts. We will denote by $l\mathsf{F}$ and $g\mathsf{F}$ the local and the global consequence relations defined by the class of frames F.

We recall the notions of generated subframe and generated submodel. Given a frame $\langle W, R \rangle$, a frame $\langle W', R' \rangle$ is said to be a *generated subframe* of the frame $\langle W, R \rangle$ if W' is an R-closed subset of W (i.e., $W' \subseteq W$ and if $x \in W'$ and xRy, then $y \in W'$) and R' is the restriction of R to W' (i.e., $R' = R \cap (W' \times W')$). A model $\langle W', R', V' \rangle$ is said to be a *generated submodel* of a model $\langle W, R, V \rangle$ if $\langle W', R' \rangle$ is a generated subframe of $\langle W, R \rangle$ and the valuation V' is such that for every propositional variable p, $V'(p) = V(p) \cap W'$. Given a model $\langle W, R, V \rangle$ and one of its points x, the *submodel generated* by x is the model $\langle W_x, R_x, V_x \rangle$ where W_x is the least subset of W that contains x and is closed under R and $\langle W_x, R_x, V_x \rangle$ is the corresponding generated submodel of $\langle W, R \rangle$. We say that $\langle W_x, R_x \rangle$ is *the subframe generated by x* of $\langle W, R \rangle$. The following lemma is well known.

Lemma 2.2   *If $\langle W', R', V' \rangle$ is a generated submodel of a model $\langle W, R, V \rangle$, then for every formula φ,*

$$V'(\varphi) = V(\varphi) \cap W'.$$

The relations between the local and the global consequence relations associated with a given class of frames are stated (under certain conditions) in the next proposition. Let us define for every formula φ and every natural number n, $\top^n \to \varphi$ as follows:

$$\top^0 \to \varphi = \varphi, \ \top^{n+1} \to \varphi = \top \to (\top^n \to \varphi).$$

Proposition 2.3   *For every class of frames F closed under generated subframes, every set of formulas Γ, and every formula φ,*

1. *if $\Gamma \models_{l\mathsf{F}} \varphi$, then $\Gamma \models_{g\mathsf{F}} \varphi$;*
2. *$\{\top^n \to \psi : n \in \omega, \psi \in \Gamma\} \models_{l\mathsf{F}} \varphi$ iff $\Gamma \models_{g\mathsf{F}} \varphi$;*
3. *if $\models_{l\mathsf{F}}$ is finitary, $\models_{g\mathsf{F}}$ is finitary as well.*

Proof   (1) follows immediately from the definitions. To prove (2), assume that $\{\top^n \to \psi : n \in \omega, \psi \in \Gamma\} \models_{l\mathsf{F}} \varphi$. Let $\langle W, R, V \rangle$ be a model based on a frame in F and suppose that $V(\Gamma) = W$. Then it is easy to see by induction on n that for every formula $\psi \in \Gamma$ and every n, $V(\top^n \to \psi) = W$. Thus, $V(\varphi) = W$. This proves the implication from left to right. To prove the other implication assume that $\Gamma \models_{g\mathsf{F}} \varphi$. Let $\langle W, R, V \rangle$ be a model based on a frame in F and let $x \in W$ be such that for every $\psi \in \Gamma$ and every n, $x \in V(\top^n \to \psi)$. It is easy to show by induction on n that for every formula δ,

$$\forall n \forall z, y \in W(zR^n y \ \& \ z \in V(\top^n \to \delta) \implies y \in V(\delta)).$$

Let us consider the submodel generated by x, $\langle W_x, R_x, V_x \rangle$, of $\langle W, R, V \rangle$. By assumption its frame belongs to F. Then for all $\psi \in \Gamma$, $V_x(\psi) = W_x$. Thus, by assumption, $V_x(\varphi) = W_x$. Therefore $x \in V(\varphi)$. Finally, (3) follows immediately from (2). $\square$

Given a class of frames F we have its local consequence relation $\models_{l\mathsf{F}}$. We can extend it by adding the (Hilbert) rule,

(N) $\varphi \vdash \top \to \varphi,$

that is, we consider the least consequence relation including $\models_{l\mathsf{F}}$ and the sequents of the form $\varphi \vdash \top \to \varphi$. We denote the logic so obtained by $s\mathsf{F}$. Notice that if $l\mathsf{F}$ is finitary, $s\mathsf{F}$ is finitary too.

Lemma 2.4   *For any class of frames F closed under subframes and such that its local consequence relation is finitary and any set of formulas $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vdash_{s\mathsf{F}} \varphi \ \textit{iff} \ \{\top^n \to \psi : n \in \omega, \psi \in \Gamma\} \models_{l\mathsf{F}} \varphi.$$

Proof   If $\{\top^n \to \psi : n \in \omega, \psi \in \Gamma\} \models_{l\mathsf{F}} \varphi$, let Δ be a finite subset of Γ and let m be such that $\{\top^n \to \psi : n \leq m, \psi \in \Delta\} \models_{l\mathsf{F}} \varphi$; they exist because $l\mathsf{F}$ is finitary. Then, using the rule (N) and the rule of $l\mathsf{F}$ just mentioned, we obtain that $\Delta \vdash_{s\mathsf{F}} \varphi$ and thus the desired result.

To prove the other implication assume that $\Gamma \vdash_{s\mathsf{F}} \varphi$. Since the rule (N) is clearly a rule of $\models_{g\mathsf{F}}$, the logic $s\mathsf{F}$ is a sublogic of the logic $g\mathsf{F}$. Thus, $\Gamma \models_{g\mathsf{F}} \varphi$. From Proposition 2.3 it follows that $\{\top^n \to \psi : n \in \omega, \psi \in \Gamma\} \models_{l\mathsf{F}} \varphi$.   □

The next corollary follows immediately from Lemma 2.4 and Proposition 2.3.

Corollary 2.5   *For any class of frames F closed under subframes and such that its local consequence relation is finitary, the logic $s\mathsf{F}$ is precisely the global consequence relation determined by F.*

3 Došen's Logic

In this section we review the system $K(\sigma)$ of [6] and prove that it is algebraizable. It was introduced by Došen to define a logic (as a set of theorems) that has the same relation to the normal modal logic **K** as intuitionistic logic has to **S4**, that is, there is a translation of the formulas of the intuitionistic language into formulas of the modal language such that a formula is a theorem $K(\sigma)$ if and only if its translation is a theorem of **K**. The translation σ considered by Došen is defined by

$$\begin{aligned}
\sigma(p) &= p \\
\sigma(\bot) &= \bot \\
\sigma(\varphi \wedge \psi) &= \sigma(\varphi) \wedge \sigma(\psi) \\
\sigma(\varphi \vee \psi) &= \sigma(\varphi) \vee \sigma(\psi) \\
\sigma(\varphi \to \psi) &= \Box(\sigma(\varphi) \to \sigma(\psi)).
\end{aligned}$$

The set $K(\sigma)$ is defined as the set of formulas φ such that $\sigma(\varphi)$ is a theorem of **K**. An axiomatization is given by the Hilbert-style calculus K_σ which we display below. A similar axiomatization is given in [4]. As usual we can associate a consequence relation with this calculus using the notion of proof with premises, that is, by declaring that a formula φ follows from a set of formulas Γ, in symbols $\Gamma \vdash_{K_\sigma} \varphi$, if there is a proof of φ in the given calculus K_σ that uses premises in Γ. This logic can be called *Došen's logic* and we will also denote it by K_σ. Its consequence relation will be denoted by $\vdash_{K_\sigma}$.

3.1 Hilbert-style calculus

3.1.1 Axioms

1. $\varphi \to \varphi$
2. $((\varphi \to \psi) \wedge (\psi \to \delta)) \to (\varphi \to \delta)$
3. $((\delta \to \varphi) \wedge (\delta \to \psi)) \to (\delta \to (\varphi \wedge \psi))$
4. $(\varphi \wedge \psi) \to \varphi$

5. $(\varphi \wedge \psi) \rightarrow \psi$
6. $\varphi \rightarrow (\varphi \vee \psi)$
7. $\psi \rightarrow (\varphi \vee \psi)$
8. $((\varphi \rightarrow \delta) \wedge (\psi \rightarrow \delta)) \rightarrow ((\varphi \vee \psi) \rightarrow \delta)$
9. $(\varphi \wedge (\psi \vee \delta)) \rightarrow ((\varphi \wedge \psi) \vee (\varphi \wedge \delta))$
10. $\bot \rightarrow \varphi$

3.1.2 Rules of inference

(MP) $\varphi, \varphi \rightarrow \psi \vdash \psi$
(W) $\varphi \vdash \psi \rightarrow \varphi$
(Ad) $\varphi, \psi \vdash \varphi \wedge \psi$

An alternative set of rules consists of (MP) and the rule

$$\varphi_2, (\varphi_1 \wedge \varphi_2) \rightarrow \psi \vdash \varphi_1 \rightarrow \psi.$$

The following rules are derived rules:

(R$_n$) $\varphi_n, \varphi_1 \wedge \cdots \wedge \varphi_n \rightarrow \psi \vdash \varphi_1 \wedge \cdots \wedge \varphi_{n-1} \rightarrow \psi$
(Pr) $\varphi \rightarrow \psi \vdash (\delta \rightarrow \varphi) \rightarrow (\delta \rightarrow \psi)$
(Sf) $\varphi \rightarrow \psi \vdash (\psi \rightarrow \delta) \rightarrow (\varphi \rightarrow \delta)$

Using the rules (Pr) and (Sf) it is easy to see that Došen's logic is an implicative logic of Rasiowa [9] and is therefore algebraizable (see Blok and Pigozzi [3]). We highlight this fact.

Theorem 3.1   *Došen's logic is algebraizable.*

We have the following completeness theorem for Došen's logic, proved in [6].

Theorem 3.2 (Došen) $\vdash_{K_\sigma} \varphi$ *if and only if φ holds in every Kripke model.*

The relation between Intuitionistic logic (Int) and the modal logic **S4** given by Gödel's translation τ is such that for every set Γ of intuitionistic formulas and every intuitionistic formula φ,

$$\Gamma \vdash_{\text{Int}} \varphi \ \ \text{iff} \ \ \tau[\Gamma] \vdash_{\textbf{S4}} \tau(\varphi),$$

where $\vdash_{\textbf{S4}}$ is the local consequence of **S4**; it can be defined syntactically by $\Gamma \vdash_{\textbf{S4}} \varphi$ if and only if φ is a theorem of **S4**, or there are $\varphi_0, \ldots, \varphi_n \in \Gamma$ such that $\varphi_0 \wedge \cdots \wedge \varphi_n \rightarrow \varphi$ is a theorem of **S4**. Došen's logic and the translation σ do not have this property with respect to the logic **K**. For instance, $p, p \rightarrow q \vdash_{K_\sigma} q$ but $p, \Box(p \rightarrow q) \not\vdash_K q$, and also $p \vdash_{K_\sigma} q \rightarrow p$ but $p \not\vdash_K \Box(q \rightarrow p)$. Here $\vdash_K$ refers to the local consequence of **K**, which can be defined syntactically as we did for the **S4** case.

In the next section we define a logic wK_σ with the property that

$$\Gamma \vdash_{wK_\sigma} \varphi \ \ \text{iff} \ \ \sigma[\Gamma] \vdash_{\textbf{K}} \sigma(\varphi),$$

and thus with the same set of theorems as K_σ.

4 Weak Došen's Logic

Although Došen's logic has as its set of theorems the formulas valid in every Kripke frame, its entailment relation does not coincide with the local consequence relation defined by the class of all Kripke frames which we simply denote by $\models_l$. For example, modus ponens is not valid for this consequence relation. The model $\langle\{a\}, \varnothing, V\rangle$, where $V(p) = \{a\}$ and $V(q) = \varnothing$, witnesses this fact. For instance,

$V(p) \cap V(p \to q) = \{a\} \nsubseteq V(q)$. Moreover, Došen's logic does not coincide with the global consequence relation defined by the class of all Kripke frames as we will show later.

