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The Decidability of the $\forall^*\exists$ Class and the Axiom of Foundation

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Abstract We show that the Axiom of Foundation, as well as the Antifoundation Axiom *AFA*, plays a crucial role in determining the decidability of the following problem. Given a first-order theory T over the language $=, \in$, and a sentence F of the form $\forall x_1, \dots, x_n \exists y F^M$ with F^M quantifier-free in the same language, are there models of T in which F is true? Furthermore we show that the Extensionality Axiom is quite irrelevant in that respect.

1 Introduction

Let $\mathcal{L}_\in$ be the first-order language with equality and a binary relation symbol $\in$. We investigate the following decision problem: given a theory T and a $\forall^*\exists$ sentence in $\mathcal{L}_\in$, namely, a sentence F of the form $\forall x_1, \dots, \forall x_n \exists y F^M$ where F^M is quantifier free, determine whether there is a model of T in which F is true. We will restrict our attention to theories in which $\in$ retains at least some of its ordinary meaning of membership relation. More precisely we will consider only theories which extend the theory *NW* whose axioms in skolemized form over $\mathcal{L}_\in$ are the nullset axiom $(N)\forall x(x \notin \emptyset)$ and the with axiom $(W)\forall x \forall y \forall z(x \in \mathbf{w}(y, z) \leftrightarrow x \in y \vee x = z)$.

Omodeo et al. [5] establishes the completeness with respect to $\exists^*\forall$-sentences of the theory *NWLER* whose axioms are, besides N and W, the axiom L for the removal of an element from a set, which in skolemized form reads as $\forall x \forall y \forall z(z \in x\,\mathbf{l}\,y \leftrightarrow z \in x \wedge z \neq y)$; the Extensionality Axiom E, $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$; and the Regularity Axiom R, $\forall x(x \neq \emptyset \rightarrow \exists y \in x \forall z \in y(z \notin x))$. Actually from the proof in [5], it is easy to recognize that, thanks to the presence of axiom R, such a completeness result also holds if axiom L is omitted. Thus *NWER* is complete with respect to $\exists^*\forall$ sentences, hence also with respect to $\forall^*\exists$ sentences, and our decision problem is solvable whenever T is an extension of *NWER*.

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On the other hand, Bellè and Parlamento [3] establishes the undecidability of the satisfiability with respect to *NW* of sentences of the form $\forall x F$ where F is a conjunction of equalities with one inequality in the language with the constant $\varnothing$ and the binary function symbol **w**. In *NWE* such sentences are equivalent to $\forall^*\exists$ sentences of $\mathcal{L}_\in$ obtained by eliminating the function symbols $\varnothing$ and **w** in favor of their definitions provided by N and W; however, as shown in [3], on the grounds of results in Bellè and Parlamento [2], their satisfiability with respect to *NWE* is decidable. Thus [5] and [3] leave it open whether the satisfiability of $\forall^*\exists$ sentences with respect to *NW* as well as *NWE* is a decidable property.

We will show that, while in both cases that problem remains undecidable, it is turned into a decidable one as soon as axiom R or else the Antifoundation Axiom called *AFA* in Aczel [1] is added.

2 *NW* + Regularity Axiom

Proposition 2.1 *The problem of establishing whether a $\forall^*\exists$-sentence is satisfiable with respect to NWR is decidable.*

Proof Every $\forall^*\exists$-sentence F admits a normal form which can be obtained as follows. By a (finite) graph with equality we mean a structure $(\{1, \dots, n\}, \in_G, =_G)$, where $=_G$ is an equivalence relation congruent with respect to the binary relation $\in_G$. Let $\hat{G}$ be the formula

$$\bigwedge_{1\le i,j\le n} (y_i \in^G_{i,j} y_j \wedge y_i =^G_{i,j} y_j)$$

where $\in^G_{i,j}$ is $\in$ if there is an edge from the node i to the node j in G, $\in^G_{i,j}$ is $\notin$ otherwise; $=^G_{i,j}$ is $=$ if $i =_G j$, $\neq$ otherwise.

