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Periodicity of Negation

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Abstract In the context of a distributive lattice we specify the sort of mappings that could be generally called "negations" and study their behavior under iteration. We show that there are periodic and nonperiodic ones. Natural periodic negations exist with periods 2, 3, and 4 and pace 2, as well as natural nonperiodic ones, arising from the interaction of interior and quasi interior mappings with the pseudocomplement. For any n and any even $s < n$, negations of period n and pace s can also be constructed, but in a rather ad hoc and trivial way.

1 Introduction

In this paper we are concerned with how the various kinds of negation behave under iteration. The motivation comes first, from the classical and linear negation which are involutions ($\neg\neg\varphi = \varphi$), second, from the intuitionistic one which collapses at the third iteration ($\neg\neg\neg\varphi = \neg\varphi$), and third, from certain less common negations such as the "cyclic negation" of Post logic [9] (see also Malinowski [7] for a more up to date presentation) or the "chaotic negation" of Mar and Grim [8]. The n-valued cyclic negation $\neg$ needs n truth values $t_0 < \cdots < t_{n-1}$ and causes a cyclic rotation of this set in the sense that $\neg t_i = t_{i+1}$, for $i < n-2$, and $\neg t_{n-1} = t_0$. $\neg$ is obviously periodic with $\neg^n\varphi = \varphi$. On the other hand, the chaotic negation $\neg$ assumes the interval $[0, 1]$ as the set of truth values and $\neg : [0, 1] \to [0, 1]$ is the mapping such that $\neg x = 1 - |1 - 2x|$. The iterates $\neg^n x$, $n \in \mathbb{N}$, for certain $x \in [0, 1]$, behave chaotically.

We shall see below that the modal intuitionistic (or modal classical) negation $\Box\neg$ closes at the fourth step; namely, $(\Box\neg)^4\varphi = (\Box\neg)^2\varphi$. On the other hand, the bimodal intuitionistic (or classical) negation $\Box_1\Box_2\neg$ is, in general, strongly nonperiodic; namely, there can be φ such that $(\Box_1\Box_2\neg)^n\varphi \neq (\Box_1\Box_2\neg)^m\varphi$, for all $m \neq n$.

But what is a negation after all? To start with, according to Gabbay [5], the basic idea behind the definition of a negation connective A^* is that a formula B

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should deduce A^* if and only if A and B together would lead to some "undesirable" conclusion. More precisely, assuming that we possess a deduction relation $\vdash$, and a class of undesirable formulas Θ, then a connective A^* is a form of negation if for any formulas A, B,

$$(\dagger) \qquad B \vdash A^* \iff \exists y \in \Theta(B, A \vdash y)$$

(see [5], p. 99, Def. D1). Variants of (†) are also studied in [5], but these concern only the meaning of 'A, B *together* imply something'—not the basic ingredient of (†) which, to our view, is '$\iff$'.

In this paper we shall examine negation in terms of algebraic semantics rather than in terms of syntax. That is, instead of formulas we shall consider a distributive bottomed lattice $A = (A, \wedge, \vee, \leq, \bot)$ whose ordering $\leq$ captures the deduction relation $\vdash$, $\wedge$ captures "together", and $\bot$ captures the "bad" formulas Θ. Then, according to Gabbay, a mapping $f : A \to A$ is a *negation* if the translation of (†) holds, that is,

$$(\ddagger) \qquad x \wedge y = \bot \iff y \leq f(x).$$

However, an f satisfying (‡) is a very special operation, namely, a *pseudocomplement*. A stronger notion is that of a *relative pseudocomplement*. A relative pseudocomplement in A is a binary operation $x \to y$ such that for all $x, y, z \in A$,

$$(\pounds) \qquad z \wedge x \leq y \iff z \leq x \to y.$$

Obviously (£) implies (‡) since $f(x) = x \to \bot$ is a pseudocomplement. A lattice A with a relative pseudocomplement is met in the bibliography under the following names: *relatively pseudocomplemented lattice*, *Brouwerian lattice*, *pseudo-Boolean algebra*, *Heyting algebra*. We shall use throughout the name 'Heyting algebra' as it seems to have been established in more recent years. A lattice A is said to be *complete* if infinite joins and meets exist, denoted $\bigvee X$, $\bigwedge X$, for every $X \subseteq A$. (Existence of either of them suffices for completeness.) It is well known (see, for example, Birkhoff [1], p. 128) that a complete lattice A is a Heyting algebra if and only if the following infinite distributivity law holds in A:

$$(\mathrm{ID}) \qquad x \wedge (\textstyle\bigvee_i y_i) = \bigvee_i (x \wedge y_i).$$

In this case the unique relative pseudocomplement is defined on A by setting $x \to y = \bigvee\{z : z \wedge x \leq y\}$. Let $-_A$ denote the induced pseudocomplement of A. We shall refer to it as the *natural* pseudocomplement of A. Every topology (set of open sets of a topological space) is a Heyting algebra with respect to $\cap$ and $\cup$. Moreover it is complete and satisfies ID.

2 Defining Negation

Let us look more closely at Gabbay's defining equivalence (‡). The direction '$\Leftarrow$' is equivalent to the condition

$$(\mathrm{N1}) \qquad x \wedge f(x) = \bot \text{ (disjointness)}.$$

For every "crisp" logic (that is, except the fuzzy and paraconsistent ones) N1 is a standard requirement. However, the implication '$\Rightarrow$' of (‡) says that $f(x)$ is the *greatest* element disjoint from x which is indeed a very special condition. For instance (‡) implies the following (see Proposition 2.2 below).

$$(\mathrm{N2}) \qquad x \leq y \Rightarrow f(y) \leq f(x) \text{ (order-inversion)},$$

(N3) $$x \leq f^2(x) \text{ (regularity)},$$

(N4) $$x \leq f(y) \Rightarrow y \leq f(x) \text{ (strong order-inversion)}.$$

Lemma 2.1 N2 + N3 $\iff$ N4.

