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Propositional Logic of Imperfect Information: Foundations and Applications

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Abstract I will show that the semantic structure of a new imperfect-information propositional logic can be described in terms of extensive forms of semantic games. I will discuss some ensuing properties of these games such as imperfect recall, informational consistency, and team playing. Finally, I will suggest a couple of applications that arise in physics, and most notably in quantum theory and quantum logics.

1 Introduction

The research Sandu and I have done ([30], [32]) describes a new propositional logic of imperfect information, or IF (independence-friendly) propositional logic. This logic can be viewed as a fragment of IF first-order logic (Hintikka [10], Hintikka and Sandu [11], Pietarinen and Sandu [26], Sandu [31]). That research addressed the relation between partial logic and the semantics for IF propositional logic in terms of imperfect information semantic games, and one of the outcomes was an alternative compositional semantics for it. Here I propose to look at these imperfect information semantic games in more detail and show that they may conveniently be viewed as extensive-form games with a partitional information structure. Such partitional structures have been studied in game theory in relation to imperfect information but they have not been applied to logic before. I will discuss a couple of issues that these games give rise to, namely, imperfect recall, information consistency, and team playing. After presenting the game-theoretic foundations for propositional IF logic I will turn to a couple of applications of this logic which are here shown to be rife in the realm of quantum theory and quantum logics.

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2 Extensive Games of Perfect Information

Let us fix a family of actions A in which a finite sequence $\langle a^i \rangle_{i=1}^n$ represents the consecutive choices of players N (no chance moves), $a^i \in A, i = \{1, \ldots, n\}$. An *extensive game* $\mathcal{G}$ of perfect information is a five-tuple $\mathcal{G}_A = \langle H, Z, P, N, (u_i)_{i \in N} \rangle$ such that

1. H is a set of finite sequences of actions $h = \langle a^i \rangle_{i=1}^n$ from A called *histories* of the game:
   (a) the empty sequence $\langle\rangle$ is in H,
   (b) if $h \in H$, then any initial segment of h is in H too: if $h = \langle a^i \rangle_{i=1}^n \in H$ then $pr(h) = \langle a^i \rangle_{i=1}^{n-1} \in H$ for all $n = 1, 2, \ldots$, where $pr(h)$ is the *immediate predecessor* of h ($= \varnothing$ for $h = \varnothing$);
2. Z is a set of *maximal histories* (complete plays) of the game: if a history $h = \langle a^i \rangle_{i=1}^n \in H$ can continue as $h' = \langle a^i \rangle_{i=1}^{n+1} \in H$, h is a nonterminal history and $(a^{n+1}) \in A$ is a nonterminal element—otherwise histories are terminal; any $h \in Z$ is terminal;
3. $P\colon H \setminus Z \to N$ is the *player function* which assigns to every nonterminal history a player N whose turn is to move;
4. each $u_i, i \in N$ is the *payoff function*, that is, a function which specifies for each maximal history the payoff for player i.

For any nonterminal history $h \in H$ define $A(h) = \{x \in A \mid h \frown x \in H\}$. A (pure) *strategy* for a player i is any function $f_i\colon P^{-1}(\{i\}) \to A$ such that $f_i(h) \in A(h)$, in which $P^{-1}(\{i\})$ is the set of all histories where player i is to move. A strategy specifies an action also for histories that may never be reached.

A *strictly competitive* game is a particular case of a game defined as above, in which $N = \{\exists, \forall\}$ and, in addition,

1. $u_\exists(h) = -u_\forall(h)$,
2. either $u_\exists(h) = 1$ or $u_\exists(h) = -1$ (i.e., $\exists$ either wins or loses),

for all terminal histories h.

3 Extensive Semantic Games of Perfect Information

An extensive form of a semantic game $\mathcal{G}(\varphi, M)$ associated with a formula φ and a model M is exactly like an extensive game $\mathcal{G}_A$ defined above, except that it has one extra element: a labeling function $L\colon H \to \mathsf{Sub}(\varphi)$ such that

1. $L(\langle\rangle) = \varphi$ (the root);
2. for every terminal history $h \in Z$, $L(h)$ is an atomic subformula $p \in \mathsf{Sub}(\varphi)$ of φ or its negation.

In addition, the components $H, L, P, u_\exists$ and $u_\forall$ satisfy the following requirements:

1. if $L(h) = \psi \vee \theta$ or $L(h) = \psi \wedge \theta$, then $h \frown \mathsf{Left} \in H$, $h \frown \mathsf{Right} \in H$, and $L(h \frown \mathsf{Left}) = \psi$, and $L(h \frown \mathsf{Right}) = \theta$;
2. if $L(h) = \psi \vee \theta$, then $P(h) = \exists$;
3. if $L(h) = \psi \wedge \theta$, then $P(h) = \forall$;
4. for every terminal history $h \in Z$:
   (a) if $L(h) = p$ and $M \models^+ p$, then $u_\exists(h) = 1$ and $u_\forall(h) = -1$;
   (b) if $L(h) = p$ and $M \models^- p$, then $u_\exists(h) = -1$ and $u_\forall(h) = 1$.

The relation '$\models^{+}$' means positive logical consequence (being true in a model) and '$\models^{-}$' means negative logical consequence (being false in a model).

The notion of strategy is defined in the same way as before. A winning strategy for i is a set of strategies f_i that leads i to $u_i(h) = 1$ no matter how the player $-i$ (the player other than i) decides to act.

Let $\mathcal{L}$ be a classical propositional language and let $\varphi \in \mathcal{L}$. Then

1. $M \models^{+}_{\text{GTS}} \varphi$ if and only if there exists a winning strategy for player $\exists$ in the game $\mathcal{G}(\varphi, M)$.
2. $M \models^{-}_{\text{GTS}} \varphi$ if and only if there exists a winning strategy for player $\forall$ in the game $\mathcal{G}(\varphi, M)$.