Given a $\forall^*\exists$ sentence F, we first transform it into the disjunctive normal form,

$$\forall y_1, \dots, y_n \bigvee_j (H_j(y_1, \dots, y_n) \wedge \exists x E_j(y_1 \dots, y_n, x)),$$

where

1. the H_js and the E_js are conjunction of literals,
2. each of the E_js contains all the atoms of the form $x \neq y_i$,
3. the variable x occurs in all the atoms of the E_js and in no atoms of the H_i,

and then into the following sentence F':

$$\bigwedge_i \forall y_1, \dots, y_n(\hat{G}_i(y_1, \dots, y_n) \to \bigvee_{j:H_j\subseteq G_i} \exists x E_j(y_1, \dots, y_n, x))$$

where $\{G_i\}$ is the collection of the n-node graphs with equality and $H \subseteq G$ if every literal in H occurs in G.

It is easy to check that F is equivalent to F'; furthermore we may assume that the satisfiability of F with respect to *NWR* is equivalent to the satisfiability with respect to *NWR* of every conjunct in the above normal form. In fact, as we are going to show, if a single conjunct of the form

$$F_0 = \forall y_1, \dots, \forall y_n(\hat{G}(y_1, \dots, y_n) \to \bigvee_j \exists x E_j(y_1, \dots, y_n, x)),$$

is satisfiable with respect to *NWR* then it is satisfiable in the collection of the hereditarily finite sets *HF*. Clearly F_0 is equivalent to the formula obtained by identifying the variables in the equivalence classes induced on $\{y_1, \ldots, y_n\}$ by $=_G$. Therefore, without loss of generality, we can assume the equivalence relation $=_G$ is the identity relation on $\{1, \ldots, n\}$.

It is immediate that if $\in_G$ is cyclic, $\hat{G}$ is not satisfiable in *HF*. In this case F_0 is trivially true in *HF*. Otherwise, since every acyclic graph G can be embedded in *HF*, there exist n distinct sets that satisfy $\hat{G}$ (Appendix A.1). Thus assuming $s_1, \ldots, s_n$ are hereditarily finite sets such that $\hat{G}(s_1, \ldots, s_n)$ holds, we have to find an s that satisfies one of the E_js.

If there exists an $E_j(y_1, \ldots, y_n, x)$ that does not contain complementary literals and whose positive literals are all of the form $y_i \in x_i$, then $s = \{s_i : y_i \in x \text{ is a literal in } E_j\} \cup \{d\}$, where $d = \{s_1, \ldots, s_n\}$ is added in order to make s different from each of the s_is, is a hereditarily finite set such that $E_j(s_1, \ldots, s_n, s)$ holds.

Otherwise a literal of the form $x \in x$ or $x \in y_i$ occurs in each of the E_js. Since there is no set in *HF* that satisfies $x \in x$, the witness for the existential variable, if there exists one, must be among the predecessors of the s_is. Let τ be a graph which extends G and is minimal with respect to the following conditions:

1. τ is acyclic;
2. τ is extensional over G, that is, for $i \neq j$, i and j have different sets of $\in_G$-predecessors.

By results in Parlamento et al. [6], τ has at most $2n - 1$ nodes; thus the set T of such τs is finite. If the collection of the E_js is such that for every τ in T there exists an E_j such that $\tau \models (\exists x E_j(y_1, \ldots, y_n, x))[y_i/i]$, we are guaranteed that at least one of the disjuncts can be satisfied. For consider a minimal differentiating set $\{d_1, \ldots, d_m\}$ for $s_1, \ldots, s_n$, that is, a subset of the set of the predecessors of $s_1, \ldots, s_n$ minimal with respect to the following property: if $s_i \neq s_j$ then there exists a d_k, $1 \leq k \leq m$ such that $d_k \in s_i$ if and only if $d_k \notin s_j$. Since $\{s_1, \ldots, s_n, d_1, \ldots, d_m\}$ with the membership relation is isomorphic to a τ in T, the required condition ensures the existence of an E_j such that $E_j[s_1, \ldots, s_n, d_k]$ for some $1 \leq k \leq m$.