Proof Suppose N2 and N3 hold and let $x \leq f(y)$. Then by N2, $f^2(y) \leq f(x)$ and, by N3, $y \leq f^2(y) \leq f(x)$. Hence $x \leq f(y) \Rightarrow y \leq f(x)$. Conversely, suppose N4 holds. Then $f(x) \leq f(x) \Rightarrow x \leq f^2(x)$, from which we get N3. Using the latter, if $x \leq y$ then $x \leq f^2(y)$ which, by N4, gives $f(y) \leq f(x)$. Thus N2 holds. □

Proposition 2.2

1. (‡) $\Rightarrow$ N1 + N2 + N3.
2. N1 + N2 + N3 $\not\Rightarrow$ (‡).

Proof (1) N1 and N3 are obvious consequences of (‡). To see N2, let $x \leq y$. Since $f(y) \wedge y = \bot$, we get $f(y) \wedge x = \bot$. Thus (‡) yields $f(y) \leq f(x)$.

(2) We shall specify A and $f : A \to A$ such that (A, f) satisfies N1, N2, N3 but not (‡). Let $(A, \cup, \cap, \subseteq, \varnothing)$ be the lattice of open subsets of $\mathbb{R}$ (or any locally compact metric space). For every $X \in A$, let $X^* = \{x \in \mathbb{R} : d(x, X) \leq \varepsilon\}$, where $d(x, y)$ is the standard metric of $\mathbb{R}$, $d(x, X) = \inf\{d(x, y) : y \in X\}$, and ε is a fixed positive real. Intuitively X^* is the set resulting from X if we add the closed strip of width ε along its border. Clearly, $X \subseteq X^*$, $X \subseteq Y \Rightarrow X^* \subseteq Y^*$, and X^* is closed. Therefore setting $f(X) = -X^*$, f is a mapping from A to A. Obviously $X \cap f(X) = \varnothing$ and $X \subseteq Y \Rightarrow f(Y) \subseteq f(X)$, that is, N1 and N2 hold. Also (‡) is false in (A, f) since clearly $f(X)$ is not the greatest element of A disjoint from X. Thus it suffices to show N3, that is,

$$X \subseteq f^2(X) = -f(X)^* = -\{x : d(x, f(X)) \leq \varepsilon\} = \{x : d(x, f(X)) > \varepsilon\},$$

or

$$x \in X \Rightarrow d(x, f(X)) > \varepsilon. \tag{1}$$

□

Claim 2.3 *For every $x \in X$ and every $y \in \partial(X^*)$, $d(x, y) > \varepsilon$.*

Proof Let $x \in X$ and $y \in \partial(X^*)$. Clearly $d(y, X) = \varepsilon$, and since $d(x, y) \geq d(X, y)$, $d(x, y) \geq \varepsilon$. Assume $d(x, y) = \varepsilon$. Since X is open, we can find $x' \in X$ such that $d(x', y) < d(x, y) = \varepsilon$. But this contradicts the fact that $d(y, X) = \varepsilon$. This proves the claim. □

Claim 2.4 $x \in X$ & $y \in \mathrm{cl}(f(X)) \Rightarrow d(x, y) > \varepsilon$.

Proof Let $x \in X$. If $y \in f(X)$, then, by the definition of $f(X)$, $d(x, y) \geq d(y, X) > \varepsilon$, and the claim holds. Suppose $y \in \partial(f(X))$. But $\partial(f(X)) = \partial(-X^*) = \partial(X^*)$. Then, by Claim 2.3, $d(x, y) > \varepsilon$. □

Proof of Equation (1) Let $x_0 \in X$. For every $y \in f(X)$, $d(y, X) > \varepsilon$, hence, since $d(x_0, y) \geq d(y, X)$, $d(x_0, y) > \varepsilon$. Thus $d(x_0, f(X)) \geq \varepsilon$. Assume $d(x_0, f(X)) = \varepsilon$. By Claim 2.4, there is no $y \in f(X)$ such that $d(x_0, y) = \varepsilon$. So for every $n > 0$, there must be a $y_n \in f(X)$ such that $\varepsilon < d(x_0, y_n) < \varepsilon + 1/n$. Clearly we can take all y_n to be, say, in the interval $[x_0 - 1, x_0 + 1]$, so, by compactness, there is a subsequence of $(y_i)_i$ converging to y^+. Then $y^+ \in \mathrm{cl}(f(X))$

and $d(x_0, y^+) = \varepsilon$. Since, however, $x_0 \in X$ and $y^+ \in \mathrm{cl}(f(X))$, by Claim 2.4, $d(x_0, y^+) > \varepsilon$, a contradiction. This proves that $d(x_0, f(X)) > \varepsilon$ as required. □

So, again, what is a negation? All we can say is that *most* negations share N1 and N2. Most but not all. Post negation, for instance, referred to at the beginning, is not order-inverting; indeed as soon as there are at least three truth values $t_0 < t_1 < t_2$, we have $t_0 < t_1$ and yet $\neg t_0 = t_1 < t_2 = \neg t_1$. On the other hand, the standard negation of fuzzy logic $\neg : [0, 1] \to [0, 1]$, such that $\neg x = 1 - x$, does not satisfy N1. (Recall that in the last case we refer to the lattice $[0, 1]$ with operations $\wedge = \min$, $\vee = \max$, and $\bot = 0$.)