The subscript GTS comes from 'game-theoretic semantics'.

4 Propositional Logic of Imperfect Information

So far the languages have been classical perfect information ones. However, there is a variant of an IF (independence-friendly) first-order language of ([10], [11]) which consists of propositional symbols Ψ, each having its own arity, and a finite set $i_1, \ldots, i_n$ of indices ranging over a set of two elements.[1]

The well-formed formulas of $\mathcal{L}_{\text{IF}}$ are defined by the following clauses:

1. if $p \in \Psi$, the arity of p is n, and $i_1, \ldots, i_n$ are indices, then $p_{i_1,\ldots,i_n}$ and $\neg p_{i_1,\ldots,i_n}$ are $\mathcal{L}_{\text{IF}}$-formulas; let us write $p_{i_1,\ldots,i_n}$ also as $p(i_1, \ldots, i_n)$;
2. if φ and ψ are $\mathcal{L}_{\text{IF}}$-formulas then $\varphi \vee \psi$ and $\varphi \wedge \psi$ are $\mathcal{L}_{\text{IF}}$-formulas;
3. if φ is an $\mathcal{L}_{\text{IF}}$-formula then $\forall i_n \varphi$ and $\exists i_n \varphi$ are $\mathcal{L}_{\text{IF}}$-formulas;
4. if φ is an $\mathcal{L}_{\text{IF}}$-formula then $(\exists i_n/U)\varphi$ is an $\mathcal{L}_{\text{IF}}$-formula (U is a finite set of indices, $i_n \notin U$).

The notions of free and bound variables are the same as in first-order logic. In $(\exists i_n/U)\varphi$ the indices on the right-hand side of the slash are free. For simplicity, the clauses for dual prefixes such as $(\forall i_n/U)$ are omitted.

The models for the language will be of the form $M = \langle I^M, (p^M)_{p\in\Psi}\rangle$ where I^M is any set with two elements, and each p^M is a set of finite sequences of indices from I^M.

With every $\mathcal{L}_{\text{IF}}$-sentence φ and a model $M = \langle I^M, (p^M)_{p\in\Psi}\rangle$ a semantic game $\mathcal{G}(\varphi, M)$ of imperfect information is associated, played by $\forall$ and $\exists$. The rules of the game are as follows.

1. $\varphi \vee \psi$ prompts a move by $\exists$ who chooses φ or ψ; the game goes on with the chosen formula.
2. $\varphi \wedge \psi$ prompts a move by $\forall$ who chooses φ or ψ; the game goes on with the chosen formula.
3. $\exists i_n \varphi$ prompts a move by $\exists$ who chooses $m_v \in I^M$ to be the interpretation of i_n.
4. $\forall i_n \varphi$ prompts a move by $\forall$ who chooses $m_v \in I^M$ to be the interpretation of i_n.
5. $(\exists i_n/U)\varphi$ prompts a move by $\exists$ who chooses $m_n \in I^M$ to be the interpretation of i_n independently of the choices corresponding to the elements in U.

The notion of "choosing independently" is explicated in the strategy functions (see below). The game ends with an atomic formula or its negation and a sequence of elements $\langle m_1, \ldots, m_n \rangle$ in which each $m_n \in I^M$. Then

1. if $\langle m_1, \ldots, m_n \rangle \in p^M$, then $\exists$ wins;
2. if $\langle m_1, \ldots, m_n \rangle \notin p^M$, then $\forall$ wins.

Let φ be an $\mathcal{L}_{\text{IF}}$-sentence and let M be a model $\langle I^M, (p^M)_{p \in \Psi} \rangle$. Then

1. $M \models^+_{\text{GTS}} \varphi$ if and only if there exists a winning strategy for player $\exists$ in the game $\mathcal{G}(\varphi, M)$.
2. $M \models^-_{\text{GTS}} \varphi$ if and only if there exists a winning strategy for player $\forall$ in the game $\mathcal{G}(\varphi, M)$.

For example, let $M = \langle I^M, (p^M)_{p \in \Psi} \rangle$, where $I^M = \{\text{Left, Right}\}$. Then

$$M \models^+_{\text{GTS}} (\forall i_1 (\exists i_2/\{i_1\})\, p_{i_1 i_2}) \quad \text{iff} \quad M \models \exists i_2 \forall i_1\, p_{i_1 i_2} \quad \text{iff}$$

$$\langle \text{Left,Left} \rangle \in p^M \quad \text{and} \quad \langle \text{Right, Left} \rangle \in p^M, \quad \text{or}$$
$$\langle \text{Left, Right} \rangle \in p^M \quad \text{and} \quad \langle \text{Right, Right} \rangle \in p^M.$$

$$M \models^-_{\text{GTS}} (\forall i_1 (\exists i_2/\{i_1\})\, p_{i_1 i_2}) \quad \text{iff} \quad M \models \exists i_1 \forall i_2 \neg p_{i_1 i_2} \quad \text{iff}$$

$$\langle \text{Left,Left} \rangle \in p^M \quad \text{and} \quad \langle \text{Left, Right} \rangle \notin p^M, \quad \text{or}$$
$$\langle \text{Right, Left} \rangle \in p^M \quad \text{and} \quad \langle \text{Right, Right} \rangle \notin p^M.$$

Formulas of $\mathcal{L}_{\text{IF}}$ can be rewritten by using infix notation when confusion about the location of imperfect information does not arise. For example,

$$\forall i_1 (\exists i_2/i_1)\, p_{i_1 i_2} \equiv (p_{a_1}\ (\vee/\wedge)\ p_{a_2}) \wedge (p_{b_1}\ (\vee/\wedge)\ p_{b_2}). \tag{1}$$

The connective $(\vee/\wedge)$ (respectively, $(\wedge/\vee)$) means that the verifier $\exists$ (the falsifier $\forall$) makes a choice of the disjunct (conjunct) without being informed of what choices have been made with respect to the conjunction (disjunction) denoted on the right-hand side of the slash.