4.1 Local consequence   The local consequence relation of all Kripke models can be axiomatized by the following Gentzen-style calculus; this is the content of the soundness and completeness theorem we will prove below. The calculus operates on sequents of the form $\Gamma \vdash \varphi$ where Γ is a (possibly empty) finite set of formulas and φ is a formula.

4.1.1 Gentzen rules

$$\frac{\Gamma \vdash \varphi}{\Gamma, \psi \vdash \varphi} \qquad\qquad \frac{\Gamma \vdash \varphi \quad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi} \qquad \frac{\Gamma, \varphi, \psi \vdash \alpha}{\Gamma, \varphi \wedge \psi \vdash \alpha} \qquad \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}$$

$$\frac{\Gamma, \varphi \vdash \alpha \quad \Gamma, \psi \vdash \alpha}{\Gamma, \varphi \vee \psi \vdash \alpha} \qquad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}$$

$$\frac{\varphi \vdash \psi}{\varnothing \vdash \varphi \to \psi} \quad (\text{DT}_0)$$

4.1.2 Gentzen Axioms

1. $(\varphi \to \psi), (\varphi \to \delta) \vdash \varphi \to (\psi \wedge \delta)$
2. $(\varphi \to \delta), (\psi \to \delta) \vdash (\varphi \vee \psi) \to \delta$
3. $(\varphi \to \psi), (\psi \to \delta) \vdash \varphi \to \delta$
4. $\varphi \vdash \varphi$

The following rules are derivable.

(Pre) $$\frac{\varphi \vdash \psi}{\delta \to \varphi \vdash \delta \to \psi}$$

(Pre$_2$) $$\frac{\varnothing \vdash \psi}{\varnothing \vdash \delta \to \psi}$$

(Suf) $$\frac{\varphi \vdash \psi}{\psi \to \delta \vdash \varphi \to \delta}$$

We can replace (DT$_0$) with (Pre) to obtain an equivalent calculus.

The above calculus defines a logic, which we denote by wK_σ, in the standard way, namely, $\Gamma \vdash_{wK_\sigma} \varphi$ if and only if there is a finite $\Delta \subseteq \Gamma$ such that the sequent $\Delta \vdash \varphi$ is derivable.

Now we prove the completeness theorem for wK_σ mentioned above. The ideas of the proof are the ideas used by Došen to prove the completeness theorem for the system K_σ.

A set Γ of formulas is a *theory* of wK_σ if it is closed under the relation $\vdash_{wK_\sigma}$. A theory Γ is *consistent* if there is some formula $\varphi \notin \Gamma$, and it is *prime* if it is consistent and for all formulas φ, ψ such that $\varphi \vee \psi \in \Gamma$, $\varphi \in \Gamma$ or $\psi \in \Gamma$.

Let $W_c = \{\Gamma : \Gamma$ is a prime theory of $wK_\sigma\}$ and define the binary relation R_c in W_c as follows:

$$\Gamma R_c \Delta \text{ iff } \forall \varphi, \psi (\varphi \to \psi \in \Gamma \text{ and } \varphi \in \Delta \Longrightarrow \psi \in \Delta).$$

Let V_c be defined by

$$V_c(p) = \{\Gamma : p \in \Gamma\},$$

for every propositional variable p. The frame $\langle W_c, R_c \rangle$ is called the *canonical Kripke frame* of wK_σ and the model $\langle W_c, R_c, V_c \rangle$ is called the *canonical Kripke model* of wK_σ.

Lemma 4.1   *For every prime theory Γ and any formula φ,*

$$\Gamma \in V_c(\varphi) \text{ iff } \varphi \text{ in } \Gamma.$$

Proof   The proof is by induction. We only deal with the implication case. If $\varphi \to \psi \in \Gamma$, it is clear that if $\Gamma R_c \Delta$ and $\varphi \in \Delta$ then $\psi \in \Delta$. Therefore, $\Gamma \in V_c(\varphi \to \psi)$. On the other hand, if $\varphi \to \psi \notin \Gamma$, we see that there is $\Delta \in W_c$ such that $\Gamma R_c \Delta$, $\varphi \in \Delta$ and $\psi \notin \Delta$. Let us consider the set

$\mathcal{F} =$

$\{\Delta : \Delta \text{ is a theory}, \varphi \in \Delta, \psi \notin \Delta \text{ and } \forall \delta, \varepsilon (\delta \to \varepsilon \in \Gamma \,\&\, \delta \in \Delta \Longrightarrow \varepsilon \in \Delta)\}.$

This set is nonempty as

$$\Gamma_0 = \{\delta : \varphi \to \delta \in \Gamma\}$$

is a theory that verifies the conditions for being an element of $\mathcal{F}$. To see this one has to use the prefixing rules. Using Zorn's lemma we obtain a maximal element Δ in $\mathcal{F}$. Let us see that Δ is prime. Assume that $\delta \vee \varepsilon \in \Delta$, that $\delta \notin \Delta$, and that $\varepsilon \notin \Delta$. Let

$$F(\Delta, \delta) = \{\varepsilon : \exists \gamma \in \Delta \,(\gamma \wedge \delta) \to \varepsilon \in \Gamma\}.$$

This set is a theory that includes $\Delta \cup \{\varphi, \delta\}$ and verifies the condition

$$\forall \delta_1, \varepsilon_1 (\delta_1 \to \varepsilon_1 \in \Gamma \,\&\, \delta_1 \in F(\Delta, \delta) \Longrightarrow \varepsilon_1 \in F(\Delta, \delta)).$$

This can be seen using the prefixing rules. Therefore, as Δ is properly included in $F(\Delta, \delta)$, $\psi \in F(\Delta, \delta)$. Analogously, the set

$$F(\Delta, \varepsilon) = \{\varepsilon_1 : \exists \gamma \in \Delta \,(\gamma \wedge \varepsilon) \to \varepsilon_1 \in \Gamma\}$$

is a theory that includes $\Delta \cup \{\varphi, \varepsilon\}$ and verifies the condition

$$\forall \delta_1, \varepsilon_1 (\delta_1 \to \varepsilon_1 \in \Gamma \,\&\, \delta_1 \in F(\Delta, \delta) \Longrightarrow \varepsilon_1 \in F(\Delta, \varepsilon)).$$

Therefore, as Δ is properly included in $F(\Delta, \varepsilon)$, $\psi \in F(\Delta, \varepsilon)$. Let us now consider $\delta_1, \delta_2 \in \Delta$ such that

$$(\delta_1 \wedge \delta) \to \psi \in \Gamma \text{ and } (\delta_2 \wedge \varepsilon) \to \psi \in \Gamma.$$

Then

$$((\delta_1 \wedge \delta_2 \wedge \delta) \vee (\delta_1 \wedge \delta_2 \wedge \varepsilon)) \to \psi \in \Gamma.$$

Therefore,

$$((\delta_1 \wedge \delta_2) \wedge (\delta \vee \varepsilon)) \to \psi \in \Gamma.$$

As $(\delta_1 \wedge \delta_2), (\delta \vee \varepsilon) \in \Delta$, we obtain that $\psi \in \Delta$, which is absurd.   □

Lemma 4.2 $\Delta \vdash_{wK_\sigma} \varphi$ *if and only if for every prime theory Γ such that $\Delta \subseteq \Gamma$,* $\varphi \in \Gamma$.

Proof   If $\Delta \nvdash_{wK_\sigma} \varphi$, then φ does not belong to the theory generated by Δ. Let us consider the set

$$\mathcal{F} = \{\Gamma : \Gamma \text{ is a theory such that } \Delta \subseteq \Gamma \text{ and } \varphi \notin \Gamma\}.$$

By Zorn's lemma there is a maximal set Γ in $\mathcal{F}$ in the inclusion order. Let us see that it is a prime theory. Assume that $\psi \vee \delta \in \Gamma$ and $\psi, \delta \notin \Gamma$. Let Δ_1 be the theory generated by $\Gamma \cup \{\psi\}$ and Δ_2 the theory generated by $\Gamma \cup \{\delta\}$. If Δ_1 is inconsistent, $\Gamma \cup \{\psi\} \vdash_{wK_\sigma} \delta$. Therefore $\Gamma \cup \{\psi \vee \delta\} \vdash_{wK_\sigma} \delta$, that is, $\Gamma \vdash_{wK_\sigma} \delta$, which is not the case. Hence, Δ_1 is consistent. Analogously it can be seen that Δ_2 is consistent. By maximality of Γ, $\varphi \in \Delta_1$ and $\varphi \in \Delta_2$. So, $\Gamma \cup \{\psi\} \vdash_{wK_\sigma} \varphi$ and $\Gamma \cup \{\delta\} \vdash_{wK_\sigma} \varphi$. Therefore, $\Gamma \cup \{\psi \vee \delta\} \vdash_{wK_\sigma} \varphi$, that is, $\Gamma \vdash_{wK_\sigma} \varphi$, which is absurd.   □

Theorem 4.3 (Soundness and Completeness)   *For every set of formulas Σ and every formula φ,*

$$\Sigma \vdash_{wK_\sigma} \varphi \;\; iff \;\; \Sigma \models_l \varphi.$$

Proof   The soundness part is routine.   The completeness part follows from Lemma 4.2.   □

Corollary 4.4   *The local consequence of the class of all frames is finitary.*

Proof   It follows from the soundness and completeness theorem because the logic wK_σ is finitary by definition. It also follows from the fact that the class of all frames is closed under ultraproducts.   □

Theorems 3.2 and 4.3 imply that the theorems of the logic wK_σ are exactly the theorems of the logic K_σ. What is the exact relation between wK_σ and K_σ? To answer this question, let us show that a restricted version of modus ponens holds in wK_σ.

Proposition 4.5   *If $\vdash_{wK_\sigma} \varphi \to \psi$ (that is, $\varphi \to \psi$ is a theorem of wK_σ), then* $\varphi \vdash_{wK_\sigma} \psi$.

Proof   Assume $\vdash_{wK_\sigma} \varphi \to \psi$. We use completeness. Assume that $\langle W, R, V \rangle$ is a Kripke model and that $x \in V(\varphi)$. To see that $x \in V(\psi)$ we construct another model. Let $\alpha \notin W$. Define $W_\alpha = W \cup \{\alpha\}$, $R_\alpha = R \cup \{\langle \alpha, z \rangle : z \in W\}$, and V_α by $V_\alpha(p) = V(p)$ for every propositional variable p. Then it is easy to show by induction that for any formula φ, $V(\varphi) = V_\alpha(\varphi) \cap W$. By assumption we have $V_\alpha(\varphi \to \psi) = W_\alpha$. Hence, $\alpha \in V_\alpha(\varphi \to \psi)$. Since $\alpha R_\alpha x$ and $x \in V(\varphi) \subseteq V_\alpha(\varphi)$ we obtain that $x \in V_\alpha(\psi)$, but since $x \in W$, $x \in V(\psi)$, as desired.   □

The proposition implies that K_σ is a proper extension of wK_σ.