In this paper we shall confine our attention to negations satisfying N1 and N2 as basic properties. N3, on the other hand, is a special property.

Lemma 2.5

1. N1 + N2 + N3 *imply that* $f(\bot)$ *is the top element of the lattice, in which case we write* $f(\bot) = \top$. *Moreover,* $f(\top) = \bot$.
2. N1 + N2 *imply that* $f(\bot)$ *is the top element of the set* $f[A]$ *and we write* $f(\bot) = \top_f$. *Then* $f(\top_f) = \bot$.

Proof (1) Since for every x, $\bot \le f(x)$, $f^2(x) \le f(\bot)$, therefore, by N3, $x \le f^2(x) \le f(\bot)$, that is, $f(\bot) = \top$ is greatest. Further, since by N1 $f(\top) \wedge \top = \bot$, necessarily $f(\top) = \bot$.

(2) For every $x \in A$, $\bot \le x$ implies $f(x) \le f(\bot) = \top_f$. Hence the first claim. In particular, $f(\top_f) \le \top_f$, so $\bot = \top_f \wedge f(\top_f) = f(\top_f)$. □

Definition 2.6 Given any lattice A, a *negation* in A is any mapping $f : A \to A$ satisfying N1 and N2. If f satisfies, in addition, N3, it is said to be *regular*. f is a *pseudocomplement* if it satisfies (‡) and a *complement* if and only if, in addition, $f^2(x) = x$ for all x. The mapping f is said to be *periodic*, if $f^m = f^n$ for some $m \ne n$. The *global period* of f is the least n for which there is $m < n$ such that $f^m = f^n$. In this case the number $s = n - m$ is said to be the *global pace* of f. Similarly we define *local periodicity*, *local period*, and *local pace* of f at a point x. The pair $(n - s, n)$ for a periodic f is the *global index* of f. If k, t is the period and the pace respectively of f at x, $(k - t, k)$ is the *local index* of f at x.

Lemma 2.7 *For every regular negation* f, $f^3 = f$.

Proof By regularity $f^2(x) \ge x$ for every $x \in A$. This implies, on the one hand, $f^3(x) \ge f(x)$ replacing x by $f(x)$, and on the other, $f^3(x) \le f(x)$ by order-inversion. □

Notice that a periodic f is periodic at each particular point but the converse is not true. The following contains some standard facts about periodicity.

Lemma 2.8 *Let* f *be any periodic (respectively, periodic at* x*) mapping with index (local index)* (m, n)*. If* $k < l$ *and* $f^k = f^l$ *(respectively,* $f^k(x) = f^l(x)$*), then* $n \le l$*,* $m \le k$*, and* $n - m | l - k$.

Proof We show the global case, the local being similar. Let $k < l$ and $f^k = f^l$. By the definition of n, $n \le l$. If $l = n$, clearly $k = m$ (otherwise $f^k = f^m$ which means that n is not the least collapsing iterate). So $n < l$ and the first claim holds. Suppose $k < m$. Let $s = n - m$ and let $p = \max\{a : l - as \leq n\}$. Then $l - ps \leq n < l - (p-1)s$, hence $k < m = n - s < l - ps < n$. But $f^{l-ps} = f^l = f^k$ which again contradicts the fact that n is the period. So the second claim holds. Finally, assume that $l - k$ is not a multiple of $s = n - m$. Define p as before and also let $q = \max\{a : k - as \leq n\}$. Then clearly $s \leq k - qs \neq l - ps \leq n$ and $f^{l-ps} = f^{k-qs}$. Since $s = k - qs$ and $n = l - ps$ cannot both be true, this is a contradiction. □

Lemma 2.9 *Let (m, n) be the global index of f and (k, l) the local index at x. Then $k \leq m$, $l \leq n$, and $l - k | n - m$. Moreover, the global pace $s = n - m$ is even.*

Proof The relation between (k, l) and (m, n) follows immediately from Lemma 2.8. In particular, by Lemma 2.5, the index of $\perp$ is always $(0, 2)$, therefore $2|n - m$, hence s is even. □

Definition 2.10 Given a Heyting algebra A, x is said to be *complemented* if there is y such that $x \wedge y = \perp$ and $x \vee y = \top$. Such a y is said to be a *complement* of x.

It is well known that for a distributive pseudocomplemented A, every $x \in A$ can have at most one complement and this is $-x$. Moreover, A is a Boolean algebra if and only if every $x \in A$ is complemented.

Proposition 2.11 *Let A be a lattice as above. There can be no periodic negation of index $(0, n)$, for $n > 2$.*

Proof Let f have index $(0, n)$. Observe first that f must be one-to-one because $f(x) = f(y)$ implies $f^n(x) = f^n(y)$, hence $x = y$. Second, we can easily see that $x < y \Leftrightarrow f(y) > f(x)$. One direction follows by order inversion and the other by the periodicity. Further f must be a dual automorphism, that is, $f(x \wedge y) = f(x) \vee f(y)$ and $f(x \vee y) = f(x) \wedge f(y)$. Indeed, obviously $f(x \wedge y) \geq f(x) \vee f(y)$. So assume $f(x \wedge y) > f(x) \vee f(y)$. Then by the preceding remarks, $f^2(x \wedge y) < f(f(x) \vee f(y)) \leq f^2(x) \wedge f^2(y)$. That is, $f^2(x \wedge y) < f^2(x) \wedge f^2(y)$. Continuing this way, since n is even, we get $f^n(x \wedge y) < f^n(x) \wedge f^n(y)$, or $x \wedge y < x \wedge y$, a contradiction.