5 Information Partition: Imperfect Information and Imperfect Recall

The semantics for slashes gives rise to expressions such as "not being informed of," "not knowing that," or "being independent of" that need to be captured. To do this, imperfect information is represented by means of extensive forms of semantic games with a partitional information structure. Let us extend $\mathcal{G}_A$ to a six-tuple $\mathcal{G}^*_A = \langle H, Z, P, N, (u_i)_{i \in N}, (\Im_i)_{i \in N} \rangle$ with an additional component $\Im_i$, an *information partition* of $P^{-1}(\{i\})$ (the set of histories in which i moves) such that for all $h, h' \in S^i_j$,

$$h \frown x \in H \text{ iff } h' \frown x \in H, x \in A,\ j = 1, \ldots, m, i = 1, \ldots, k, m \leq k.$$

S^i_j is an *information set* $S^i_j \in \Im_i$.

In imperfect information games, the strategy function is required to be defined on indistinguishable histories:

$$\text{if } h, h' \in S^i_j \in \Im_i \text{ then } f_i(h) = f_i(h'),\ \text{ for } i \in N.$$

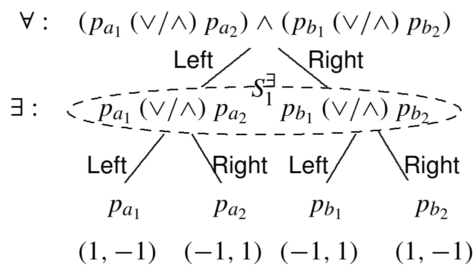

Figure 1 An extensive-form semantic game $\mathcal{G}^*(\phi, M)$ for $\phi = (p_{a_1}\,(\vee/\wedge)\,p_{a_2}) \wedge (p_{b_1}\,(\vee/\wedge)\,p_{b_2})$, with one nontrivial information set $S_1^{\exists}$ for ∃.

Figure 1 illustrates an extensive form of a semantic game for the right-hand side formula in the equivalence (1). Let $\preccurlyeq$ be a partial order on the tree structure of extensive form games. An extensive-form game satisfies *non-absentmindedness*, if $h, h' \in S_j^i$, and if $h \preccurlyeq h'$ then $h = h'$. Let a depth $d(Q)$ of a logical component Q in an $\mathcal{L}_{\text{IF}}$-formula φ be defined inductively in a standard way. It can be observed that $\mathcal{G}^*(\varphi, M)$ for $\mathcal{L}_{\text{IF}}$-formulas φ satisfy nonabsentmindedness. This follows from the fact that all logical components Q in any $\mathcal{L}_{\text{IF}}$-formula φ have a unique depth $d(Q)$, and so every subformula of φ has a unique position in $\mathcal{G}^*$ given by $\mathcal{L}(h)$. Thus for any two subformulas of φ at $h, h' \in H$ within S_j^i, $h \not\preccurlyeq h'$ and $h' \not\preccurlyeq h$.

In other words, for any $\mathcal{L}_{\text{IF}}$-formulas φ and ψ, if ψ is a subformula of φ at $h \in H$ and η is a subformula of ψ at $h' \in H$, and ψ and η labeled along the same history $h \in Z$, if $\psi \in S_j^i$ and $\eta \in S_l^k$, then $i = k$. (Since there is no $(\Im_i)_{i \in N}$ in $\mathcal{G}$, it vacuously satisfies non-absentmindedness.)

Let $Z(h)$ be a set of plays that pass through any $h \in H$, if h becomes a subsequence of any $h' \in Z(h)$. Likewise, let $Z(a^i)$ be a set of plays that pass through an action $a^i \in A$ (or a sequence of actions $\langle a^i \rangle_{i=1}^n$), if $a^i \in h' \in Z(a^i)$. Define a precedence relation $<^*$ between any two information sets S_j^i, S_k^i as follows:

$$\text{If } h, h' \in S_j^i \times S_k^i \text{ such that } h \prec h', \text{ then } S_j^i <^* S_k^i.$$

Thus $S_j^i <^* S_k^i$ says that there exists a play $h'' \in Z$ passing through h and h'. If non-absentmindedness holds then any $h'' \in Z$ passes through S_j^i or S_k^i at most once.

Let $P^{-1}(\{i\})$ be the set of histories where i moves playing a strategy f_i. S_j^i is relevant for f_i if $S_j^i \cap P^{-1}(\{i\})$ is nonempty.

The property of recall is now characterized so that a game $\mathcal{G}_A^*$ has *perfect recall*$_1$ if S_j^i is relevant for f_i implies $S_j^i \subset P^{-1}(\{i\})$ for all f_i, $S_j^i \in \Im_i$. There is also an alternative way of characterizing perfect recall. A game $\mathcal{G}_A^*$ has *perfect recall*$_2$ if $S_j^i <^* S_k^i$ implies the existence of a sequence of actions $\langle a^i \rangle_{i=1}^n \in A$ available from S_j^i such that $Z(S_k^i) \subseteq Z(\langle a^i \rangle_{i=1}^n)$.