Proposition 4.6 K_σ *is a proper extension of wK_σ.*

Proof   First we observe that

$$\varphi_1, \ldots, \varphi_n \vdash_{wK_\sigma} \psi \;\; \text{iff} \;\; \vdash_{K_\sigma} (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \psi. \tag{1}$$

This holds because $\varphi_1, \ldots, \varphi_n \vdash_{wK_\sigma} \psi$ if and only if $\varphi_1 \wedge \cdots \wedge \varphi_n \vdash_{wK_\sigma} \psi$ if and only if $\vdash_{wK_\sigma} (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \psi$ (by (DT$_0$) and the previous proposition) if and

only if $\vdash_{K_\sigma} (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \psi$ (since by the completeness theorem K_σ and wK_σ have the same theorems). Therefore, if $\Gamma \vdash_{wK_\sigma} \varphi$, then there is a finite $\Gamma' \subseteq \varphi$ such that $\Gamma' \vdash_{wK_\sigma} \varphi$. By (1) and the fact that (MP) and (Ad) are rules of K_σ, we obtain that $\Gamma' \vdash_{K_\sigma} \varphi$. Therefore $\Gamma \vdash_{K_\sigma} \varphi$. Thus K_σ is an extension of wK_σ. That it is proper follows from the fact that (MP) is not a rule of wK_σ as is implied by the comment at the beginning of the section and the soundness part of Theorem 4.3. □

The relation between K_σ and wK_σ established in (1) of the preceding proof allows us to establish the relation, given by the translation σ considered by Došen, between wK_σ and the local consequence relation associated with the modal logic **K** that we denote by $\vdash_{lK}$.

Theorem 4.7   *For every set of subintuitionistic formulas Γ and every subintuitionistic formula ψ,*

$$\Gamma \vdash_{wK_\sigma} \psi \ \ iff \ \ \sigma[\Gamma] \vdash_{lK} \sigma(\psi).$$

Proof   By (1) we have

$$\varphi_1, \ldots, \varphi_n \vdash_{wK_\sigma} \psi \ \ \text{iff} \ \ \vdash_{K_\sigma} (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \psi.$$

Therefore, by Došen's result,

$$\varphi_1, \ldots, \varphi_n \vdash_{wK_\sigma} \psi \ \ \text{iff} \ \ \vdash_{lK} \sigma((\varphi_1 \wedge \cdots \wedge \varphi_n) \to \psi),$$

where σ is the translation of the subintuitionistic language into the modal language considered by Došen. So,

$$\varphi_1, \ldots, \varphi_n \vdash_{wK_\sigma} \psi \ \ \text{iff} \ \ \vdash_{lK} \Box(\sigma(\varphi_1 \wedge \cdots \wedge \varphi_n) \to \sigma(\psi)).$$

Since $\Box\delta/\delta$ is an admissible rule of lK (i.e., if $\Box\delta$ is a theorem of lK then δ is a theorem of lK) we obtain that

$$\varphi_1, \ldots, \varphi_n \vdash_{wK_\sigma} \psi \ \ \text{iff} \ \ \vdash_{lK} \sigma(\varphi_1) \wedge \cdots \wedge \sigma(\varphi_n) \to \sigma(\psi).$$

Thus,

$$\varphi_1, \ldots, \varphi_n \vdash_{wK_\sigma} \psi \ \ \text{iff} \ \ \sigma(\varphi_1), \ldots, \sigma(\varphi_n) \vdash_{lK} \sigma(\psi).$$

From this the desired result easily follows.   □

The content of Theorem 4.7 can be described in the following way: wK_σ is the strict implication fragment (with $\wedge$, $\vee$, and $\bot$) of the local consequence associated with the normal modal logic **K**. A semantic proof can be given by showing that for every Kripke model $\langle W, R, V \rangle$ and every formula φ, if we denote by V^* the extension of V to the modal language, $V(\varphi) = V^*(\sigma(\varphi))$.

   We know that the logic wK_σ does not satisfy modus ponens. In fact, as we will see, there is no formula $\delta(p, q)$ such that $p, \delta(p, q) \vdash_{wK_\sigma} q$ and $\vdash_{wK_\sigma} \delta(p, p)$ and indeed no set $\Delta(p, q)$ of formulas with these properties. This means that the logic wK_σ is not protoalgebraic (cf. [5]). The proof of the following theorem uses the algebra that Suzuki, Wolter, and Zacharyaschev use in [15] to prove the analogous result for the logic **BPL**, which is an extension of wK_σ.

Theorem 4.8   *The logic wK_σ is not protoalgebraic.*

Proof   Let us consider the algebra A whose universe A is $\{1, a, 0\}$, the infimum and supremum are defined according to the linear order $0 < a < 1$, and the operation $\to^A$ is the constant function 1. It is easy to check that the matrices $\langle A, \{1\} \rangle$ and $\langle A, \{1, a\} \rangle$ are models of wK_σ. Now, $\mathbf{\Omega}_A(\{1\})$ is the congruence that only identifies

a and 0 and $\mathbf{\Omega}_A(\{1, a\})$ is the congruence that only identifies a and 1. Therefore, $\mathbf{\Omega}_A(\{1\}) \nsubseteq \mathbf{\Omega}_A(\{1, a\})$. Since $\{1\} \subseteq \{1, a\}$ we conclude that wK_σ is not protoalgebraic. $\square$

4.2 Global consequence   The global consequence relation defined by the class of all Kripke frames will be denoted simply by $\models_g$. From Proposition 2.3 we obtain the following.

Proposition 4.9   *For every set of formulas Γ and every formula φ,*

$$\{\top^n \to \psi : n \in \omega, \psi \in \Gamma\} \vdash_{wK_\sigma} \varphi \ \text{ iff } \ \Gamma \models_g \varphi.$$

The proposition implies that the global consequence defined by the class of all Kripke frames is a finitary consequence relation. Let us denote by sK_σ the logic axiomatized by all the Hilbert rules of wK_σ plus the rule (N). Recall that this rule is $\varphi \vdash \top \to \varphi$. Thus $sK_\sigma = wK_\sigma + $ (N). Corollary 2.5 and Theorem 4.3 imply the following.

Theorem 4.10   *sK_σ is the global consequence defined by the class of all Kripke frames.*

Another axiomatization of sK_σ can be given by adding the rule (W) to wK_σ. In fact, modulo wK_σ, the rule (W) and the rule (N) are equivalent.

Proposition 4.11 $sK_\sigma = wK_\sigma + $ (W).

Proof   It is clear that rule (N) follows from (W). On the other hand, by completeness of wK_σ one obtains that $\top \to \varphi \vdash_{wK_\sigma} \psi \to \varphi$ (this can also be deduced using (Suf)). Hence, by the rule (N) we obtain (W). $\square$

At this point a warning is in order. Adding the rule (W) to wK_σ spoils the closure of the logic so obtained under (some of) the Gentzen rules of the calculus used to define wK_σ. For instance, the rule of introduction of the disjunction on the left does not hold. For example, $p \models_g (\top \to p) \vee (\top \to q)$ and $q \models_g (\top \to p) \vee (\top \to q)$ but $p \vee q \nvDash_g (\top \to p) \vee (\top \to q)$. Thus sK_σ does not have what in some contexts is called the property of disjunction. In addition, the rule (DT_0) does not hold. We have $p \models_g \top \to p$, but $\nvDash_g p \to (\top \to p)$.

Since the new rule (N) of sK_σ is an instance of the rule (W) of K_σ we obtain that K_σ is an extension of sK_σ too. In fact, it is a proper extension since, as we will see in the next proof, modus ponens is not a rule of sK_σ. The exact extension relation between the three logics considered thus far is the following.

Theorem 4.12 $wK_\sigma < sK_\sigma < K_\sigma$.

Proof   It is clear that (N) is not a rule of wK_σ because, for example, the model

$$\langle \{a, b\}, \{(a, b)\}, V \rangle$$

where $V(p) = \{a\}$ is such that $a \in V(p)$ but $a \notin V(\top \to p)$. Moreover, modus ponens is not a sound rule of sK_σ because the model $\langle \{a\}, \varnothing, V \rangle$ where $V(p) = \{a\}$ and $V(q) = \varnothing$ is such that $V(p) = V(p \to q) = \{a\}$ but $V(q) \neq \{a\}$. $\square$

But not only is modus ponens not a sound rule of sK_σ, sK_σ is not protoalgebraic.

Theorem 4.13   *The logic sK_σ is not protoalgebraic.*

Proof   The matrices used in the proof of Theorem 4.8 are clearly models of sK_σ since the rule (N) is sound for them.   □

To conclude the section let us see how sK_σ is related to the normal modal logic **K** via the translation σ. Let us denote by $\vdash_{gK}$ the global consequence relation associated with **K**.

Theorem 4.14   *For every set of subintuitionistic formulas Γ and every subintuitionistic formula φ,*

$$\Gamma \vdash_{sK_\sigma} \varphi \ \textit{iff} \ \sigma[\Gamma] \vdash_{gK} \sigma(\varphi).$$

Proof   The following equivalences show it:

$$
\begin{aligned}
\Gamma \vdash_{sK_\sigma} \varphi \quad &\text{iff} \quad \{\top^n \to \psi : \psi \in \Gamma, n \in \omega\} \vdash_{wK_\sigma} \varphi \\
&\text{iff} \quad \{\sigma(\top^n \to \psi) : \psi \in \Gamma, n \in \omega\} \vdash_{lK} \sigma(\varphi) \\
&\text{iff} \quad \{\Box^n \sigma(\psi) : \psi \in \Gamma, n \in \omega\} \vdash_{lK} \sigma(\varphi) \\
&\text{iff} \quad \sigma[\Gamma] \vdash_{gK} \sigma(\varphi).
\end{aligned}
$$

The third equivalence holds because $\sigma(\top \to \psi)$ is equivalent to $\Box\sigma(\psi)$, and the fourth because the relation between gK and lK is, as is well known, the following:

$$\Gamma \vdash_{gK} \varphi \ \text{iff} \ \{\Box^n \psi : \psi \in \Gamma, n \in \omega\} \vdash_{lK} \varphi,$$

where $\Gamma \cup \{\varphi\}$ is now a set of formulas in the modal language.   □

Theorem 4.14 shows that sK_σ is the strict implication fragment (with $\wedge$, $\vee$, and $\bot$) of the global consequence associated with the normal modal logic **K**.

The global and local consequence relations for the class of models with a reflexive and transitive relation and a R-persistent valuation coincide, and both are the intuitionistic consequence. With respect to the global consequence relation $gS4$ associated with **S4** and Intuitionistic consequence, Gödel's translation τ has the same property as σ with respect to the global consequence associated with **K** and the logic sK_σ because

$$
\begin{aligned}
\varphi_1, \ldots, \varphi_n \vdash_{\text{Int}} \varphi \quad &\text{iff} \quad \vdash_{\text{Int}} \varphi_1 \wedge \cdots \wedge \varphi_n \to \varphi \\
&\text{iff} \quad \vdash_{\textbf{S4}} \tau(\varphi_1 \wedge \cdots \wedge \varphi_n \to \varphi) \\
&\text{iff} \quad \vdash_{\textbf{S4}} \Box(\tau(\varphi_1) \wedge \cdots \wedge \tau(\varphi_n) \to \tau(\varphi)) \\
&\text{iff} \quad \vdash_{\textbf{S4}} \Box(\tau(\varphi_1) \wedge \cdots \wedge \tau(\varphi_n)) \to \tau(\varphi) \\
&\text{iff} \quad \tau(\varphi_1), \ldots, \tau(\varphi_n) \vdash_{gS4} \tau(\varphi).
\end{aligned}
$$

The reason why the last equivalence holds is because of the deduction theorem for $gS4$, and the reason why the penultimate equivalence holds is that the set of values of the translation of an intuitionistic formula is always R-persistent, and we are in **S4**. Therefore, the local and the global consequence relations associated with **S4** are the same when restricted to the Gödel translations of intuitionistic formulas. The analogous situation does not hold for **K** and σ. Thus, to obtain real analogs of the intuitionistic situation for the translation σ and the system **K** we have to consider two logics instead of one, wK_σ and sK_σ; neither of them is the logic we called Došen's logic.

5 Modus Ponens and Reflexivity

In this section we will study the logics obtained when one adds the rule of modus ponens to the logics of the previous section, and the connections between modus ponens and reflexive frames.