Claim 2.12 *For every complemented element $x \in A$, $f(x) = -x$ and $f(-x) = x$, hence $f^2(x) = x$.*

Proof If x is a complemented element, then by the previous comments, $f(x) \wedge f(-x) = \perp$ and $f(x) \vee f(-x) = \top$, that is, $f(x)$ and $f(-x)$ are complements of each other, or

$$(*) \qquad -f(x) = f(-x).$$

Now $f(x) \leq -x$, hence $-f(x) \geq x$. Therefore, in view of $(*)$, $f(-x) \geq x$. Since also $f(-x) \leq x$, we get $f(-x) = x$ and $f(x) = -x$. This proves the claim. □

By assumption there is at least one element $x \in A$, such that x, $f(x), \ldots, f^{n-1}$ are all distinct and $n \geq 4$. Then $f(x \vee f(x)) = f(x) \wedge f^2(x) = \perp = f(\top)$. Since f is one-to-one, $x \vee f(x) = \top$. Therefore x is complemented with complement $f(x)$. It follows from the claim that $f^2(x) = x$ which contradicts our assumption. □

The following result shows that negations of any global index (m, n), with $n-m$ even and $0 < m$ are possible. A set $X \subseteq A$ is said to be an *antichain* if for any distinct $x, y \in X$, $x \not\leq y$ and $y \not\leq x$.

Proposition 2.13 *Let A be a Heyting algebra containing antichains of length n. Then for every $2 \leq m < n$ such that $n-m$ is even, there is a negation in A of index (m, n). If A contains n pairwise disjoint elements, then the same holds for every m with $0 < m < n$, $n-m$ even.*

Proof By the hypothesis we can choose an antichain $C = \{c_0, \ldots, c_{n-1}\}$ of cardinality n. Set $f(c_i) = c_{i+1}$ for every $0 \leq i \leq n-2$ and $f(c_{n-1}) = c_m$. It is easy to see that the index of c_k for $k \leq m$ is $(m-k, n-k)$ whereas for $m \leq k \leq n-1$ the index of c_k is $(0, n-m)$.

Set $f(\bot) = \top$ and $f(\top) = \bot$. Next for every $x \in A$, let $C_x = \{c_i \in C : x \leq c_i\}$. If $C_x = \varnothing$, we set $f(x) = \bot$. If $C_x \neq \varnothing$, we set $f(x) = \bigvee f(C_x)$. Observe that if $C_x \neq \varnothing$, there is no c_i such that $c_i \leq x$; otherwise we would have $c_i \leq x \leq c_j$ which is impossible by the fact that C is an antichain.

Now if $C_x = \varnothing$, $f(x) = \bot$, so the index of x is $(1, 3)$. If C_x is a singleton, say $C_x = \{c_i\}$, then $f(x) = f(c_i)$ hence $f^n(x) = f^n(c_i) = f^m(c_i) = f^m(x)$, that is, the index of x is (m, n). But if $|C_x| \geq 2$, and $C_x \neq \{c_{m-1}, c_{n-1}\}$, then, because of the antichain condition, $\bigvee f(C_x) \not\leq c_i$ for any c_i, hence $f(x) = \top$ and $f^2(x) = \bot$; thus the index of x is $(2, 4)$.

Let $x, y \in A$. Then clearly $x \leq y \Rightarrow C_y \subseteq C_x$. Therefore, if $C_x = \varnothing$, then $C_y = \varnothing$, too and $f(x) = f(y) = \bot$; hence $f(y) \leq f(x)$. If $C_x \neq \varnothing$ and $C_y = \varnothing$, then $f(y) = \bot \leq f(x) = \bigvee f(C_x)$. If $C_x \neq \varnothing$ and $C_y \neq \varnothing$, then $C_y \subseteq C_x$ clearly implies $\bigvee f(C_y) \leq \bigvee f(C_x)$, that is, $f(y) \leq f(x)$. Therefore f is order-inverting.

Concerning N1, if $f(x) = \bot$ the property holds trivially. Otherwise, $x \leq \bigwedge C_x$ and $f(x) = \bigvee f(C_x)$. By distributivity and the fact that $f(c_i) \wedge c_i = \bot$ for all $c_i \in C$, we easily see that $(\bigwedge C_x) \wedge (\bigvee f(C_x) = \bot$. Hence also $x \wedge \bigvee f(C_x) = \bot$ or $x \wedge f(x) = \bot$.

Finally, since the local indexes of the elements are $(0, 2)$, $(1, 3)$, or $(2, 4)$, and $(m-k, n-k)$ for $k \leq m$, it follows that the global index is (m, n). By Lemma 2.9, $m \geq 2$.

If in the above construction the elements of C are pairwise disjoint, then we easily check that there are no elements of local index $(2, 4)$. These elements were the only reason to require $m \geq 2$. So m can be taken to be just > 0. □

In contrast to the preceding result, not allowing global indexes of the form $(0, n)$, $n > 2$, there exist negations allowing elements to have local index $(0, n)$, even for odd n.