It can now be observed that games $\mathcal{G}^*(\varphi, M)$ for $\mathcal{L}_{\text{IF}}$-formulas φ do not satisfy perfect recall$_i$, $i = 1, 2$. This is because any $\mathcal{L}_{\text{IF}}$-formula φ containing a subformula $\psi = Q_1 i_1, \ldots, (Q_2 i_n / i_1)$, $Q_1, Q_2 = \exists$ or $Q_1, Q_2 = \forall$ and $d(Q_1) < d(Q_2)$, gives rise to a partition in which the subformula of ψ beginning with Q_2 induces

S_k^i and the subformula of φ beginning with Q_1 induces S_j^i such that $S_j^i <^* S_k^i$. Thus the actions A that the player $i \in \{\exists, \forall\}$ chooses for $Q_1 i_1$ are available from S_j^i, but then $Z(\langle a^i \rangle_{i \in 1}^n) = Z(S_k^i)$. On the other hand, perfect recall$_1$ depends on allowing "nonstandard" information sets that are not relevant for players' strategies f_i. But any such information set violates perfect recall$_1$, in which all $P^{-1}(\{i\})$ are $\mathcal{L}(P^{-1}(\{i\}))$.

Thus it is possible to study fragments of imperfect information logic in which perfect recall holds only in some restricted sense, in which case we need to study aspects of *bounded recall* (see Lehrer [20] for bounded recall in game theory).

6 Information Consistency and Team Players

In order to understand partitional information structure for semantic games some qualifications are needed. First, the game-theoretic notion of information or *time consistency* says that the notion of time has an unambiguous meaning in games: the information partition distinguishes between the past, present, and future choices in an unambiguous way. This is something that can be tried to be captured by assuming that information sets are partially ordered in the sense of $<^*$.

Apart from partially-ordered information sets, time inconsistencies may arise in situations in which the same information set may be visited more than once during a play of the game. This was described in Section 5 in terms of histories which lie within the same path but are included into the same information set. The resulting property of absentmindedness informally says that a player may not be able to recall his or her own location in a game.

Absentmindedness describes rather erratic player behavior and may give rise to certain game-theoretic paradoxes (Piccione and Rubinstein [24]). Therefore it is reasonable that the partition $(\Im_i)_{i \in N}$ in any $\mathcal{G}^*(\varphi, M)$ is time consistent, that is, all $S_j^i \in \Im_i$ are partially ordered and non-absentminded. Now we have seen that there is no absentmindedness in $\mathcal{L}_{\mathrm{IF}}$, and so the question arises whether the time consistency holds for all IF propositional formulas. It indeed turns out that time consistency holds, for example, for formulas of IF first-order logic, but surprisingly, in propositional fragments there, in fact, are formulas in which information sets may not always be time consistent. An example is provided by

$$(p_1\ (\vee/\wedge)\ p_2) \wedge (q_1 \vee q_2). \tag{2}$$

What is the corresponding information partition for this formula? One cannot just draw an information set around the histories labeled with $p_1\ (\vee/\wedge)\ p_2$ and $q_1 \vee q_2$ because there is no imperfect information on the right-hand side conjunct.

Two options seem to be available. Either the information set gives rise to a time inconsistent game in which two consecutive histories may form an information set, or else the definition of information sets needs to be revised. Neither option is particularly attractive. In the former case, the initial move in the game becomes ambiguous, since a decision has to be made between $\forall$'s choice of a conjunct or $\exists$'s choice of a disjunct for the left-hand side conjunct, but this indecisiveness has been precluded by the above remarks. In the latter case, the modified definition would say that information sets do not form equivalence classes but are asymmetric instead (yet transitive and reflexive). This could mean that at the history labeled with $p_1\ (\vee/\wedge)\ p_2$, $\exists$ has imperfect information and does not know whether Left or Right has been chosen for conjunction, whereas at the history labeled with $q_1 \vee q_2$ information is not hidden.

Thus even the existence of imperfect information alone can be conditioned by single actions taken in the game.[2]

The second remark concerns forward-looking reasoning essential in propositional games. Players can sometimes recover knowledge about their location within an information set by looking at the available choices. This happens, for instance, in cases in which some of the histories within an information set are terminal. An example is provided by

$$(p_1 \ (\vee/\wedge) \ p_2) \wedge p_3. \tag{3}$$

In this and similar cases one may apply the idempotence law to make the otherwise terminal histories nonterminal by adding superfluous moves so that a player repeatedly chooses p_3. How this is done in terms of extensive games is straightforward.

There is also a dual notion to the phenomenon of imperfect recall. One can characterize learning or information increase that happens with, say

$$\forall i_1 (\exists i_2/i_1) \exists i_3 \ p_{i_1 i_2 i_3}. \tag{4}$$

Of course, in one formula both imperfect recall and learning can happen, for consider

$$\forall i_1 (\exists i_2/i_1)(\exists i_3/i_2) \ p_{i_1 i_2 i_3}. \tag{5}$$

Game-theoretically, such an information increase reminds us of *screening games* (Rasmusen [28]) in which the first player is uninformed of certain aspects of the game and the second player, being fully informed, can screen his or her actions. In contrast, in *signaling games*, the informed player moves first and may signal previous features of the game to the subsequent uninformed player. In the former case, if the types of the first and the second player are the same, then screening amounts to learning, and likewise, signaling means that some (higher level) information is being forgotten.

To make more precise sense of these aspects of information fluctuation in logic, a convenient way to understand imperfect recall is to view players as teams of agents. A team is a set of noncoordinating players $i = \{1, \ldots, n\}$ who have identical payoffs $u_i(h)$ but who act individually.[3] The teams $\exists$ and $\forall$ have a finite number of individual members $\exists_l \in \exists$ and $\forall_k \in \forall$, $l, k \leq n$. The members of a team are not allowed to communicate because this destroys team's ability, if viewed as one player, to forget information. The members of the same team all receive the payoff $u_i(h)$ as soon as the outcome of a play is solved.