We have thus proved that F_0 is satisfiable if one of the following holds:

1. $\in_G$ is cyclic,
2. all the positive literals of one of the E_js are of the form $y_i \in x$,
3. for every τ in T there exists an E_j such that $\tau \models \exists x E_j(y_1, \ldots, y_n, x)[y_i/i]$.

In order to show that these conditions are also necessary assume F_0 is satisfiable in a model M of *NWR*. We prove that if (1) and (2) do not hold then (3) holds. Since (1) does not hold, T is not empty. Let τ be a graph in T. Since τ is acyclic and extensional over G, there is a map * from τ onto *HF* which is an isomorphism on τ^* and faithful on G, that is, all the members of the sets in the image of G are in the image of τ (see Appendix A.1).

Let $s_1, \ldots, s_n, d_1, \ldots, d_m$ with $m \leq n - 1$ be the images of the nodes in τ. Since *HF* is isomorphically embedded as an $\in$-initial part in every model of *NW* and in particular in M, we can consider $s_1, \ldots, s_n, d_1, \ldots, d_m$ as elements in M. Since F_0 and $G[s_1, \ldots, s_n]$ are true in M there must be a disjunct E_j and an element $s \in M$ such that $M \models E_j[s_1, \ldots, s_n, s]$. Since (2) does not hold and $s \notin^M s$, due to the fact that $M \models R$, s must be among the predecessors of $s_1, \ldots, s_n$. Therefore, since * is faithful on G, $s = d_k$ for some $1 \leq k \leq m$. Finally, since * is an isomorphism on $M|_{\{s_1,\ldots,s_n,d_1,\ldots,d_m\}}$ we conclude that $\tau \models E_j[1,\ldots,n,k]$ for some $1 \leq k \leq m$ and (3) holds.

Thus the effective construction of T and the verification of conditions (1), (2), and (3) above, provide a correct and complete decision test for the satisfiability in *HF* of a conjunct in F. By applying it to every such conjunct we obtain a decision test for the satisfiability with respect to *NWR* of F. □

The following proposition follows as an immediate byproduct of the previous proof.

Proposition 2.2

1. *A $\forall^*\exists$ sentence is satisfiable with respect to NWR if and only if it is true in $(HF, \in)$.*
2. *A $\forall^*\exists$ sentence is satisfiable with respect to NWR if and only if it is satisfiable with respect to NWER so that the same decision test provided in the proof of Proposition 2.1 applies to the theory NWER as well.*
3. *NWER is a conservative extension of NWR as far as $\exists^*\forall$ sentences are concerned.*

Notation 2.3 The conservativeness result stated in the previous proposition does not hold for $\forall^*\exists$ sentences, since axiom E is (logically) equivalent to a $\forall\forall\exists$ sentence and, as it is easy to see, there are nonextensional models of *NWR*.

3 *NW* + Antifoundation Axiom

Let us recall that the antifoundation axiom named *AFA* in [1] states that every graph (G, R) has a unique decoration, namely, a function f whose domain is G and such that $\forall a \in G,\ f(a) = \{f(b) : bRa\}$. The uniqueness of the decoration whose existence is stated in *AFA*, entails a strong form of extensionality that can be analyzed using the notion of bisimulation, which we define as follows. A binary relation R is a bisimulation if

$$aRb \Rightarrow \forall x \in a(x \in b \vee \exists y \in b(xRy)) \wedge \forall y \in b(y \in a \vee \exists x \in a(xRy)).$$

Furthermore we say that R is proper if it contains at least one pair (a, b) with $a \neq b$.

This is a slight variant of the definition of bisimulation given in [1]. Whereas the two are equivalent in ZF-Regularity Axiom (ZF$^-$), only the present one is appropriate when working with weak theories like *NW* (see Appendix A.2).