Proposition 2.14 *In a Heyting algebra, for every $n > 0$, there are negations with local index $(0, n)$ at some point.*

Proof Let $c_0, c_1, \ldots, c_{n-1}$ be pairwise disjoint. Define f as follows. $f(c_i) = c_{i+1}$ for $i < n-1$ and $f(c_{n-1}) = c_0$. If $x \not\leq c_i$ for every i, we set $f(x) = \bot$. If $x = \bot$ we set $f(x) = \top$. Otherwise there is a unique c_i such that $x \leq c_i$. Then we set $f(x) = f(c_i)$. As in the previous proposition, it is easy to check that

1. The index of every c_i is $(0, n)$.
2. For $\bot \neq x < y_i$, the index of x is $(1, n+1)$.

3. The indexes of $\perp$ and $\top$ are $(0, 2)$.
4. For $\top \neq x \not\leq c_i$, the index is $(1, 3)$.

The global pace s of f is the least common multiplier of the local paces of the elements, that is, $s = \mathrm{lcm}(2, n)$. Thus the global index of f is $(1, s+1)$. □

Observe that whenever we have a point with local index $(m, m+s)$, we have also a point with index $(0, s)$, that is, a fixed point for the mapping f^s. It is well known that if a continuous real mapping $f : I \to I$, where I is an interval $[a, b]$, has a point of index $(0, s)$, for *odd* s, then f has points of index $(0, k)$ for every even k, as well as for every odd $k > s$. Especially for $s = 3$, f is chaotic. This follows from the nice theorem of Šarkovski (e.g., Block and Coppel [2] or Li and Yorke [6]). The theorem reveals the tremendous difference between the periods (=paces) 2 and 3.

Although our setting is quite remote from that of analysis, we can still see the great difference between even and odd pace. In Proposition 2.13 we constructed negations of any pace, but for most $x \in A$, the orbit of x contains $\perp$. This makes the situation a bit trivial. If we require the orbit of x, and also the orbits of $-x$, $--x$, $f(-x)$, and so on, not to contain $\perp$, then we can show that, if f is periodic at x, the pace is even.

Definition 2.15 Given a negation $f : A \to A$, let $\{-, f\}^*$ be the the set of mappings which is a word of the alphabet $\{-, f\}$. The set

$$N_f = \{x : (\exists h \in \{-, f\}^*)(h(x) = \perp)\}$$

is said to be the *nucleus* of f.

Lemma 2.16 *Let A be a Heyting algebra and let f be any negation on A. Then for every $x \notin N_f$ and every $k \in \mathbb{N}$,*

1. $f^{2k}(x) \wedge x \neq \perp$ *and*
2. $f^{2k+1}(x)$ *and x are incomparable.*

Proof Let $x|y$ mean 'x, y are incomparable'. Fix some $x \notin N_f$, that is, for all $h \in \{-, f\}^*$, $h(x) \neq \perp$. Clearly, for every such h, $h(x) \notin N_f$. We prove (1) and (2) by simultaneous induction on k. Let $k = 0$. Then (1) follows from the fact that $x \neq \perp$. If (2) is not true, then either $x \leq f(x)$ or $f(x) \leq x$. In the first case $x \leq f(x) \leq -x$, whence $x = \perp$ and in the second case $f(x) \leq x$ and $f(x) \leq -x$, hence $f(x) \leq \perp$. Thus the claim follows from the fact that $x, f(x) \neq \perp$. Therefore it suffices to assume

(a_k) $(\forall x \notin N_f)(f^{2k}(x) \wedge x \neq \perp)$ and
(b_k) $(\forall x \notin N_f)(f^{2k+1}(x)|x)$,

and to prove

(a_{k+1}) $(\forall x \notin N_f)(f^{2k+2}(x) \wedge x \neq \perp)$ and
(b_{k+1}) $(\forall x \notin N_f)(f^{2k+3}(x)|x)$.

Proof of (a_{k+1}) Assume that (a_{k+1}) is false, that is, for some $x \notin N_f$, $f^{2k+2}(x) \wedge x = \perp$. Then $f^{2k+2}(x) \leq -x$. On the other hand, $f(x) \leq -x$ implies $f^2(x) \geq f(-x)$, whence, applying the order-preserving f^{2k}, we get $f^{2k+2}(x) \geq f^{2k+1}(-x)$. Thus $f^{2k+1}(-x) \leq f^{2k+2}(x) \leq -x$, that is, $f^{2k+1}(-x) \leq -x$, which, since $-x \notin N_f$ contradicts (b_k). □

Proof of (b_{k+1}) Assume (b_{k+1}) is false. Then for some $x \notin N_f$ either $x \leq f^{2k+3}(x)$ or $f^{2k+3}(x) \leq x$. □

Case 1 Let $x \leq f^{2k+3}(x)$. $f(x) \leq -x$ again implies (applying f^{2k+2}) $f^{2k+3}(x) \leq f^{2k+2}(-x)$. Therefore $x \leq f^{2k+2}(-x)$. But $f^{2k+2}(-x) \wedge f^{2k+1}(-x) = \bot$; hence $x \wedge f^{2k+1}(-x) = \bot$ or $f^{2k+1}(-x) \leq -x$,which contradicts (b_k) because $-x \notin N_f$.

Case 2 Let $f^{2k+3}(x) \leq x$. Since $f(x) \wedge x = \bot$, the preceding inequality implies $f^{2k+3}(x) \wedge f(x) = \bot$ or $f^{2k+2}(f(x)) \wedge f(x) = \bot$. Since $f(x) \notin N_f$, this contradicts (a_{k+1}). □

Lemma 2.17 *Let f be a negation and $x \notin N_f$.*

1. *If $f^k(x) = f^l(x)$, then $|k - l|$ is even.*
2. *Let $f^k(x) = f^{k+s}(x)$.*

Then the set $Y = \{f^k(x), f^{k+1}(x), \ldots, f^{k+s-1}(x)\}$ is an antichain (provided its elements are all distinct).

Proof (1) Let $k < l$ and $l = k + s$. Then $f^k(x) = f^{k+s}(x)$, or for $y = f^k(x)$, $y = f^s(y)$. Since $x \notin N_f$, clearly $y \notin N_f$. By Lemma 2.16(2), for s, odd y and $f^s(y)$ should be incomparable. Therefore s must be even.