More precisely, then, whenever a move associated with the team $\exists$ or the team $\forall$ is regarded as independent of the move made by the same team, a new member $\exists_l \in \exists$ or $\forall_k \in \forall$, $0 < i, j \leq n$ makes the new move. Therefore, in the second step in evaluating $\mathcal{G}^*(\phi, M)$, in which

$$\phi = \exists i_1 (\exists i_2/i_1) \ p_{i_1 i_2}, \tag{6}$$

a new member $\exists_2$ makes a decision, and this time she does not have the information $\exists$ had when making the first move in the game. This is the way to prevent $\exists$ signaling her first choice further, and a consequence could be that, unlike in classical case, a winning strategy does not exist for $\exists$. The information for individual team members remains persistent although the teams, viewed as single players, do not forget information.

To see another example, consider

$$\exists i_1 \forall i_2 (\exists_{i_2}/i_1) \ p_{i_1 i_2 i_3}. \tag{7}$$

Here the move for $(\exists i_2/i_1)$ is made in ignorance of the first move for $\exists i_1$. Clearly, this game has both imperfect information and imperfect recall and the latter is realized by splitting $\exists$ into $\exists_1$ and $\exists_2$.

The idea of team or multiperson games for imperfect recall goes back to von Neumann and Morgenstern [35], Strotz [33], and Isbell [15]. Whereas this approach is not unproblematic, it immediately provides a way to understand the behavior of semantic games.[4] It should be emphasized that the team approach is a device to understand the information flow in imperfect recall games rather than a technical necessity. Viewing imperfect recall as team games aims at explaining what happens when information is dispelled from player's memory, and since information should be persistent for decision makers, this team approach provides a way to understand semantic games for IF logic. When there is complete independence between logical components, what one gets is an *agent normal form* of extensive games in which each information set belongs to a separate player.

The phenomenon of multiple players puts the games here broadly within Team Theory, which sees teams as groups of agents with identical interests but individual actions and individual information (Kim and Roush [16]). In team games, strategies may be based on previous information in a game but not on information the other members of the team might have. A connection is provided by the result that basic solution concepts for two-person zero-sum games hold also for games played by teams (Ho and Sun [12]). It has also been argued in Koller and Megiddo [18] that imperfect recall games should use strategies more appropriate than just the traditional mixed ones, such as team-maxmin strategy profiles. A logical representation of teams, of course, has scores of potential applications in system and organization theory as well as in distributed computing.

7 Applications to Quantum Physics and Logic

The logic $\mathcal{L}_{\text{IF}}$, and its game-theoretic interpretation, has some interesting applications and correlates in the field of quantum physics and quantum logics.

7.1 Skirmish To start with an example of classical "mechanics," consider the following game with two flesh-and-blood players: Player 1 is hiding behind a small bump and Player 2 is armed with a ball. Player 1's goal is to run and get home without being hit by the ball, and Player 2's goal is to prevent 1 from getting home by trying to hit him with the ball. There are four actions: Player 2 waits or throws, and Player 1 hides or runs. Both players fix their choice of action independently and simultaneously. The winning condition for 1 is '*get home*', and the winning condition for 2 is '*not get home*', so I assume that if 1 hides and 2 waits, nothing happens and 1 never gets home.

The point here is that even this simple game needs $\mathcal{L}_{\text{IF}}$ for its adequate modeling. This is because players make concurrent and independent choices. By taking φ = '2 waits', $\neg\varphi$ = '2 throws', ψ = '1 hides', and $\neg\psi$ = '1 runs':

$$\phi = (\varphi\ (\vee/\wedge)\ \neg\varphi) \wedge (\psi\ (\vee/\wedge)\ \neg\psi). \tag{8}$$

If we look away from the possibility of partially interpreted models, the game $\mathcal{G}^*(\phi, M)$ is given in Figure 2, with one nontrivial information set $S_1^{\exists}$ for $\exists$. Consequently, the apparent sequential format of a game of imperfect information game does not have to correspond to the real or physical notion of time. Because of

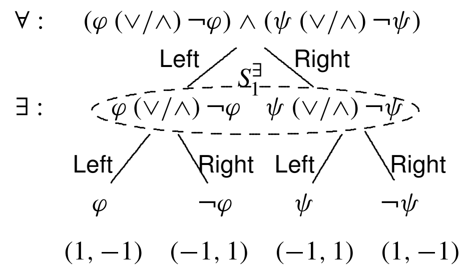

Figure 2

imperfect information, the games of this kind are not determined and thus the law of excluded middle fails.

Further analysis of this game would take into account repeated plays that occur when the game is drawn back to its initial state. Games with repetition would then involve probability distributions over strategies. In addition, winning strategies would have memories, that is, carry information from previous plays to future ones, in order to verify that certain probabilities are consistently used. Improper randomization may give strategic advantages to contestants.

7.2 Quantum mechanics Apart from the macro-world decision problems outlined above, we can relate our logic to quantum theory and quantum logic in particular.

As soon as one moves from classical to quantum mechanics, certain logical properties have to be given up. The step involves a transition from Boolean algebra (represented by a complemented distributive lattice) to non-Boolean algebra isomorphic to the lattice structure describing the properties of a physical system, which are closed subspaces of a Hilbert space (Bub [5]).

Quantum mechanical properties cannot be approached in terms of classical Boolean logic, however, and hence some new logics and logical laws are inevitable. It is nonetheless possible to describe at least some quantum phenomena in terms not particularly far removed from classical logic.