Given the logic wK_σ and several (Hilbert) rules $R_1, \ldots, R_n$, we denote by

$$wK_\sigma(R_1, \ldots, R_n)$$

the logic defined by the Gentzen calculus obtained by adding as axioms the rules $R_1, \ldots, R_n$ to the Gentzen calculus $\mathcal{G}$ used to define wK_σ.

The first observation concerning modus ponens and reflexivity is the following.

Proposition 5.1 *Let $\langle W, R \rangle$ be a Kripke frame. Then R is reflexive if and only if modus ponens holds in $\langle W, R \rangle$.*

Proof If R is reflexive and V is a valuation on $\langle W, R \rangle$ such that $x \in V(\varphi) \cap V(\varphi \to \psi)$, then, as $x R x$, $x \in V(\psi)$. On the other hand, if modus ponens holds in $\langle W, R \rangle$, let $x \in W$ and consider a valuation V such that $V(p) = \{x\}$ and $V(q) = R(x)$. Then, since $R(x) \cap V(p) \subseteq V(q)$, $x \in V(p \to q)$. By (MP), $x \in V(q)$. Therefore, $x R x$. □

In [4], Corsi considers the set of theorems of the logic obtained by adding to K_σ the axiom

(R) $\varphi \wedge (\varphi \to \psi) \to \psi$

and shows that a formula is a theorem of the logic so obtained if and only if it is valid in every reflexive Kripke frame. But the formula $p \wedge (p \to q) \to q$ does not correspond to reflexivity on Kripke frames. Indeed, it is valid in every reflexive Kripke frame but is also valid in some nonreflexive Kripke frames.

Proposition 5.2 *Although $p \wedge (p \to q) \to q$ is valid in every reflexive Kripke frame, there are nonreflexive Kripke frames where it is valid.*

Proof Consider the frame $\langle \{a\}, \varnothing \rangle$. Then, since $R(a) = \varnothing$, for every valuation V, every conditional formula is valid; hence our formula is valid. □

Propositions 5.1 and 5.2 show that from the point of view of the correspondence of properties of Kripke frames with the validity on them of formulas/sequents in the subintuitionistic language, the sequents are much more well behaved. This phenomena will appear again when we consider transitivity.

We turn now to the consideration of extensions of the logic wK_σ with modus ponens. At least two natural extensions come to mind. One is the logic $wK_\sigma + (MP)$; the other is the logic $wK_\sigma(MP)$. The first of the following results show that these two logics are not the same; in fact, the second is a proper extension of the first one.

Proposition 5.3

1. $wK_\sigma + (MP)$ *and* K_σ *have the same theorems.*
2. $wK_\sigma(MP)$ *and* K_σ *do not have the same theorems;* $p \wedge (p \to q) \to q$ *is a theorem of the first but not of the second.*
3. $wK_\sigma + (MP) < wK_\sigma(MP)$.
4. $wK_\sigma + (MP) < K_\sigma$.

Proof (1) The set of theorems of wK_σ and of K_σ are the same but this set is closed under modus ponens; this implies that $wK_\sigma + \text{(MP)}$ and K_σ have the same theorems.

(2) The formula $p \wedge (p \to q) \to q$ is a theorem of $wK_\sigma\text{(MP)}$ since by (MP), the rules for conjunction and Cut, $p \wedge (p \to q) \vdash q$ is derivable, which by (DT_0) implies that $\varnothing \vdash p \wedge (p \to q) \to q$ is derivable too. Now it is easy to construct a Kripke model where $p \wedge (p \to q) \to q$ is not valid, thus showing that the formula is not a theorem of K_σ. Consider just two different points a and b, with a related to b, and a valuation where p is true only at b and q is false everywhere. In this model the formula under consideration is not true at a.

(3) By (1) and (2), $wK_\sigma + \text{(MP)}$ and $wK_\sigma\text{(MP)}$ cannot be equal. Moreover, it is clear that every rule of any axiomatization of $wK_\sigma + \text{(MP)}$ is a rule of the logic $wK_\sigma\text{(MP)}$ so this last logic is a proper extension of the other one.

(4) It follows from the fact that all the rules of $wK_\sigma + \text{(MP)}$ are valid in any reflexive Kripke model but there are reflexive Kripke models where the rule (W), which is a rule of K_σ, is not valid. We leave it to the reader to find one. $\square$

A consequence of the proposition is the frame incompleteness result below. We say that a logic is *frame complete* if every rule and every formula valid in all the frames where the rules of the logic are valid is a rule of the logic or a theorem of the logic.

Theorem 5.4 *The logic $wK_\sigma + \text{(MP)}$ is frame incomplete.*

Proof It follows from Proposition 5.1 that the frames where all the rules of $wK_\sigma + \text{(MP)}$ are valid are the reflexive frames. The formula $p \wedge (p \to q) \to q$ is valid in all reflexive frames, but, by (2) and (4) of Proposition 5.3, it is not a theorem of $wK_\sigma + \text{(MP)}$. $\square$

The frame completion of $wK_\sigma + \text{(MP)}$, namely, the logic of the class of frames where the sequents of the logic $wK_\sigma + \text{(MP)}$ are valid, is $wK_\sigma\text{(MP)}$, as follows from the next theorem. We want to prove that the logic $wK_\sigma\text{(MP)}$ is precisely the local consequence defined by the class of all reflexive frames. To do so we can perform a completeness proof analogous to that of Theorem 4.3 for wK_σ and show that the relation of the canonical model is reflexive.

Theorem 5.5 $wK_\sigma\text{(MP)}$ *is the local consequence defined by the class of all reflexive Kripke frames.*

Proof The canonical Kripke frame for $wK_\sigma\text{(MP)}$ is reflexive: if $\varphi \to \psi, \varphi \in \Gamma$, by (MP), $\psi \in \Gamma$. Therefore $\Gamma R_c \Gamma$. $\square$

Now we turn to sK_σ and the global consequence defined by the class of all reflexive Kripke frames.

Theorem 5.6 $sK_\sigma + \text{(MP)} = K_\sigma$.

Proof Since $sK_\sigma < K_\sigma$ and (MP) is a rule of K_σ, $sK_\sigma + \text{(MP)} \leq K_\sigma$. On the other hand, every axiom of the axiomatization of K_σ is a theorem of sK_σ. Clearly (Ad) is a rule of sK_σ. Moreover, since sK_σ is the global consequence defined by the class of all Kripke frames, it is clear that the rule (W) is a rule of sK_σ. Hence all the

rules of the given axiomatization of K_σ are rules of $sK_\sigma + (\text{MP})$. We conclude that $sK_\sigma + (\text{MP}) = K_\sigma$. $\qquad \square$

By Corollary 2.5 the logic obtained by adding the rule (N) to $wK_\sigma(\text{MP})$, or alternatively (W) (we can reason as in Proposition 4.11 to show this), is the global consequence relation defined by the class of all reflexive Kripke frames.

Theorem 5.7   *The logic $wK_\sigma(\text{MP}) + (\text{W})$ is the global consequence defined by the class of all reflexive Kripke frames.*

This last logic has the same theorems as $wK_\sigma(\text{MP})$; therefore it is not equal to K_σ but is one of its extensions. The reason is that every axiom of K_σ is a theorem of $wK_\sigma(\text{MP})$ and also every rule of K_σ is one of its rules.

Proposition 5.8 $K_\sigma < wK_\sigma(\text{MP}) + (\text{W})$.

Summarizing we have

$$wK_\sigma < wK_\sigma + (\text{MP}) < wK_\sigma(\text{MP}) < wK_\sigma(\text{MP}) + (\text{W}),$$

and

$$wK_\sigma + (\text{MP}) < sK_\sigma + (\text{MP}) = K_\sigma < wK_\sigma(\text{MP}) + (\text{W}).$$

We now move to the consideration of the logics obtained by adding (R) as an axiom to the logics already considered.

Proposition 5.9

1. *The logics $wK_\sigma(\text{MP})$, $wK_\sigma(\text{MP}) + (\text{W})$, and $wK_\sigma + (\text{W}) + (\text{MP}) + (\text{R})$, which is $K_\sigma + (\text{R})$, have the same theorems.*
2. $wK_\sigma(\text{MP}) = wK_\sigma(\text{MP}) + (\text{R}) = wK_\sigma(\text{MP}, \text{R})$.
3. $wK_\sigma(\text{MP}) + (\text{W}) = wK_\sigma(\text{MP}) + (\text{W}) + (\text{R}) =$
   $wK_\sigma + (\text{MP}) + (\text{W}) + (\text{R})(= K_\sigma + (\text{R}))$.

Proof   (1)   By Corsi's results in [4] the theorems of $K_\sigma + (\text{R})$ are the formulas valid in every reflexive Kripke frame and these are precisely the theorems of $wK_\sigma(\text{MP})$ and $wK_\sigma(\text{MP}) + (\text{W})$.

(2)   It holds because (R) is valid in every reflexive Kripke frame and therefore is a theorem of $wK_\sigma(\text{MP})$.

(3)   The first equality holds because (R) is a theorem of $wK_\sigma(\text{MP})$. We prove now that $wK_\sigma(\text{MP}) + (\text{W}) = K_\sigma + (\text{R})$. That $wK_\sigma(\text{MP}) + (\text{W})$ extends $K_\sigma + (\text{R})$ holds by the same reason. To prove that $K_\sigma + (\text{R})$ extends $wK_\sigma(\text{MP}) + (\text{W})$, it suffices to show that if $\psi_0, \ldots, \psi_{n-1} \vdash_{wK_\sigma(\text{MP})} \varphi$, then $\psi_0, \ldots, \psi_{n-1} \vdash_{K_\sigma+(\text{R})} \varphi$. Assume that $\psi_0, \ldots, \psi_{n-1} \vdash_{wK_\sigma(\text{MP})} \varphi$. Using the rules for $\wedge$ and (DT_0) of the Gentzen calculus we obtain that $\vdash_{wK_\sigma(\text{MP})} \psi_0 \wedge \cdots \wedge \psi_{n-1} \to \varphi$. Thus $\psi_0 \wedge \cdots \wedge \psi_{n-1} \to \varphi$ is a theorem of $K_\sigma + (\text{R})$. Using the inference rules of the axiomatization of K_σ it follows that $\psi_0, \ldots, \psi_{n-1} \vdash_{K_\sigma+(\text{R})} \varphi$. $\qquad \square$

Theorem 5.10   *The logics $wK_\sigma + (\text{R})$ and $sK_\sigma + (\text{R})$ are non-protoalgebraic. Moreover, all the logics with (MP) are protoalgebraic.*

Proof   The matrices used in the proof of Theorem 4.8 show that $wK_\sigma + (\text{R})$ and $sK_\sigma + (\text{R})$ are non-protoalgebraic. For the logics with (MP), in each one holds that $p \to p$ is a theorem and that modus ponens is a rule, this is known to imply protoalgebraicity. $\qquad \square$

Theorem 5.11   *The logics $K_\sigma + $ (R) and wK_σ (MP) + (W) are algebraizable.*

Proof   They are extensions of the algebraizable logic K_σ. $\square$

Proposition 5.12   *The axiomatic extensions by* (R) *of the logics wK_σ, sK_σ, $wK_\sigma + $ (MP), and $sK_\sigma + $ (MP) are proper extensions.*

Proof   The four logics wK_σ, sK_σ, $wK_\sigma + $ (MP), and $sK_\sigma + $ (MP) have the same theorems, namely, the formulas valid in all Kripke frames, and we know that (R) is not a theorem of $sK_\sigma + $ (MP). $\square$

Proposition 5.13

1. $wK_\sigma + $ (R) $< wK_\sigma + $ (W) + (R) $< wK_\sigma + $ (W) + (MP) + (R).
2. $wK_\sigma + $ (R) $< wK_\sigma + $ (MP) + (R) $< wK_\sigma$ (MP).

To prove the proposition it is convenient to present first an auxiliary lemma. To state it we introduce the next definition.