(2) Note that $f^k(x) = f^{k+s}(x)$ implies $f^{k+i}(x) = f^{k+i+s}(x) = f^{k+i+js}(x)$, for every i, j. By Lemma 2.16(2), all elements of Y which are an odd number of steps apart are incomparable. So it suffices to show that this is also the case for elements of Y which are an even number of steps apart. Two such elements are of the form $f^{k+i}(x)$ and $f^{k+i+2j}(x)$, for suitable i, j. Suppose they are comparable. Then either $f^{k+i}(x) < f^{k+i+2j}(x)$ or $f^{k+i}(x) > f^{k+i+2j}(x)$. Assume the first. $f^{k+i}(x) < f^{k+i+2j}(x)$ implies $f^{k+i+2j}(x) < f^{k+i+4j}(x)$, and continuing this way we shall get $f^{k+i}(x) < f^{k+i+2j}(x) < f^{k+i+2sj}(x)$. But $f^{k+i+2sj}(x) = f^{k+i}(x)$, hence $f^{k+i}(x) < f^{k+i}(x)$, a contradiction. The case $f^{k+i}(x) > f^{k+i+2j}(x)$ is similar. □

Remark 2.18 Are there negations without periodic points except $\bot$ and $\top$? Note that for every negation f, the mappings f^{2n} are order-preserving, hence, if A is complete, by Tarski's Fixed Point Theorem (see [1]), for every x such that $x \leq f^{2n}(x)$ there is a point $a \geq x$ such that $f^{2n}(a) = a$. However, the proof of this theorem does not guarantee either that $a \neq \bot, \top$ or that $2n$ is the least k such that $f^k(a) = a$.

3 Negations Induced by Interiors

Definition 3.1 A mapping $i : A \to A$ in the lattice A is said to be an *interior operator* on A (or just an *interior*), if

1. $i(x \wedge y) = i(x) \wedge i(y)$,
2. $i(x) \leq x$, and
3. $i^2 = i$.

i is a *quasi interior* if (1) and (2) hold and is a *weak interior* if (2) holds and i is order preserving.

The dual definitions of a closure, quasi closure and weak closure $c : A \to A$ read in the obvious way with the preceding conditions replaced with: (1) $c(x \vee y) = c(x) \vee c(y)$, (2) $x \leq c(x)$, and (3) $c^2 = c$.

It follows easily that every quasi interior i (quasi closure c) is order-preserving; therefore every quasi interior (closure) is a weak interior (closure) but not vice versa.

Notice that the classes of quasi and weak interiors (closures) are closed under composition. We often write fg instead of $f \circ g$ and ix, cx instead of $i(x)$, $c(x)$.

Proposition 3.2 *Let f be a regular negation. For every interior i (closure c) the mapping $g = if$ (respectively, $g = fc$) is a negation such that $g^4 = g^2$; that is, g has period at most 4. Moreover, there are i, f, c as above such that if and fc have period 4.*

Proof That if and fc are negations is obvious. We show the claim for $g = if$, the other being similar. We have $ifx \leq fx$, hence $fifx \geq f^2x$. By assumption $f^2x \geq x$, therefore $fifx \geq x$. This implies $ififx \geq ix$, and setting ix for x, $g^2(ix) = ifif(ix) \geq i^2x = ix$. Therefore the restriction g_1 of g to $i[A]$ is regular; hence as in Lemma 2.7, we see that $g_1^3 = g_1$. Thus

$$g^4(x) = g^3(if(x)) = g_1^3(ifx) = g_1(ifx) = g(ifx) = g^2(x).$$

Further we give examples of f and i such that if is of period exactly 4 (in fact f will be a complement). Let i be the interior operator in $\mathbb{R}$ with respect to the usual metric and let $-X$ be the complement in the Boolean algebra $P(\mathbb{R})$. Consider the mapping $g(X) = i - X$. Clearly g satisfies the conditions of (2) above, so $g^4 = g^2$. We show that $g^3 \neq g$, that is, for some $X \subseteq \mathbb{R}$, $g^3(X) \neq g(X)$. Let

$$X = \{0\} \cup (1, 2].$$

Then

$$\begin{array}{rcl} rcl - X & = & (-\infty, 0) \cup (0, 1] \cup (2, \infty). \\ g(X) & = & i - X = (-\infty, 0) \cup (0, 1) \cup (2, \infty). \\ -g(X) & = & \{0\} \cup [1, 2]. \\ g^2(X) & = & i - g(X) = (1, 2). \\ -g^2(X) & = & (-\infty, 1] \cup [2, \infty). \\ g^3(X) & = & i - g^2(X) = (-\infty, 1) \cup (2, \infty) \neq g(X). \\ -g^3(X) & = & [1, 2]. \\ g^4(X) & = & i - g^3(X) = (1, 2) = g^2(X). \end{array}$$

Note that if f is a complement, i an interior, and set $c = fif$, then $fc = if$, so the above example provides also a closure c with the required property. □

A Heyting algebra A endowed with an interior i is a *topological* Heyting algebra (tHA). Recall that the logical analogue of i is a necessity operator $\Box$. Augmenting the language of Intuitionistic Propositional Logic (IPL) with $\Box$, let IML (for Intuitionistic Modal Logic) consist of the usual axioms of IPL plus the modal axioms $\Box\varphi \to \varphi$, $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, and $\Box\varphi \to \Box\Box\varphi$, and the rules Modus Ponens and Necessitation. It is well known (see, for example, Font [4]) that tHAs form sound and complete algebraic semantics for IML. Due to this correspondence, Proposition 3.2 yields immediately the following.