One of the characteristic features of the classical universe is that the algebra of variables forms a commutative algebra of real-valued functions on a phase space S, and properties or propositions become isomorphic to Borel subsets of S. The law of commutativity does not hold in the quantum realm, however, and the non-Boolean lattice properties form a noncommutative structure. Hence quantum logic of propositions that represent the non-Boolean structure conflicts with the classical law of commutativity.

An explanation of the failure of commutativity is foreseeable, however. One way of looking at it is as a manifestation of the overall failure of perfect information in the associated semantic evaluation games for quantum logical propositions. In preliminary terms, noncommutativity boils down to the phenomenon that in certain situations requiring simultaneous actions or decisions by one of the players, the order of the components of a connective cannot be arbitrarily changed. This, in turn, is so because such a change in the order of propositions would prevent certain strategies to be applicable, since game theory requires that when actions are planned, players observe their available choices and make inferences on the basis of that information.

If a player can detect an irregularity in the order of available actions in cases in which there are simultaneous or hidden moves in the game, the states are no longer indistinguishable from each other and the imperfectness of information vanishes. I will return to the nature of the laws of propositional logic later.

7.3 Nonlocality Let us consider next the prima facie striking, but by now rather well-researched, phenomenon of quantum theoretic nonlocality (Einstein et al. [7]). Nonlocality is a property of "entangled" photon pairs. Two spacelike separated photons originating from the same source can exist in an entangled state such that when one photon is manipulated by an interaction with a polarization filter changing the polarization of the photon, there is a 100% anticorrelation to that polarization in the other photon as well. This is because the polarized photon has to maintain its original correlations with the other photon in the entangled system.

Again, this translates into simultaneous action. Reading the propositions in (9) as follows,

1. φ: the measurement outcome of a photon x being left-polarized,
2. ψ: the measurement outcome of a photon x not polarized,
3. θ: the measurement outcome of a photon y anticorrelated (right-polarized),
4. χ: the measurement outcome of a photon y not anticorrelated (not polarized),

there is an $\mathcal{L}_{\mathrm{IF}}$-formula corresponding to nonlocality:

$$(\varphi\ (\vee/\wedge)\ \psi) \wedge (\theta\ (\vee/\wedge)\ \chi). \tag{9}$$

In terms of (9), one may throw some light on the phenomenon of nonlocality and its role in quantum mechanics and quantum logics. For what being spacelike separated but correlated means is that no physical information is allowed to pass between two quantum subsystems, and in this sense the two particles are separated. However, entanglement means that the outcome of the measurement on one of the particles is not independent of how the measurement is chosen to be performed on the other, separated particle. Game-theoretically, when simultaneous action has to be represented, the outcome of one of the actions determines the winner on which the winning strategies, and hence the truth-values, of the propositions are based, which in turn depends on other actions in the game (irrespectively of whether the actions are taken to be hidden or not).

Thus the barriers needed for the information trespassing in the evaluation of (9) are brought out by the information encapsulation, which implies that the information regarding ∃'s choice of the two disjunctions may not be used when ∀ plans a decision between the conjuncts (and vice versa).

This kind of information encapsulation is of course not a sufficient reason to account for the nonlocality. In entangled systems, some further effect such as a quantum field is needed to correlate the separated systems. In logical terms, however, the answer to what nonlocality means is surprisingly simple. It can be directly derived from the truth-value of the formula representing the entangled system and its nonlocality. This is because in order to make (9) true, one has to be able to make at least one atomic formula in both conjuncts true, and this in turn requires that, despite the hidden information regarding the conjunct that has been chosen at the other history, both conjuncts that represent the states of the two separated systems are needed in establishing its truth.

Nothing in our argument—purported here to show that logic of quantum mechanics goes beyond not only the purviews of classical but to some extent also received quantum logics—hinges on this traditional EPR formulation of nonlocality as action at distance concerning two separated but correlated particle systems. Similar remarks are valid also for other descriptions of nonlocality. Consequently, the argument does not hinge on the Bell inequalities either (Bell [1]).

To see another example, consider the Greenberger-Horne-Zeilinger (GHZ) experiment for entangled systems introduced in Greenberger et al. [8]. Following the presentation in Mermin [21], the GHZ-experiment involves three spin-1/2 particles extending to different directions from the common source. There are two possible measurements to be performed on each particle: one can measure the particle's X-spin or its Y-spin. Now quantum theory predicts with certainty that if the Y-spin of two of the three particles and the X-spin of the third particle are measured, an even number of measurements will have the outcome "spindown," whereas if the X-spin of all three particles is measured, an odd number of measurements will have that outcome. The particularity of this experiment in distinction to its traditional EPR counterpart is that it ascribes no Y-spin values to the third unmeasured particle before the measurement of the other two particles is performed. Like in its traditional EPR-version, these values are determined after the measurement has been performed to a sufficient number of other particles in the system.

The logical structure of the GHZ setup would then be of the following form (the key—X_1: 'X-spin of particle 1'; Y_2: 'Y-spin of particle 2'; S_e: 'an even number of particles has a spindown state'):

$$(((X_1(\wedge/\vee)Y_2(\wedge/\vee)Y_3) \vee (Y_1(\wedge/\vee)X_2(\wedge/\vee)Y_3) \vee \\ (Y_1\,(\wedge/\vee)\,Y_2\,(\wedge/\vee)\,X_3)) \rightarrow S_e) \wedge ((X_1 \wedge X_2 \wedge X_3) \rightarrow S_e^{\perp}). \tag{10}$$

The symbol '$^{\perp}$' marks a singular orthocomplementation operation which corresponds to the game-theoretic negation. The hidden disjunctions refer to either disjunction.