Definition 5.14   Let $\langle W, R, V \rangle$ be a model, $F \subseteq \{V(\varphi) : \varphi \in Fm\}$ and $\langle \Gamma, \varphi \rangle$ be a Hilbert-style rule. We will say that $\langle \Gamma, \varphi \rangle$ is F-valid in $\langle W, R, V \rangle$ if for every substitution instance $\langle e[\Gamma], e(\varphi) \rangle$ of $\langle \Gamma, \varphi \rangle$ such that $\{V(e(\psi)) : \psi \in \Gamma\} \subseteq F$ it holds that $V(e(\varphi)) \in F$.

Lemma 5.15   *Let $\{R_i : i \in I\}$ be a set of Hilbert-style rules, $\mathscr{S}$ a finitary logic, and $\langle W, R, V \rangle$ a model where all the Hilbert-style rules $\langle \Gamma, \varphi \rangle$ such that $\Gamma \vdash_\mathscr{S} \varphi$ are valid (i.e., $V(\Gamma) \subseteq V(\varphi)$). For any set $F \subseteq \{V(\varphi) : \varphi \in Fm\}$ closed under intersections, upward closed by the inclusion relation, and with the property that every rule R_i is F-valid in $\langle W, R, V \rangle$, it holds that for every set of formulas Γ such that $\{V(\psi) : \psi \in \Gamma\} \subseteq F$ and every formula φ, if $\Gamma \vdash_{\mathscr{S}+\{R_i : i \in I\}} \varphi$, then $V(\varphi) \in F$.*

Proof   The proof is by induction on the length of the proofs. $\square$

Proof of Proposition 5.13   (1)   If $\Gamma \vdash_{wK_\sigma + (R)} \varphi$, then in any reflexive Kripke model $\mathcal{M}$, if the formulas in Γ are true at a point x, then φ is true at x. This holds because (R) is valid in any reflexive Kripke model, and wK_σ is the local consequence of all Kripke models. But there are reflexive Kripke models where (W) does not hold. Let $W = \{a, b\}$, $R = \{\langle a, a \rangle, \langle b, b \rangle, \langle a, b \rangle\}$ and consider any valuation such that $V(p) = \{a\}$. Then $V(p) \not\subseteq V(\top \to p)$. Thus $p \not\vdash_{wK_\sigma + (R)} \top \to p$. Hence $wK_\sigma + $ (R) $< wK_\sigma + $ (W) + (R). To prove the other inequality it is clear that $wK_\sigma + $ (W) + (MP) + (R), which is $K_\sigma + $ (R), is an extension of $wK_\sigma + $ (W) + (R) and that they are different follows from the fact that the first one is non-protoalgebraic and the second one is algebraizable.

(2)   There are Kripke frames where (MP) is not valid but (R) is; for instance, the frame considered in the proof of Proposition 5.2. This proves the first inequality. That $wK_\sigma + $ (MP) + (R) $\leq wK_\sigma$ (MP) is obvious since (R) is a theorem of wK_σ (MP). To prove that these two logics are different we will see that the formula $\delta := \top \to ((p \land (p \to q)) \to q)$, which is clearly a theorem of wK_σ (R) and therefore of wK_σ (MP), is not a theorem of $wK_\sigma + $ (MP) + (R). To this end we will use Lemma 5.15. We consider the Kripke frame $\mathcal{F} = \langle W, R \rangle$ where $W = \{w_1, w_2, w_3\}$ and $R = \{\langle w_1, w_1 \rangle, \langle w_1, w_2 \rangle, \langle w_1, w_3 \rangle, \langle w_2, w_2 \rangle, \langle w_2, w_1 \rangle\}$. Let A be the set whose elements are $0 := \varnothing$, $1 := W$, $a := \{w_3\}$, and $b := \{w_2, w_3\}$. This set is closed under intersections, unions, and the operation

that serves to interpret $\to$, namely, the operation $\Rightarrow_R$ on the powerset of W defined by $X \Rightarrow_R Y = \{x \in W : R(x) \cap X \subseteq Y\}$, for every $X, Y \in \mathcal{P}(W)$ which, restricted to A, is displayed in the following table.

$\Rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	a	1	1
1	a	a	a	1

Consider the set $F = \{\{w_2, w_3\}, W\}$. It is easy to check that for any $X, Y \in A$, $X \cap (X \Rightarrow Y) \Rightarrow Y \in F$ and that if $X, (X \Rightarrow Y) \in F$, then $Y \in F$. Thus (R) and (MP) are F-valid in any model $\langle W, R, V \rangle$ on the frame $\mathcal{F}$ whose valuation V takes values in A. Since $V(\top \to ((p \wedge (p \to q)) \to q)) = a \notin F$ it follows by Lemma 5.15 that $\top \to ((p \wedge (p \to q)) \to q)$ is not a theorem of $wK_\sigma + (\text{MP}) + (\text{R})$.[1]

$\square$

Finally, let us see which logics are obtained when (R) is added as an axiom to the Gentzen calculus used to define wK_σ. We will consider the logics $wK_\sigma(\text{R})$, $wK_\sigma(\text{R}) + (\text{MP})$, $wK_\sigma(\text{R}) + (\text{W})$, and $wK_\sigma(\text{R}) + (\text{W}) + (\text{MP})$; $wK_\sigma(\text{MP, R})$ has already been considered. We have the situation displayed in the next proposition. To prove it we will use the following fact, proved by Bou.

Fact 5.16 (Bou) $wK_\sigma(\text{R})$ can be axiomatized by adding to the logic wK_σ the axioms of the form $\top^n \to \varphi$ where φ is a formula of the form $\psi \wedge (\psi \to \delta) \to \delta$, that is, $wK_\sigma(\text{R})$ is the logic $wK_\sigma + \{\varnothing \vdash \top^n \to ((p \wedge (p \to q)) \to q) : n \in \omega\}$.

This fact can be proved by syntactic means and its proof is beyond the scope of this paper.

Proposition 5.17

1. $wK_\sigma + (\text{R}) < wK_\sigma(\text{R}) < wK_\sigma(\text{R}) + (\text{MP}) < wK_\sigma(\text{MP})$.
2. $wK_\sigma + (\text{R}) + (\text{MP}) < wK_\sigma(\text{R}) + (\text{MP})$.
3. $wK_\sigma + (\text{W}) + (\text{R}) = wK_\sigma(\text{R}) + (\text{W}) < wK_\sigma(\text{R}) + (\text{W}) + (\text{MP})$.
4. $wK_\sigma(\text{R}) < wK_\sigma(\text{R}) + (\text{W})$.

Proof (1)   The first inequality follows from (2). The second inequality follows because there are models of (R) where (MP) does not hold. For instance, the model used in the proof of Proposition 5.2. To prove the third inequality, since (R) is a theorem of $wK_\sigma(\text{MP})$, it is enough to show that $wK_\sigma(\text{R}) + (\text{MP})$ and $wK_\sigma(\text{MP})$ are different. We will use Lemma 5.15. Let us consider the frame $\mathcal{F} = \langle W, R \rangle$ where $W = \{w_1, w_2, w_3\}$ and $R = \{\langle w_1, w_2 \rangle, \langle w_3, w_3 \rangle\}$. Consider the set A whose elements are $0 = \varnothing$, $a = \{w_1\}$, $b = \{w_2\}$, $c = \{w_1, w_2\}$, and $1 = W$. This set is closed under unions, intersections, and the operation $\Rightarrow_R$ that interprets $\to$, displayed in the following table.

$\Rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	0	1	1
b	0	c	1	1	1
c	0	0	0	1	1
1	0	0	0	c	1

Let $F = \{c, 1\}$. It is easy to check that for every $X, Y \in A$, $X \cap (X \Rightarrow_R Y)$ $\Rightarrow_R Y \in F$, $W \Rightarrow_R X \in F$, and that if $X, X \Rightarrow_R Y \in F$, then $Y \in F$. Thus for any valuation V taking values in A, modus ponens and the rules $\varnothing \vdash \top^n \to (p \wedge (p \to q) \to q)$ are F-valid in $\langle W, R, V \rangle$. Now we consider a valuation V such that $V(p) = V(r) = a$ and $V(q) = b$. Then, $V(p \vee q), V(p \vee (q \to r)) \in F$ but $V(p \vee r) \notin F$. Hence, by Lemma 5.15 and Fact 5.16, we obtain that $p \vee q, p \vee (q \to r) \nvdash_{wK_\sigma(\text{R})+(\text{MP})} p \vee r$. But it is easy to prove that $p \vee q, p \vee (q \to r) \vdash_{wK_\sigma(\text{MP})} p \vee r$.

(2) In the proof of (2) of Proposition 5.13 we have shown that $\top \to ((p \wedge (p \to q)) \to q)$ is not a theorem of $wK_\sigma + (\text{R}) + (\text{MP})$. But this formula is easily seen to be a theorem of $wK_\sigma(\text{R})$, thus of $wK_\sigma(\text{R}) + (\text{MP})$.

(3) The equality $wK_\sigma + (\text{W}) + (\text{R}) = wK_\sigma(\text{R}) + (\text{W})$ follows from the fact that $wK_\sigma(\text{R}) = wK_\sigma + \{\varnothing \vdash \top^n \to (p \wedge (p \to q) \to q) : n \in \omega\}$. Thus $wK_\sigma(\text{R}) + (\text{W})$ is $wK_\sigma + (\text{W}) + (\text{R})$ because from (R) and (W) all the formulas of the form $\top^n \to (p \wedge (p \to q) \to q)$ can be proved. To show the inequality, it is clear that $wK_\sigma + (\text{W}) + (\text{MP}) + (\text{R})$, which is $K_\sigma + (\text{R})$, is an extension of $wK_\sigma + (\text{W}) + (\text{R})$. That they are different follows from the fact that the second logic is non-protoalgebraic and the first one is algebraizable.

(4) If $wK_\sigma(\text{R}) = wK_\sigma(\text{R}) + (\text{W})$, $wK_\sigma(\text{R}) + (\text{MP}) = wK_\sigma(\text{R}) + (\text{W}) + (\text{MP})$. This last logic is $wK_\sigma(\text{MP}) + (\text{W})$. We know from (1) that $wK_\sigma(\text{R}) + (\text{MP}) < wK_\sigma(\text{MP})$ and we also know that $wK_\sigma(\text{MP}) < wK_\sigma(\text{MP}) + (\text{W})$. Thus we have (4). $\square$

Proposition 5.18   *The logics $wK_\sigma(\text{R})$ and $wK_\sigma(\text{R}) + (\text{W})$ are non-protoalgebraic. The logics with (MP) are protoalgebraic and the logic $wK_\sigma(\text{R}) + (\text{W}) + (\text{MP})$ is algebraizable.*

Proof   The matrices in the proof of Proposition 4.8 can also be used in this case to show that $wK_\sigma(\text{R})$ and $wK_\sigma(\text{R}) + (\text{W})$ are non-protoalgebraic. The reason why the logics with (MP) are protoalgebraic is the same as the one given in Proposition 5.10. Finally, $wK_\sigma(\text{R}) + (\text{W}) + (\text{MP})$ is algebraizable because it is an extension of the algebraizable logic K_σ. $\square$

To conclude this section let us state the relation that the logics $wK_\sigma(\text{MP})$ and $wK_\sigma(\text{MP}) + (\text{W})$ have to the normal modal logic **KT**. Let us denote by *lKT* the local consequence in the modal language associated with **KT** and by *gKT* the global consequence; they are, respectively, the local and global consequence relations determined by the class of all reflexive frames in the language of modal logic.