Corollary 3.3 IML *proves* $(\Box\neg)^4\varphi \leftrightarrow (\Box\neg)^2\varphi$ *for every* φ*, whereas* $(\Box\neg)^3\varphi \leftrightarrow \Box\neg\varphi$ *is not provable.*

Proof By Proposition 3.2, the interpretation of $(\Box\neg)^4\varphi \leftrightarrow (\Box\neg)^2\varphi$ is true in every (A, i), hence, by completeness, the formula is provable. On the other hand, the interpretation of $(\Box\neg)^3\varphi \leftrightarrow \Box\neg\varphi$ is false in some algebra, so, by soundness, the equivalence in question is unprovable. □

This result is not new. Došen [3] and [4] have shown that in the system IML above there are only thirty-one non(provably)-equivalent modalities, that is, strings of words $s_1, \dots, s_n$, with $s_i \in \{\neg, \Box\}$.

Given an interior i, call *modality* any function $f \in \{-, i\}^*$, that is, f is a word of the alphabet $\{-, i\}$. It is natural to ask which of the modalities are negations, and, if periodic, of what index.

Lemma 3.4 *Let* A *be as before,* i *be an interior in* A*, and* $h : A \to A$ *be a modality with respect to* i *and* $-$*. If* h *is a negation then* h *is one of the following:* $-$*,* $i-$*, and* $--i-$*. These are periodic with indexes* $(1, 3)$*,* $(2, 4)$*, and* $(2, 4)$ *respectively.*

Proof Note that $-$ regular, hence $(-)^3 = -$. Consequently, by Proposition 3.2, $(i-)^4 = (i-)^2$. Let $g = --i-$. Since g contains three $-$s, g is order-inverting. On the other hand, $i-x \le -x$, hence $-i-x \ge --x \ge x$. Hence $gx \le --i-x \le -x$. Thus g is a negation. Moreover, $g^2 = --i-i-$, $g^3 = --i-i-i-$, and $g^4 = --i-i-i-i-- = --i-i- = g^2$ (by the periodicity of $i-$). Hence g is of period 4. Note that $ig = i--i-$ is also a negation. However, $ig = i-$. Indeed, since $y \le --y$, $i-x \le --i-x \le -x$, whence $i-x \le i--i-x \le i-x$, Therefore $i-x = i--i-x$ for every x.

We show that there are no other negations formed from $-$ and i. Let h be such a negation. Since h is order-inverting, it must contain an odd number of $-$s, say $2k+1$.

1. For $k = 0$, all possible words are $-$, $i-$, $-i$, $i-i$, of which only the first two are negations and these are contained in our list.
2. For $k = 1$, the possible words are $--i-$, $i--i-$, $i--i-i$, $-i--$, $i-i--$, $-i--i$, $i-i--i$, $-i-i-$, $i-i-i-$, $-i-i-i$, $i-i-i-i$. It is tedious to check that from these only the first two are negations which also belong to our list (the second being equal to $i-$).
3. For $k = 2$ (five $-$s), if the word contains $i--i-$ or $i-i-i-$, it collapses to one with fewer $-$s, since $i--i- = i-$ and $i-i-i- = i-$ as we saw above. So the only nonreducing words of this type are $--i-i--$, $-i-i-i--$, and $--i-i--i$. But none of these is a negation.
4. For $k > 2$, clearly the word contains some of the patterns $i--i-$, $i-i-i-$, so it eventually collapses to one of the previous cases. Thus all the possible negations are captured in steps (1)–(3). □

Can we characterize *all* negations of a Heyting algebra in terms of quasi or weak interior operators and the pseudocomplement? The next result says that this is true for Boolean algebras but not for Heyting ones.

Lemma 3.5 *Let* A *be a lattice with pseudocomplement* $-$*. For every negation* f *in* A*, there is a weak interior* i *such that* $i- \le f \le -$*, hence* $if = i-$*. If* A *is Boolean,* $f = i-$*.*

Proof Let f be a negation. Then $fx \le -x$. Set $ix = x \wedge f - x$. Then $ix \le x$. Further, let $x \le y$. Then $f - x \le f - y$, hence $x \wedge f - x \le y \wedge f - y$ or $i(x) \le i(y)$. It follows that i is a weak interior. Also from $- - x \ge x$ we get $f - -x \le fx$, therefore $i - x = f - -x \le fx \le -x$. Thus $i- \le f \le -$, and applying i to the latter, $i- = if$. If A is Boolean, $i - x = f - -x = fx$, that is, $i- = f$. □

4 Nonperiodic Negations

Strangely enough, Proposition 3.2 does not generalize to $n > 4$. Period 4 and Pace 2 seem to be the highest barriers for "naturally defined" periodic negations. Higher periods and paces can be obtained by the method of Proposition 2.13.

The only reasonable way to generalize Proposition 3.2 seems to be by combining two or more interior operators (modalities), for example, considering the negation $g = ij-$, where i, j are interiors. But $i \circ j$ is no longer an interior; it is a *quasi* interior, as already pointed out in the last section (even if $-$ is a complement). But when i is a quasi interior, the mapping $g = i-$ is, in general, nonperiodic. More strongly, it can be nonperiodic *at a point*. In this section we give an example of such a negation.

In the lattice $P(\mathbb{N})$ consider the mappings

$$j(X) = \{x \in X : x + 1 \in X\} \text{ and } h(X) = j - X.$$

It is easy to see that j is a quasi interior. Let also $C : \mathbb{N} \to \mathbb{N}$ be the predecessor mapping of $\mathbb{N}$, that is, $C(x) = x - 1$, if $x > 0$ and $C(0) = 0$. Given X and $x \in X$, x is said to be *isolated in* X if $x - 1$ (when it exists) and $x + 1$ do not belong to X.