As seen from (10), some information encapsulation is again needed in order to account for nonlocality. But such encapsulation is lacking in traditional logics for quantum theory. In fact, nonlocality is information encapsulation: a certain density matrix exhibits nonlocality precisely when there is no way of explaining the correlation between two spacelike separated particles A and B in terms of information extant at particle A or information extant at particle B. The probability distributions that the density matrix assigns to the observables then give rise to the correlations formulated in Bell inequalities [1] (or, alternatively, to the correlations that Kochen and Specker formulated by dispensing with the quantum probability formalism in [17]).

These EPR- and GHZ-experimentation schemes nonetheless have much wider repercussions than just the need for a reconsideration of the logic of nonlocality. Namely, these implications pertain to the indispensability of the nonlocality phenomenon and its versions under the different interpretations of quantum theory and in experimental testing,[5] and also to the arguments for the essentially incomplete nature of quantum theory. These traits will not be pursued here any further, however.

7.4 Quantum logic Further connections between the logic of imperfect information and quantum theory surface in quantum logic. Quantum logic was introduced in Birkhoff and von Neumann [3]. The key observation in that work was that one can start with a Hilbert space formalism of quantum mechanics and make observables or subspaces of Hilbert space correspond to propositions about a quantum physical system. This procedure constitutes an algebraic lattice-theoretic structure. The laws of quantum logic differ from the laws of classical logical, however, and have sometimes argued to derive their justification from empirical considerations about the physical reality (Putnam [27]).

One important classically valid propositional rule is commutativity, which is not valid in quantum logic. As previously was remarked, it is possible to spell out the reasons for its failure, using the game-theoretic semantics of imperfect information. In such games, commutativity is constrained by a highly nontrivial principle of the existence of nonsingleton information sets. Those connectives that are influenced by imperfect information, that is, the locations in which the partition of histories into information sets affect outcome actions, do not permit commutation to be applied to actions. This is because in such histories, the strategy functions have to satisfy that all the actions have to be the same for all the histories within the same information set.

However, similar remarks may be voiced on distributivity:

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r). \tag{11}$$

In quantum logical terms, distributivity holds just in case the propositions are not members of a common sublattice, that is, they denote incompatible subspaces (observables). Then the left-hand side sentence and right-hand side sentence mean that the subspaces in question are different, and hence not equivalent.

How can this be explained in game-theoretic terms? In short, the explanation is that distributivity changes the order in which players make their moves. Hence the left- and right-hand sides of (11) cannot be taken to exemplify the same logical situation. In other words, incompatible subspaces simply cannot be conjoined—starting from $p \wedge (q \vee r)$ one cannot infer $(p \wedge q) \vee (p \wedge r)$ because the pairs $\{p, q\}$ and $\{p, r\}$ are mutually incompatible. Indeed, these laws do not hold in quantum theoretic algebra that is non-Boolean.

Consider also modularity which is weaker than distributivity. If $p \leq r$ then

$$p \wedge (q \vee r) = (p \wedge q) \vee r. \tag{12}$$

Again, modularity illustrates an imperfect information phenomenon. It boils down to the following:

$$p \wedge (q\,(\vee/\wedge)\,r) = (p\,(\wedge/\vee)\,q) \vee r. \tag{13}$$

For if you choose disjunction independently of conjunction, you can as well go ahead and choose it before conjunction.

However, in quantum logic with infinite-dimensional Hilbert space one is interested in orthomodular structures which have the following order between elements in a lattice:

$$\text{If } a \leq b, \text{ then } b = a \vee (b \wedge a^{\perp}). \tag{14}$$

The relation '$\leq$' means that a is a subspace of b. Logically, orthomodularity replaces distributivity since it does not try to form conjunctions of mutually incompatible proposition (propositions that are not members of a common Boolean sublattice). The conjugation is legitimate only in the case any two propositions in $p \wedge (q \vee r)$ are complements of each other, and then distributivity would be retained.

Like modularity, orthomodularity illustrates a relative independence phenomenon, although in a weaker sense than full modularity. In brief, orthomodular nondistributive lattices are models for certain first-order sentences with imperfect information that are not first-order representable.

What is the lattice structure of the set of subspaces of a Hilbert space? An alternative to orthomodularity is to characterize it as a partial Boolean algebra (Hughes [13]). A partial Boolean algebra $\mathcal{B}_i = \langle B_i, \bigvee_i, \bigwedge_i, {}^{\perp_i}, 0_i, 1_i\rangle$ with B a set with at least two elements can be characterized with two conditions: it is an algebra that (1) forms a Boolean manifold (Hughes [14], p. 192) and (2) satisfies the following: For all elements $a, b, c \in \cup\{B_i\}$, if there are $i, j, k \in I$ such that $a, b \in B_i$, $b, c \in B_j$, $c, a \in B_k$, then there exists an $m \in I$ such that $a, b, c \in B_m$.

The part (2) of the definition expresses quantifier structure that cannot be symbolized using traditional first-order logic. Instead, one needs imperfect information for quantifiers:

$$\forall a \forall b \forall c (\exists i/c)(\exists j/a)(\exists k/b)\,(P_1 abcijk \to \exists m\, P_2 abcm). \tag{15}$$

This sentence expresses what is sometimes called the coherence condition on Boolean algebras (Hardegree and Frazer [9]).