Theorem 5.19   *For every set of subintuitionistic formulas Γ and every subintuitionistic formula φ,*

$$\Gamma \vdash_{wK_\sigma(\text{MP})} \varphi \;\; \text{iff} \;\; \sigma[\Gamma] \vdash_{lKT} \sigma(\varphi) \;\; \text{and} \;\; \Gamma \vdash_{wK_\sigma(\text{MP})+(\text{W})} \varphi \;\; \text{iff} \;\; \sigma[\Gamma] \vdash_{gKT} \sigma(\varphi).$$

Proof   The proof of the first equivalence is analogous to the semantic proof of Theorem 4.7 and the proof of the second one to the proof of Theorem 4.14. $\square$

6 Transitivity

In this section we turn to the study of the logics determined by the class of transitive frames. We consider the rule

(RT) $\varphi \to \psi \vdash \delta \to (\varphi \to \psi)$

and the axiom

(T) $(\varphi \to \psi) \to (\delta \to (\varphi \to \psi)).$

Notice that the rule (RT) is a particular case of the rule (W) so the logics $wK_\sigma + (W)$ and $wK_\sigma(W)$ have this rule. Moreover, by (DT_0), the formulas of the form (T) are theorems of $wK_\sigma(W)$.

First we set up the relation between transitivity, (RT) and (T).

Proposition 6.1   *Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. Then R is transitive if and only if the rule* (RT) *holds in $\mathcal{F}$.*

Proof   It is very easy to check that if R is transitive (RT) holds in the frame. To prove the converse, assume that (RT) holds in $\langle W, R \rangle$. If xRy and yRz, to see that xRz let us consider any valuation V such that $V(p) = R(y)$ and $V(q) = R(x)$. Then $R(x) \cap V(p) \subseteq V(q)$, so $x \in V(p \to q)$. Therefore, $x \in V(\top \to (p \to q))$. Hence, $R(x) \cap W \subseteq V(p \to q)$. Therefore, as xRy, $y \in V(p \to q)$. Hence, $R(y) \cap V(p) \subseteq V(q)$. So $R(y) \subseteq R(x)$. Thus, since yRz, xRz.   □

Proposition 6.2   (T) *is valid in all transitive Kripke frames but it is also valid in some nontransitive Kripke frame.*

Proof   The first part is easily proved. To prove the second part consider the frame $\langle \{a, b, c\}, \{\langle a, b \rangle, \langle b, c \rangle\} \rangle$, which is nontransitive. Tedious checking shows that (T) is valid in that frame.   □

Now we can proceed to consider the different logics we obtain by adding (RT) to the logics of the above sections. As in the case of reflexivity we find an incompleteness phenomena.

Lemma 6.3   *The set of theorems of the logic wK_σ is closed under the rule* (RT). *Therefore $wK_\sigma + (RT)$ has the same theorems as wK_σ.*

Proof   If $\varphi \to \psi$ is a theorem, then for every Kripke model $\langle W, R, V \rangle$, $V(\varphi \to \psi) = W$. Let us see that $V(\delta \to (\varphi \to \psi)) = W$. Let $x \in W$. Then, as $V(\varphi \to \psi) = W$, $R(x) \cap V(\delta) \subseteq V(\varphi \to \psi)$. Hence, $x \in V(\delta \to (\varphi \to \psi)) = W$. This proves the first part of the lemma. To prove the second part recall that $wK_\sigma + (RT)$ is defined as the logic obtained by the Hilbert calculus whose rules are all the rules of wK_σ plus the rule (RT). It is enough to see that any proof of a theorem of $wK_\sigma + (RT)$ produces a theorem of wK_σ because the set of theorems of wK_σ is closed under (RT).   □

Theorem 6.4   *The logic $wK_\sigma + (RT)$ is frame incomplete.*

Proof   The frames in which all the rules of the logic are valid are the transitive frames. The formula $(p \to q) \to (r \to (p \to q))$ is valid in every transitive frame but is not a theorem of $wK_\sigma + (RT)$ since it is not a theorem of wK_σ: there are nontransitive models where it is not valid.   □

In the sequel we deal first of all with $wK_\sigma(\mathrm{RT})$, which is the local consequence relation of all the transitive frames. The proof of this fact is analogous to the completeness proof for wK_σ. The only adjustment needed is to check that the canonical model is transitive.

Theorem 6.5 $wK_\sigma(\mathrm{RT})$ *is the local consequence defined by the class of all transitive frames.*

Proof The canonical Kripke frame for $wK_\sigma(\mathrm{RT})$ is transitive. Assume that $\Gamma_1 R_c \Gamma_2$ and $\Gamma_2 R_c \Gamma_3$. To show that $\Gamma_1 R_c \Gamma_3$, assume that $\varphi \to \psi \in \Gamma_1$ and $\varphi \in \Gamma_3$. Then by (RT), $\top \to (\varphi \to \psi) \in \Gamma_1$. Since $\top \in \Gamma_2$, $\varphi \to \psi \in \Gamma_2$. Since $\Gamma_2 R_c \Gamma_3$ we conclude that $\psi \in \Gamma_3$. $\qquad\square$

Notice that although (RT) is a rule of $wK_\sigma + (\mathrm{W})$, the logic $wK_\sigma(\mathrm{RT}) + (\mathrm{W})$ is not equal to $wK_\sigma + (\mathrm{W})$. The point is that (T) is a theorem of $wK_\sigma(\mathrm{RT}) + (\mathrm{W})$, but is not a theorem of $wK_\sigma + (\mathrm{W})$. As in the case of wK_σ and $wK_\sigma(\mathrm{MP})$ we can consider the logic $wK_\sigma(\mathrm{RT}) + (\mathrm{W})$ (alternatively $wK_\sigma(\mathrm{RT}) + (\mathrm{N})$). By Corollary 2.5 it is the global consequence defined by the class of all transitive Kripke frames.

Theorem 6.6 *The logic* $wK_\sigma(\mathrm{RT}) + (\mathrm{W})$ *is the global consequence defined by the class of all transitive Kripke frames.*

Summarizing the relations between the logics treated up to now in this section we obtain the following proposition.

Proposition 6.7
1. $wK_\sigma + (\mathrm{RT}) < wK_\sigma(\mathrm{RT}) < wK_\sigma(\mathrm{RT}) + (\mathrm{W})$.
2. $wK_\sigma + (\mathrm{RT}) < wK_\sigma + (\mathrm{RT}) + (\mathrm{W}) = wK_\sigma + (\mathrm{W}) < wK_\sigma(\mathrm{RT}) + (\mathrm{W})$.

Proof (1) As we have seen in the proof of Theorem 6.4, the formula $(p \to q) \to (r \to (p \to q))$ is not a theorem of $wK_\sigma + (\mathrm{RT})$, but using (DT_0) it is clear that it is a theorem of $wK_\sigma(\mathrm{RT})$. Moreover since (W) is not valid in the class of all transitive frames the other inequality follows.

(2) The first inequality holds because there are transitive models where (W) is not valid, and all the rules of $wK_\sigma + (\mathrm{RT})$ are valid in transitive models. The equality is clear since (RT) is a special case of (W). The last inequality follows because the theorems of $wK_\sigma + (\mathrm{W})$ are the theorems of wK_σ and the formulas of the form (T) are theorems of $wK_\sigma(\mathrm{RT}) + (\mathrm{W})$. $\qquad\square$

It is clear that the formulas of the form (T) are theorems of the logics $wK_\sigma(\mathrm{RT})$ and $wK_\sigma(\mathrm{RT}) + (\mathrm{W})$ since they are valid in all transitive Kripke frames. It makes sense to consider also the logics $wK_\sigma + (\mathrm{T})$, $wK_\sigma + (\mathrm{RT}) + (\mathrm{T})$, and $wK_\sigma + (\mathrm{W}) + (\mathrm{T})$. For these logics the known situation is the following.

Proposition 6.8
1. $wK_\sigma + (\mathrm{T}) < wK_\sigma + (\mathrm{RT}) + (\mathrm{T}) < wK_\sigma + (\mathrm{W}) + (\mathrm{T})$.
2. $wK_\sigma + (\mathrm{RT}) + (\mathrm{T}) < wK_\sigma(\mathrm{RT})$.

Proof (1) The first inequality follows from the fact that there are frames where (T) is valid but (RT) is not. The frame in the proof of Proposition 6.2 is one of them. The second inequality follows from the fact that all the rules of $wK_\sigma + (\mathrm{RT}) + (\mathrm{T})$

are valid in transitive models and the existence of transitive models where (W) is not valid.

(2) That $wK_\sigma + (RT) + (T) \leq wK_\sigma(RT)$ is immediate. To show that these logics are different we will use Lemma 5.15. Let $W = \{w_1, w_2, w_3\}$ and $R = \{\langle w_1, w_1\rangle, \langle w_2, w_1\rangle, \langle w_1, w_3\rangle\}$. Consider the set $A = \{0, 1, a, b, c\}$ where $0 = \varnothing$, $1 = W$, $a = \{w_1, w_3\}$, $b = \{w_2, w_3\}$, and $c = \{w_3\}$. This set is closed under unions, intersections, and the operation $\Rightarrow_R$ that interprets $\rightarrow$ whose table when restricted to A is

$\Rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	b	1	1	1	1
a	c	c	1	c	1
b	b	1	1	1	1
1	c	c	1	c	1

It is clear that for every $X, Y, Z \in A$, if $X \Rightarrow_R Y = W$, then $Z \Rightarrow_R (X \Rightarrow_R Y) = W$ and moreover, for every $X, Y, Z \in A$, $(X \Rightarrow_R Y) \Rightarrow_R (Z \Rightarrow_R (X \Rightarrow_R Y)) = W$. Thus for any valuation V taking values in A, the rules (RT) and (T) are $\{1\}$-valid. Consider any valuation taking values in A such that $V(p) = c$, $V(q) = 0$, and $V(r) = a$. Then $V((p \rightarrow q) \vee r) = 1$ and $V((\top \rightarrow (p \rightarrow q)) \vee r) = a$. Hence by Lemma 5.15, $(p \rightarrow q) \vee r \nvdash_{wK_\sigma + (RT) + (T)} (\top \rightarrow (p \rightarrow q)) \vee r$. But it is easy to see that $(p \rightarrow q) \vee r \vdash_{wK_\sigma(RT)} (\top \rightarrow (p \rightarrow q)) \vee r$.[2]   □

Another logic worth considering is the logic $K_\sigma + (T)$, that is, $wK_\sigma + (MP) + (W) + (T)$. The set of its theorems has been studied in [4] where it is proved that it is the set of formulas valid in every transitive frame. Thus this logic has the same theorems as $wK_\sigma(RT)$. In this paper we will not explore other combinations of the rules (MP) and (RT) and the axioms (R) and (T).

   Concerning the classification of the logics considered in this section let us state that none of them except $K_\sigma + (T)$, which is algebraizable, is protoalgebraic. $K_\sigma + (T)$ is algebraizable because it is an extension of the algebraizable logic K_σ.

Proposition 6.9 *None of the logics considered with the rule* (RT) *or the axiom* (T)*, besides $K_\sigma + (T)$, which is algebraizable, is protoalgebraic.*

Proof Using the matrices in the proof of Theorem 4.8 it can be shown that none of the logics $wK_\sigma + (W) + (T)$, $wK_\sigma(RT) + (W)$, and $wK_\sigma + (W) + (T)$ is protoalgebraic. Each of the other logics has $wK_\sigma(RT) + (W)$ or $wK_\sigma + (W) + (T)$ as an extension. So they cannot be protoalgebraic either because any extension of a protoalgebraic logic is protoalgebraic.   □

To conclude this section, we state in the next theorem the relation between the modal logic **K4** and the logics $wK_\sigma(RT)$ and $wK_\sigma(RT) + (W)$ established by the translation σ. Let us denote by *lK4* the local consequence relation associated with **K4** and by *gK4* the corresponding global consequence relation; they are, respectively, the local and global consequence relations determined by the class of all transitive frames in the modal language.