We formulate the strong extensionality axiom *SE* as follows:

$$(SE) \text{ there are no proper weak bisimulations;}$$

and *AFA*$'$ as the conjuntion of *SE* and

$$(AFA_1) \text{ every graph has at least one decoration.}$$

Note that in *NW*, *SE* entails E; in fact if a and b are different and have the same predecessors then $\{(a, b)\}$, which exists in *NW*, is a proper weak bisimulation.

Proposition 3.1 *The problem of establishing whether a $\forall^*\exists$ sentence is satisfiable with respect to $NW + AFA_1$ is decidable.*

Proof The role played by *HF* in the proof of Proposition 2.1 is now played by the structure V_f of the hereditarily finite hypersets (Appendix A.3); in particular, we will show that if a formula is satisfiable then it is satisfiable in V_f. As in the proof of the corresponding result for $NW + R$ we may restrict our attention to sentences of the form

$$F_0 = \forall y_1, \ldots, \forall y_n (\hat{G}(y_1, \ldots, y_n) \to (\bigvee_j \exists x E_j(y_1, \ldots, y_n, x)),$$

where G is a graph on n nodes, x occurs in every atom of the conjunctions of literals E_j and each of the E_js contains all the atoms $x \neq y_j$.

Let T' be the finite collection of the graphs τ which extend G and are minimal with respect to the property of being strongly extensional over G, that is, there is no proper bisimulation R on G such that if $(i, j) \in R$ then i and j have the same predecessors in $\tau \setminus G$.

We claim that F_0 is satisfiable with respect to $NW + AFA_1$ if and only if one of the following conditions holds:

(2′) one of the E_j does not contain atoms of the form $x \in y_i$;

(3′) for every τ in T' there exists an E_j such that $\tau \models (\exists x E_j(y_1, \ldots, y_n, x))[y_i/i]$.

(⇐) Let $s_1, \ldots, s_n$ be distinct hereditarily finite hypersets such that $\hat{G}[s_1, \ldots, s_n]$ is true in V_f. If (2′) holds we have that the following hyperset s satisfies $E_j(s_1, \ldots, s_n, x)$:

$$s = \begin{cases} \{s_i : y_i \in x \text{ occurs in } E_j\} \cup \{d\} & \text{if } x \in x \text{occurs in } E_j, \\ \{s_i : y_i \in x \text{ occurs in } E_j\} \cup \{d\} \cup \{s\} & \text{otherwise,} \end{cases}$$

where d is a hyperset different from $s_1, \ldots, s_n$ and from the members of any of the s_is. A d that satisfies that condition exists since there is an infinite number of hypersets and every (finite) hyperset has only a finite number of predecessors.

In case (3′) holds we proceed as follows. Let $\{d_1, \ldots, d_m\}$ be a minimal differentiating set for $s_1, \ldots, s_n$, that is, a subset of the set of the predecessors of $s_1, \ldots, s_n$ minimal with respect to the following property: if B is a proper bisimulation on $\{s_1, \ldots, s_n\}$ and $(s_i, s_j) \in B$ then there exists a d_k, $1 \leq k \leq m$ such that $d_k \in s_i$ if and only if $d_k \notin s_j$. It is straightforward to see that $(\{s_1, \ldots, s_n, d_1, \ldots, d_m\}, \in)$ is isomorphic to a τ in T. Hence, from the existence of an E_j such that $\tau \models (\exists x E_j(y_1, \ldots, y_n, x))[y_i/i]$ it follows that $E_j(s_1, \ldots, s_n, d_k)$ is true in V_f for some $1 \leq k \leq m$.