Let us make two final remarks here. It was seen above that certain laws of propositional logic have their counterparts in extensive forms of games. It is an old problem in game theory, going back to at least Thompson [34], how to conduct certain transformations on extensive games so that the strategic aspects of the games are preserved (Osborne and Rubinstein [22] summarize these transformations). These proposed transformations indeed reflect propositional laws: an interchange of moves corresponds to commutativity, an addition of superfluous moves corresponds to idempotence, and so on. In general, the games in question are ones of imperfect information and hence, when they are associated with formulas, generalize ordinary propositional logic. But so does quantum logic, the central concepts of which, such as orthocomplemented quasi-modular lattices, provide models for a generalized (imperfect information) propositional logic. Not only does the law of excluded middle fail in both, but also the logical laws of quantum logic have their correlates in laws that are used in manipulating these imperfect-information games.

Second, the existence of imperfect information explains the partial nature of connectives and truth-values of complex sentences by making some of the games nondetermined. But partiality is a common property of quantum logics in which operations are based on the algebraic structure of the lattice of projections of a Hilbert space as partial Boolean algebras, that is, operations that ascribe properties to systems in a partial way. The partiality proposed here means that there will be propositions in a lattice that are not given the truth-value True, and they are not given the truth-value False. This kind of partiality can, however, be generated without assuming any partial assignment of atomic propositions or ultrafilters taking each proposition to a truth-value. It is the game-theoretic failure of perfect information that suffices to provide a semantic ground for partiality, games in which from the existence of a winning strategy for one of the players it does not follow that there exists a winning strategy for the other, opposing player.

7.5 Quantum computation Let us finally describe yet another novel logical perspective to quantum phenomena. In quantum computation, the notion of quantum interference is important in performing computations by quantum logic gates. One can think of such quantum logical switches as randomizing devices mapping $\{0, 1\}^n$ into $\{0, 1\}^m$. That is, each of the four possibilities for a particle (say, a photon) has an identical probability of 0.5. However, when two such identical machines are concatenated the net effect of the combined system is the logical complementation operation instead of the randomization. This surprising phenomenon contradicts the usual additivity of the received probability calculus, because the probability of the combined event is not the sum of two mutually exclusive constituent events (Deutsch et al. [6]).

What is going on? In [6, p. 269] it is claimed that there exists no corresponding operator (or a priori mathematical construction) in logic that could capture the nature of these randomizing devices, and hence they cannot really exist. Yet physicists have directly observed exactly this type of single-particle interference behavior.

Contrary to these pessimistic sentiments, the game-theoretic framework advocated in this paper is capable of throwing some light on this issue. First of all, the simultaneous nature of single-photon trajectories in quantum interference devices takes place inside the quantum gates; there is no interaction between the gates and environment. Now if one interprets these simultaneous actions as uncertainty coding, or imperfect information in the sense of game theory, one sees that the third choice of action in the concatenated system, even though simultaneous with respect to the second choice, is not simultaneous with respect to the choice of action made at the first interference gate of the combined system. Thus the third action can carry information concerning the first action, and hence is capable of complementing the input signal. The corresponding game can be constructed as a three-stage game with imperfect information with respect to the second and the third move, but not with respect to the first and the third move, representing the logical operation or connective describing quantum interference. Syntactically, this connective is symbolized by '$\sqrt{\neg}$' in the literature. However, its meaning in concatenated quantum system consisting of two quantum interference gates can be captured by

$$\forall i_1 \exists i_2 (\exists i_3 / i_2)\, p_{i_1 i_2 i_3}. \tag{16}$$

The associated extensive semantic game is drawn in Figure 3.

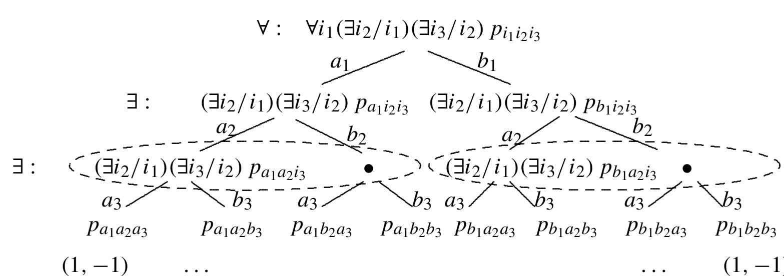

Figure 3 The semantic game for a concatenated quantum system consisting of two quantum interference devices.

8 Conclusion

Recently, various logical methods have been employed in analyzing games, and novel game-theoretic methods have been used in evaluating logics. Only a small portion of the usefulness of games in the overall logical study can be illustrated here, however. The latter connection can nonetheless be made tighter than before, an example of which has here been propositional logic with nonsequential (partial or nontransitive) information flow, interpreted via extensive forms of semantic games with imperfect information. These logics and associated games throw light on some logical anomalies in quantum theory and in quantum computation.[6] That such an enterprise is increasingly important has recently been shown in Boukas [4] in which classical von Neumann-Morgenstern games are given a quantum theoretic formulation by game moves that are associated with eigenvalues of a self-adjoint operator in a Hilbert space.

Notes

1. A more detailed exposition of this new logic, together with discussion as to why propositional imperfect information of this type is meaningful, and an alternative compositional semantics for it, is given in [30] and [32].

2. There are perhaps more feasible examples of such intensionalized imperfect information in games with quantifier moves, in which choices of an individual can conditionalize the existence of later information loss while retaining equivalence relations between histories.

3. Thus coalition games, which assume coordination, do not provide proper models for understanding imperfect recall. Indeed, they have not been considered in relation to imperfect recall in the game-theoretic literature.

4. But see Binmore [2] who criticizes multiself games because they lack realistic physical applications.

5. The GHZ-scheme has recently been experimentally tested and observed to verify the predictions of quantum mechanics in Pan et al. [23]. Sackett et al. [29] report experiments with four atom entangled systems by a method that in principle allows any number of atoms to be entangled.

6. The paper Pietarinen [25] explores further some of the game-theoretic issues that rear their heads in quantum logic and quantum theory.

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