Theorem 6.10   *For every set of subintuitionistic formulas Γ and every subintuition-istic formula φ,*

$$\Gamma \vdash_{wK_\sigma(\mathrm{RT})} \varphi \ \textit{iff} \ \sigma[\Gamma] \vdash_{lK4} \sigma(\varphi) \ \textit{and} \ \Gamma \vdash_{wK_\sigma(\mathrm{RT})+(\mathrm{W})} \varphi \ \textit{iff} \ \sigma[\Gamma] \vdash_{gK4} \sigma(\varphi).$$

Proof   The proof of the first equivalence is analogous to the semantic proof of Theorem 4.7 and the proof of the second one to the proof of Theorem 4.14. $\square$

7 Reflexivity plus Transitivity

We can go on with different combinations of the rules (RT), (MP), and the axioms (T) and (R), adding them to wK_σ and sK_σ to obtain new logics or adding them to the Gentzen calculus for wK_σ with the same purpose. In this section we will concentrate mainly on two of them, the logics that correspond to the local and to the global consequence relations of the class of all reflexive and transitive frames.

From Propositions 5.1 and 6.1 and the proofs of Theorems 5.5 and 6.5 the next theorem easily follows.

Theorem 7.1   *The logic $wK_\sigma(\mathrm{MP}, \mathrm{RT})$ is the local consequence of the class of all reflexive and transitive frames.*

In the same way as we did for the logics wK_σ, $wK_\sigma(\mathrm{MP})$, and $wK_\sigma(\mathrm{RT})$ we can prove that adding (W) to $wK_\sigma(\mathrm{MP}, \mathrm{RT})$ we obtain the global consequence of the class of all reflexive and transitive frames.

Theorem 7.2   *The logic $wK_\sigma(\mathrm{MP}, \mathrm{RT}) + (\mathrm{W})$ is the global consequence of the class of all reflexive and transitive frames.*

We consider in addition the logic $wK_\sigma + (\mathrm{MP}) + (\mathrm{RT})$. It has the same theorems as wK_σ because the set of theorems of this logic is closed, as we already know, under (MP) and (RT). So here we also have an incompleteness phenomena: The logic $wK_\sigma + (\mathrm{MP}) + (\mathrm{RT})$ is not frame complete. Its class of frames is the class of all reflexive and transitive frames, and neither $(p \wedge (p \to q)) \to q$ nor $(p \to q) \to (r \to (p \to q))$ belong to the set of its theorems.

Theorem 7.3   *The logics considered in this section are protoalgebraic.*

Proof   All of them have (MP). $\square$

Theorem 7.4   *The logic $wK_\sigma(\mathrm{MP}, \mathrm{RT})$ is finitely equivalential with the set of equivalence formulas $\{p \to q, q \to p\}$.*

Proof   The set of formulas $\{p \to q, q \to p\}$, which we abbreviate by $p \leftrightarrow q$, is a set of equivalence formulas for $wK_\sigma(\mathrm{MP}, \mathrm{RT})$. We prove it by showing that it has the required syntactical properties. We have (MP), therefore, $p, p \leftrightarrow q \vdash_{wK_\sigma(\mathrm{MP},\mathrm{RT})} q$. $p \to p$ is a theorem, thus

$$\vdash_{wK_\sigma(\mathrm{MP},\mathrm{RT})} p \leftrightarrow p.$$

Finally it is easy to check that

$$p \leftrightarrow p', q \leftrightarrow q' \vdash_{wK_\sigma(\mathrm{MP},\mathrm{RT})} p \to q \leftrightarrow p' \to q'$$

holds. To prove it semantically one has to make an essential use of transitivity. Moreover, it is clear that

$$p \leftrightarrow p', q \leftrightarrow q' \vdash_{wK_\sigma(\mathrm{MP},\mathrm{RT})} p \vee q \leftrightarrow p' \vee q'$$

and

$$p \leftrightarrow p', q \leftrightarrow q' \vdash_{wK_\sigma(\text{MP,RT})} p \wedge q \leftrightarrow p' \wedge q'.$$

<div align="right">□</div>

Theorem 7.5   *The logic $wK_\sigma(\text{MP, RT})+(\text{W})$ is algebraizable with the set of equivalence formulas $\{p \rightarrow q, q \rightarrow p\}$ and the set of defining equations $\{p \approx \top\}$.*

Proof   By the above theorem, the logic $wK_\sigma(\text{MP, RT})$ is equivalential with the set of equivalence formulas $\{p \rightarrow q, q \rightarrow p\}$. Thus, since $wK_\sigma(\text{MP, RT}) + (\text{W})$ is an extension of $wK_\sigma(\text{MP, RT})$, it is equivalential with the same set of equivalence formulas. To prove that it is algebraizable we can show that it has the syntactical properties required in the definition we gave in Section 2. Since it is equivalential with $\{p \rightarrow q, q \rightarrow p\}$ we need only to prove that

$$p \vdash_{wK_\sigma(\text{MP,RT})+(\text{W})} p \leftrightarrow \top$$

and

$$p \leftrightarrow \top \vdash_{wK_\sigma(\text{MP,RT})+(\text{W})} p.$$

The second statement follows by (MP) and the fact that $\top$ is a theorem. The first one follows because $\top$ is true at every point of every model and $wK_\sigma(\text{MP, RT}) + (\text{W})$ is the global consequence relation of the class of all reflexive and transitive models.   □

As in the above sections we can state the relationship given by the translation σ between the logics $wK_\sigma(\text{MP, RT})$ and $wK_\sigma(\text{MP, RT}) + (\text{W})$ and a modal logic. Let us denote by *lS4* the local consequence relation between modal formulas associated with **S4**, and by *gS4* the corresponding global consequence relation.

Theorem 7.6   *For every set of subintuitionistic formulas Γ and every subintuitionistic formula φ,*

$$\Gamma \vdash_{wK_\sigma(\text{MP,RT})} \varphi \ \ \textit{iff} \ \ \sigma[\Gamma] \vdash_{lS4} \sigma(\varphi)$$

and

$$\Gamma \vdash_{wK_\sigma(\text{MP,RT})+(\text{W})} \varphi \ \ \textit{iff} \ \ \sigma[\Gamma] \vdash_{gS4} \sigma(\varphi).$$

To conclude, notice that $wK_\sigma(\text{MP, RT})+(\text{W})$ is not Intuitionistic logic. The formula $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$ is not one of its theorems since there are reflexive and transitive models in which it is not valid.

8 Visser's Propositional Logic

In [17], by means of a natural deduction calculus, Visser defined a logic he called Basic Propositional Logic and which we will call, following [13], Visser's Propositional Logic, VPL for short. He proved that it is the local consequence relation defined by the class of all transitive models with a valuation V with the property that for every propositional variable p, if $x \in V(p)$ and y is accessible from x then $y \in V(p)$. VPL turns out to be the logic defined by the Gentzen calculus for wK_σ plus the following rule

$$(\text{DT}_1) \qquad \frac{\varphi, \psi \vdash \delta}{\varphi \vdash \psi \rightarrow \delta}$$

Let us call the logic defined by this Gentzen calculus $wK_\sigma(DT_1)$. We will prove that it is VPL and will study some of its properties. $wK_\sigma(DT_1)$ does not have (MP) as a derivable rule nor (R) as a theorem. Notice that if we add (DT_1) to the Gentzen calculus $\mathcal{G}$, then the rule (W) is derivable: the sequent $\varphi, \delta \vdash \varphi$ is derivable, so by (DT_1) we obtain the sequent $\varphi \vdash \delta \to \varphi$.

We say that a Kripke model $\langle W, R, V \rangle$ is a *model of a Gentzen-style rule*

$$\frac{\Gamma_0 \vdash \varphi_0, \ldots, \Gamma_{n-1} \vdash \varphi_{n-1}}{\Gamma \vdash \varphi}$$

if, for every substitution e, $V(e[\Gamma]) \subseteq V(e(\varphi))$ whenever for every $i < n$, $V(e[\Gamma_i]) \subseteq V(e(\varphi_i))$, and we say that *the rule is valid in a frame* if every model in the frame is a model of the rule.

A valuation V on a frame $\langle W, R \rangle$ is said to be *R-persistent* (or *R*-increasing, or *R*-upclosed) if for every formula φ, $V(\varphi)$ is an *R*-persistent set, that is, if $x \in V(\varphi)$ and xRy, then $y \in V(\varphi)$. It is a well-known fact that if a frame is transitive and V is a valuation such that for every propositional variable p, $V(p)$ is *R*-persistent, then V is *R*-persistent. However, there are *R*-persistent valuations in nontransitive frames. We will see that if $\langle W, R, V \rangle$ is a Kripke model of the rule (DT_1) then V is *R*-persistent. Moreover, any Kripke model satisfying this condition is a Kripke model of the rule. The same situation holds for the rule (W).

Proposition 8.1   *The following conditions are equivalent for any Kripke model* $\mathcal{M} = \langle W, R, V \rangle$:

1. $\mathcal{M}$ *is a model of* (DT_1);
2. $\mathcal{M}$ *is a model of* (W);
3. *for every formula* φ, $V(\varphi)$ *is R-persistent.*

Proof $(1) \Longrightarrow (2)$ Because (W) follows from (DT_1).

$(2) \Longrightarrow (3)$ Assume that $\mathcal{M}$ is a model of (W). Then $V(\varphi) \subseteq V(\top \to \varphi)$. In order to see that $V(\varphi)$ is *R*-persistent let $x \in V(\varphi)$ and let xRy. Then, as $x \in V(\top \to \varphi)$ and $y \in V(\top)$, $y \in V(\varphi)$.

$(3) \Longrightarrow (1)$ Let $\mathcal{M}$ be a model such that for every formula φ, $V(\varphi)$ is *R*-persistent. Assume that $V(\varphi) \cap V(\psi) \subseteq V(\delta)$. Let $x \in V(\varphi)$ and let $y \in V(\psi)$ be such that xRy. Since $V(\varphi)$ is *R*-persistent, $y \in V(\varphi)$. Therefore $y \in V(\delta)$. Hence $x \in V(\psi \to \delta)$. So we can conclude that (DT_1) holds. $\square$

Proposition 8.2   *The following conditions are equivalent for any Kripke frame* $\langle W, R \rangle$:

1. (DT_1) *is valid on* $\langle W, R \rangle$;
2. (W) *is valid on* $\langle W, R \rangle$;
3. *all valuations on the frame are R-persistent.*

Proof   The proof follows from Proposition 8.1. If (DT_1) is valid in $\langle W, R \rangle$, then every model on $\langle W, R \rangle$ is a model of (DT_1), so a model of (W). Hence (W) is valid on the frame. If this holds, every valuation is *R*-persistent. On the other hand, if this last property holds, (DT_1) must be valid in the frame. $\square$

Corollary 8.3   *The only frames in which* (DT_1) *(equivalently* (W)*) is valid are the frames whose relation is a subset of the identity relation.*

Proof   If aRb and a is different from b, no valuation with $V(p) = \{a\}$ is R-persistent. $\square$

This corollary shows that when we are interested in the condition 'the valuation is R-persistent', the frame perspective is not the right one. It is sensible to restrict the valuations on frames to the R-persistent ones. Then we have the following completeness result.

Theorem 8.4 $\Gamma \vdash_{wK_\sigma(\mathrm{DT}_1)} \varphi$ *if and only if for every Kripke frame $\langle W, R \rangle$ and any R-persistent valuation V, $V(\Gamma) \subseteq V(\varphi)$.*

Proof   The implication from left to right follows from Proposition 8.2. To prove the other implication it is enough to argue as in the proof of the completeness theorem for the logic wK_σ, proving that the canonical model obtained is such that V_c is R_c-persistent. This follows from the fact that for every formula φ, $\varphi \vdash_{wK_\sigma(\mathrm{DT}_1)} \top \to \varphi$. Assume that $\Gamma \in V_c(\varphi)$ and $\Gamma R_c \Gamma'$. Then $\varphi \in \Gamma$. Therefore, $\top \to \varphi \in \Gamma$. As $\top \in \Gamma'$, $\varphi \in \Gamma'$. Hence, $\Gamma' \in V_c(\varphi)$. $\square$

The same proof works for the logic $wK_\sigma(\mathrm{W})$.