(⇒) Assume F_0 is satisfiable in a model M of $NW + AFA_1$. We prove that if (2′) does not hold then (3′) holds. Let τ be a graph in T' (T' is nonempty). Since τ is strongly extensional over G there is a map * from τ onto V_f which is an isomorphism on τ^* and it is faithful on G (see Appendix A.4). Let $s_1, \ldots, s_n, d_1, \ldots, d_m$ be the images of the nodes in τ. Since V_f is isomorphically embedded as an $\in$-initial part in every model of $NW + AFA_1$ and in particular in M, we can consider $s_1, \ldots, s_n, d_1, \ldots, d_m$ as elements in M. Since F_0 and $G[s_1, \ldots, s_n]$ are true in M there must be a disjunct E_j and an element $s \in M$ such that $M \models E_j[s_1, \ldots, s_n, s]$. Since (2′) does not hold and * is faithful on G, s must be among $d_1, \ldots, d_m$. Finally, since * is an isomorphism on $M|_{\{s_1,\ldots,s_n,d_1,\ldots,d_k\}}$ we conclude that $\tau \models E_j[1, \ldots, n, k]$ and (3′) holds. □

The following proposition is an immediate byproduct of the previous proof.

Proposition 3.2

1. *A ∀*∃ sentence is satisfiable with respect to $NW + AFA_1$ if and only if it is true in $(V_f, \in_F)$.*

2. *A $\forall^*\exists$ sentence is satisfiable with respect to $NW + AFA_1$ if and only if it is satisfiable with respect to $NW + AFA'$, thus the same decision test provided in the proof of Proposition 3.1 applies to the theory $NWE + AFA'$.*
3. *$NW+AFA'$ is a conservative extension of $NW+AFA_1$ as far as $\exists^*\forall$ sentences are concerned.*

4 Undecidability of *NW*, *NW* + *E*, and *NW* + *SE*

Proposition 4.1 *The problem of establishing whether a $\forall\exists \wedge \forall^*$ sentence is satisfiable with respect to NW, $NW + E$, or $NW + SE$ is undecidable.*

Proof The unsolvability of the $\forall\exists \wedge \forall^*$ class with respect to the theory *NW* and its extensions with constraints on equality is obtained by reducing to it the satisfability problem for the wider class $\forall\exists\forall^*(0, 1)$ in the pure logic without equality known to be undecidable (Lewis [4]).

To every $\forall\exists\forall^*$ formula F whose matrix F^M contains only a binary predicate symbol P, we can effectively associate a $\forall\exists \wedge \forall^*$ formula G, whose matrix contains a binary predicate symbol $\in$, such that F is satisfiable if and only if G is satisfiable with respect to any one of NW, $NW + E$, and $NW + SE$. Actually we will prove that if F is satisfiable then G is satisfiable in a model of $NW + SE$—that is the strongest of our theories—and, on the other hand, that the satisfiability of F follows from the satisfiability of G in the pure logic.

The basic idea is to associate to a given Herbrand model H_F, for the skolem form of F, over a constant c and a monadic function symbol f, a structure for $\{=, \in\}$ which satisfies a formula in $\forall\exists \wedge \forall^*$, obtained from F by substituting atomic formulas of F with atomic formulas, and then to show that such a structure can be characterized through the satisfiability of a $\forall\exists$ formula in $\in$ and expanded into a model of $NW + SE$.

To each element in the domain of H_F we make correspond a 7-tuple of elements of the domain of the structure to be built.

Let A be a countable set $\{a_i\}$ and

$$\begin{aligned} R_0 \quad = \quad & \{(a_i, a_i) : i \in N \setminus \{0\}\} \cup \{(a_i, a_{i+1}) : i \in N\} \cup \\ & \{(a_{7n+1}, a_{7n+7}) : i \in N\} \cup \{(a_{7n+7}, a_{7n+1}) : i \in N\}. \end{aligned}$$

This structure permits us to distinguish sequences that arise from a single element in H_F from the others, using the following schema and interpreting the membership $\in$ as R_0:

$$E(y_1, \dots, y_7) = \bigwedge_{1 \le i \le 7} y_i \in y_i \bigwedge_{1 \le i \le 6} (y_i \in y_{i+1} \wedge y_{i+1} \notin y_i) \wedge y_1 \in y_7 \wedge y_7 \in y_1.$$

To the binary relation induced on HF by the application of the function symbol f, we make correspond the relation defined by the following schema:

$$S(y_1, \dots, y_{14}) = E(y_1, \dots, y_7) \wedge E(y_8, \dots, y_{14}) \wedge y_7 \in y_8 \wedge y_8 \notin y_7.$$