Lemma 4.1

1. *If* 0 *is not isolated in* X*, then* $C(X) \subseteq h^2(X)$.
2. *If* $-X$ *does not contain isolated elements, then* $h^2(X) \subseteq C(X)$.

Proof (1) Let 0 be nonisolated in X. Then either $0 \notin X$, or $0 \in X$ and $1 \in X$. We have to show that $x \in X \Rightarrow C(x) \in h^2(X)$. Let $x = 0$ and $x \in X$. By the nonisolation of 0, $1 \in X$. We verify that $C(0) = 0 \in h^2(X)$. Notice that by $j - X \subseteq -X$ we get $-j - X \supseteq X$ for every X. So $0, 1 \in X$ implies $0, 1 \in -j - X$. Thus by the definition of j, $0 \in j - j - X = h^2(X)$.

Let now $x \in X$ and $x \neq 0$. We shall show that $C(x) \in h^2(X)$.

Case 1 $x - 1 \in X$. Then $x - 1, x \in -j - X$, and by the definition of j, $x - 1 \in j - j - X = h^2(X)$, or $C(x) \in h^2(X)$.

Case 2 $x - 1 \notin X$. Then $x - 1 \in -X$. Since $x \notin -X$, $x - 1 \notin j - X$, hence $x - 1 \in -j - X$. So again $x - 1, x \in -j - X$, and as before $C(x) \in h^2(X)$.

(2) Let X be as stated and let $x \in h^2(X)$. We must show that $x \in C(X)$ or $x + 1 \in X$. Now $x \in j - j - X$ implies $x + 1 \in -j - X$. To reach a contradiction, assume $x + 1 \notin X$. Then $x + 1 \in -X$, and by assumption, either $x \in -X$ or $x + 2 \in -X$. Assume the first. Then $x, x + 1 \in -X$, hence $x \in j - X$. But this contradicts the fact that $x \in -j - X$. Assume $x + 2 \in -X$. Then $x + 1, x + 2 \in -X$, hence $x + 1 \in j - X$. But also $x + 1 \in -j - X$, a contradiction. This proves the claim and the lemma. □

Proposition 4.2

1. *There is $X \subseteq \mathbb{N}$ such that h above is nonperiodic at X, that is, $h^n(X) \neq h^m(X)$ for all $m \neq n$.*
2. *For every $n > 0$ there is $X \subseteq \mathbb{N}$ such that h is periodic at X with local index $(2n-2, 2n)$.*

Proof (1) Consider a partition of $\mathbb{N}$ into disjoint intervals I_k, J_k, $k \geq 1$ such that

(i) $I_1 < J_1 < I_2 < \cdots < I_k < J_k < \cdots$.

(ii) $|I_k| = k$, while $|J_k| = 2$ for all $k \geq 1$.

Thus, $I_1 = \{0\}$, $J_1 = [1,2]$, $I_2 = [3,4]$, $J_2 = [4,5]$, $I_3 = [6,7,8]$, and so on. Let $X = \cup_{k\geq 1} I_k$. Then $-X = \cup_{k\geq 1} J_k$ and $h(X) = j - X = \cup_{k\geq 1} J'_k$, where $J'_k = J_k - \{\max J_k\}$. By Lemma 4.1, for every $k > 0$, $h^{2k}(X)$ is either $C^k(X)$ or $C^k(X) - \{0\}$, and $h^{2k+1}(X)$ is either $C^k(h(X))$ or $C^k(h(X)) - \{0\}$. Since C pushes X leftward, and X is a union of intervals of increasing length, for every m there is k such that $C^k(X)$ contains only intervals of length $\geq m$. Similarly the sets $C^k(h(X))$ contain bigger and bigger gaps between their elements. So the reader can easily verify that for no $m < n$, $h^m(X) = h^n(X)$.

(2) For $n = 1$, just take $X = \varnothing$. Then $h^2(\varnothing) = \varnothing$. For $n = 2$ let $X = \{0\}$. Then $h^2(X) = \varnothing$, $h^3(X) = \mathbb{N}$, and $h^4(X) = \varnothing = h^2(X)$. For $n > 2$ let again $X = [0, n-2]$. Since neither 0 is isolated in X nor $-X$ contains isolated elements, by Lemma 4.1, $h^2(X) = C(X) = [0, n-3]$. Inductively $h^{2(k-2)}(X) = [0, n-k]$; hence $h^{2(n-2)}(X) = \{0\}$ and $h^{2n}(X) = h^4(\{0\}) = \varnothing = h^2(\{0\}) = h^{2n-2}(X)$. Therefore for every $n \geq 2$, h is of period $2n$ and pace 2 at $X = [0, n-2]$. □

Remark 4.3 The quasi interior j used in the preceding example is the composition of two interiors i_o, i_e defined in $P(\mathbb{N})$ as follows:

$$i_o(X) = X^{\mathrm{ev}} \cup \{x \in X^{\mathrm{od}} : x+1 \in X\},\ i_e(X) = X^{\mathrm{od}} \cup \{x \in X^{\mathrm{ev}} : x+1 \in X\},$$

where X^{ev} and X^{od} are the subsets of X of even and odd elements, respectively. Then $j = i_o \circ i_e = i_e \circ i_o$. The topology generated by i_o (or i_e) is called the Hjalmar Ekdal Topology (see Steen and Seebach [10], p. 78).

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