Theorem 8.5 $\Gamma \vdash_{wK_\sigma(\mathrm{W})} \varphi$ *if and only if for every Kripke frame $\langle W, R \rangle$ and any R-persistent valuation V, $V(\Gamma) \subseteq V(\varphi)$.*

Corollary 8.6   *The logics $wK_\sigma(\mathrm{DT}_1)$ and $wK_\sigma(\mathrm{W})$ are the same.*

Two interesting properties of the canonical model for $wK_\sigma(\mathrm{DT}_1)$ are the following.

Proposition 8.7   *The relation R_c of the canonical model for $wK_\sigma(\mathrm{DT}_1)$ is transitive and is included in the subset relation, that is, if $\Gamma R_c \Delta$, then $\Gamma \subseteq \Delta$.*

Proof   First we show that R_c is transitive. Assume $\Gamma_1 R_c \Gamma_2$ and $\Gamma_2 R_c \Gamma_3$. Then assume that $\varphi \to \psi \in \Gamma_1$ and $\varphi \in \Gamma_3$. Now, $\varphi \to \psi \vdash_{wK_\sigma(\mathrm{DT}_1)} \top \to (\varphi \to \psi)$; therefore $\top \to (\varphi \to \psi) \in \Gamma_1$. Moreover, $\top \in \Gamma_2$. Hence we have $\varphi \to \psi \in \Gamma_2$. Therefore, $\psi \in \Gamma_3$. We conclude that $\Gamma_1 R_c \Gamma_3$.

   To see that R_c is included in the subset relation, assume that $\Delta R_c \Gamma$ and $\varphi \in \Delta$, then since $\varphi \vdash_{wK_\sigma(\mathrm{DT}_1)} \top \to \varphi$, $\top \to \varphi \in \Delta$. As $\top \in \Gamma$, $\varphi \in \Gamma$. $\square$

The fact that the relation of the canonical model of $wK_\sigma(\mathrm{DT}_1)$ is transitive shows that $wK_\sigma(\mathrm{DT}_1)$ is complete relative to the class of models $\langle W, R, V \rangle$ where R is transitive and V is R-persistent. Moreover, in transitive frames it holds that a valuation V is R-persistent if and only if for every propositional variable p, $V(p)$ is R-persistent. Let us call a valuation with this property *R-persistent for the propositional variables*. Thus we have the following completeness results.

Theorem 8.8   *The following conditions are equivalent:*
   1. $\Gamma \vdash_{wK_\sigma(\mathrm{DT}_1)} \varphi$;
   2. *for every transitive Kripke frame $\langle W, R \rangle$ and any R-persistent valuation V, $V(\Gamma) \subseteq V(\varphi)$;*
   3. *for every transitive Kripke frame $\langle W, R \rangle$ and any valuation V that is R-persistent for the propositional variables, $V(\Gamma) \subseteq V(\varphi)$.*

The class of models for VPL considered by Visser is the class of transitive Kripke models with a valuation R-persistent for the variables. He proved that VPL is exactly the local consequence relation defined by this class of models.

Corollary 8.9 *The logic $wK_\sigma(\mathrm{DT}_1)$ is Visser's propositional logic.*

Now we turn to the characterization of the global consequence defined by the class of Kripke models where V is R-persistent.

Proposition 8.10 $\{\top^n \to \varphi : n \in \omega, \varphi \in \Gamma\} \vdash_{wK_\sigma(\mathrm{DT}_1)} \psi$ *if and only if for every Kripke model $\langle W, R, V \rangle$ with V R-persistent and such that $V(\Gamma) = W$, it holds that $V(\varphi) = W$.*

Proof The proof is as in the proof of item 2 in Proposition 2.3. □

Theorem 8.11 $wk_\sigma(\mathrm{DT}_1)$ *is also the global consequence defined by the class of Kripke models with an R-persistent valuation.*

Proof Because (N) is a rule of $wK_\sigma(\mathrm{DT}_1)$ we have

$$\{\top^n \to \varphi : n \in \omega, \varphi \in \Gamma\} \vdash_{wK_\sigma(\mathrm{DT}_1)} \psi \text{ iff } \Gamma \vdash_{wK_\sigma(\mathrm{DT}_1)} \psi.$$

Therefore, by Proposition 8.10, we obtain the desired conclusion. □

Now we turn to the study of the logic $wK_\sigma(\mathrm{DT}_1, \mathrm{MP})$, the logic defined by the Gentzen calculus $wK_\sigma(\mathrm{DT}_1)$ augmented with the modus ponens rule. First we observe that the logics $wK_\sigma(\mathrm{DT}_1, \mathrm{MP})$ and $wK_\sigma(\mathrm{DT}_1) + (\mathrm{MP})$ are equal.

Proposition 8.12 *The logics $wK_\sigma(\mathrm{DT}_1, \mathrm{MP})$ and $wK_\sigma(\mathrm{DT}_1) + (\mathrm{MP})$ are equal.*

Proof Using Theorem 8.8 it can be seen that if $\varphi_1, \ldots, \varphi_n \vdash_{wK_\sigma(\mathrm{DT}_1)} \psi$ then for every formula δ, $\delta \to \varphi_1, \ldots, \delta \to \varphi_n \vdash_{wK_\sigma(\mathrm{DT}_1)} \delta \to \psi$. Then, using a standard argument it can be proved that (DT_1) holds for $wk_\sigma(\mathrm{DT}_1) + (\mathrm{MP})$, namely, that if $\varphi, \psi \vdash_{wK_\sigma(\mathrm{DT}_1)+(\mathrm{MP})} \delta$, then $\varphi \vdash_{wK_\sigma(\mathrm{DT}_1)+(\mathrm{MP})} \psi \to \delta$. This implies that $wk_\sigma(\mathrm{DT}_1, \mathrm{MP}) = wK_\sigma(\mathrm{DT}_1) + (\mathrm{MP})$. □

Proposition 8.13 *The relation R_c of the canonical model of the logic $wK_\sigma(\mathrm{DT}_1, \mathrm{MP})$ is reflexive. Moreover, $R_c = \subseteq$.*

Proof Assume $\varphi \to \psi \in \Gamma$ and $\varphi \in \Gamma$. By (MP), $\psi \in \Gamma$. Therefore $\Gamma R_c \Gamma$. We conclude that R_c is reflexive. Moreover, we know that R_c is included in the inclusion relation. Assume now that $\Gamma \subseteq \Delta$. If $\varphi \to \psi \in \Gamma$ and $\varphi \in \Delta$, since $\varphi \to \psi \in \Delta$ too, by (MP), $\psi \in \Delta$. Hence $\Gamma R_c \Delta$. Thus, $R_c = \subseteq$. □

Using the above results this proposition has the following consequence.

Theorem 8.14
1. $wK_\sigma(\mathrm{DT}_1, \mathrm{MP})$ *is the local consequence relation defined by the class of reflexive Kripke models with an R-persistent valuation.*
2. $wK_\sigma(\mathrm{DT}_1, \mathrm{MP})$ *is the local consequence relation defined by the class of reflexive and transitive Kripke models with a valuation which is R-persistent for the propositional variables.*

Since, by Theorem 8.14(2), $wK_\sigma(\mathrm{DT}_1, \mathrm{MP})$ is precisely intuitionistic logic, we obtain that the class of models $\langle W, R, V \rangle$ with R reflexive and V R-persistent is a sound and complete semantics for Intuitionistic logic too.

Theorem 8.15 $wK_\sigma(\mathrm{DT}_1, \mathrm{MP})$, *namely, intuitionistic logic, is also the global consequence defined by the class of all reflexive Kripke models with an R-persistent valuation.*

Proof Because (W) is a rule of $wK_\sigma(\text{DT}_1, \text{MP})$ and we can argue as in Proposition 8.10 and Theorem 8.11. $\qquad\square$

We have seen that $wK_\sigma(\text{DT}_1)$ is the logic $wK_\sigma(\text{W})$. As a final observation let us state the relationship between $wK_\sigma(\text{W})$, $wK_\sigma + (\text{W})$, $wK_\sigma + (\text{W}) + (\text{R})$, $wK_\sigma(\text{W}) + (\text{R})$, and $wK_\sigma + (\text{W}) + (\text{MP})$.

Proposition 8.16

1. $wK_\sigma + (\text{W}) < wK_\sigma(\text{W}) < wK_\sigma(\text{W}) + (\text{R})$.
2. $wK_\sigma + (\text{W}) < wK_\sigma + (\text{W}) + (\text{R}) < wK_\sigma(\text{W}) + (\text{R})$.
3. $wK_\sigma(\text{W}) < wK_\sigma(\text{W}) + (\text{MP})$.
4. $wK_\sigma(\text{W}) + (\text{R}) < wK_\sigma(\text{W}) + (\text{MP})$.

Proof (1) The second inequality holds because (R) is not a theorem of $wK_\sigma(\text{W})$. The model with $W = \{a, b\}$, $R = \{\langle a, b\rangle\}$, and valuation such that $V(p) = \{b\}$ and $V(q) = \varnothing$ shows it. To prove the first inequality, we recall that the logics wK_σ and $wK_\sigma + (\text{W})$ have the same theorems. The formula $p \to (q \to p)$ is a theorem of $wK_{\text{W})}$ (use (W) and (DT$_0$)) but it is not valid in every Kripke model. Take $W = \{a, b, c\}$, $R = \text{Id}_W \cup \{\langle a, b\rangle, \langle b, c\rangle\}$, and V any valuation such that $V(p) = \{b\}$ and $V(q) = \{c\}$. Then $a \notin V(p \to (q \to p))$ because $b \in R(a) \cap V(p)$ and $R(b) \cap V(q) \nsubseteq V(p)$, which implies that $b \notin V(q \to p)$.

(2) That $wK_\sigma + (\text{W}) < wK_\sigma + (\text{W}) + (\text{R})$ was already proved in Section 5 and also follows from (1). Moreover, $p \to (q \to p)$ is not a theorem of $wK_\sigma + (\text{W}) + (\text{R})$ since the theorems of this logic are valid in every reflexive Kripke model and $p \to (q \to p)$. There are reflexive Kripke models where this formula is not valid, for instance the model used in (1).

(3) It holds because (MP) is not valid in every transitive Kripke model with an R-persistent valuation, as is easily shown.

(4) It is enough to find a model $\langle W, R, V\rangle$ such that for every set of formulas Γ and every formula φ such that $\Gamma \vdash_{wK_\sigma(\text{W})+(\text{R})} \varphi$, $V(\Gamma) \subseteq V(\varphi)$ and where (MP) does not (locally) hold. Let $W = \{a, b\}$, $R = \{\langle a, b\rangle, \langle b, b\rangle\}$, and V be such that $V(p) = W$ and $V(q) = \{b\}$. Since this model is transitive and R-persistent, if $\Gamma \vdash_{wK_\sigma(\text{W})} \varphi$, $V(\Gamma) \subseteq V(\varphi)$. An easy calculation shows that (R) is valid in this model. But, $V(p \to q) = W$. Thus, $V(p) \cap V(p \to q) \nsubseteq V(q)$. Hence (MP) does not hold. $\qquad\square$

To conclude we state the result proved by Suzuki, Wolter, and Zakharyashev on the non-protoalgebraicity of VPL.

Theorem 8.17 *The logic $wk_\sigma(\text{DT}_1)$ (i.e., VPL) is non-protoalgebraic.*

Proof The matrices in the proof of Proposition 4.8 show that VPL is not protoalgebraic. $\qquad\square$

Notes

1. The proof of the second inequality is due to Bou.

2. The proof of item 2 is due to Bou.

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