We then define

$$\begin{aligned} R \quad = \quad & R_0 \cup \{(a_{7m+1}, a_{7n+4}) : H_F \models P(f^m(c), f^n(c))\} \cup \\ & \{(a_{7n+4}, a_{7m+1}) : H_F \models P(f^m(c), f^n(c))\}, \end{aligned}$$

as we intend to let

$$P'(y_1, \dots, y_7, z_1, \dots, z_7) =_{\text{def}} y_1 \in z_4$$

be the correlate of $P(y, z)$. Despite the definition of P', the presence in R of both the pairs (a_{7m+1}, a_{7n+4}) and (a_{7m+4}, a_{7n+1}) is necessary because the addiction of the pair (a_{7m+1}, a_{7n+4}) alone might give rise to a 7-tuple of elements in the domain which satisfy the schema $E(y_1, \ldots, y_7)$ without being the correspondent of any element in H_F.

Assuming that F has the form $\forall x_1 \exists x_2 \forall x_{k+2} F^M(x_1, \ldots, x_{k+2})$, it is easy to check that the structure (A, R) satisfies the following universal sentence:

$$G_1 = \forall \underline{x}_1, \underline{x}_2 (E(\underline{x}_1) \wedge E(\underline{x}_2) \wedge S(\underline{x}_1, \underline{x}_2)) \rightarrow \\ F^M\{Px_i x_j / P'(\underline{x}_i, \underline{x}_j) : 1 \leq i, j \leq k+2\}$$

where $\underline{x}_i$ denote the 7-tuple of variables $x_{7i+1}, \ldots, x_{7i+7}$.

Furthermore (A, R) is a strongly extensional structure since, as can be proved by induction on n, a_n cannot be bisimulated with any other element of A (Appendix A.5). (A, R) can be expanded into a model of $NW + SE$ as follows: close (A, R) under the addition of singletons (i.e., the operation w (for with) defined as $w(a, b) = a \cup \{b\}$) starting with R, thought of as the membership relation on A, and then take the strongly extensional quotient of the structure obtained in this way, that is, the quotient with respect to the maximal bisimulation [1]. Let $M = (D_M, \overline{R})$ be the resulting structure. Clearly $M \models NW + SE$ and furthermore $\overline{R}$ restricted to the elements in D_M corresponding to those in A is (isomorphic to) R. It is easy to check that outside such elements $\overline{R}$ forms no cycles. Note that, having defined the elements of A as self-loops, we are able to distinguish the elements corresponding to those in A from the other sets.

It follows that $M \models G_1$ and we can conclude that the satisfiability of F implies the satisfiability of G_1 in a model of $NW + SE$.

However, the satisfiability of G_1 does not entail the satisfiability of F since, given a model M of $NW + SE + G_1$, the structure (A, R_0) needed to reconstruct a model for F may be lacking in M. Yet, as we will see, in order to guarantee its existence as a substructure of any model of NW it suffices to require that such a model satisfies the following ∀∃ sentence:

$$G_0 = \forall y \exists x (x \in x \wedge (y \in y \rightarrow (y \in x \wedge x \notin y))) \wedge \\ \forall y_1, \ldots, \forall y_{13} ((\textstyle\bigwedge_{1 \leq i \leq 12} (y_i \in y_i \wedge y_i \in y_{i+1} \wedge y_{i+1} \notin y_i)) \\ \rightarrow \triangledown_{1 \leq i \leq 7} (y_i \in y_{i+6} \wedge y_{i+6} \in y_i)).$$

where $\triangledown$ denotes the exclusive or. Note that G_0 is clearly satisfied in (A, R_0) and also in the model M built above, since no cycle is present in M outside A. This ensures that if we let the $\forall\exists \wedge \forall^*$ sentence G associated with F be $G = G_0 \wedge G_1$, G is satisfiable with respect to $NW + SE$ whenever F is satisfiable.

To show that the converse also holds, assume G is satisfiable, let H_G be a Herbrand model over a constant c_g and a monadic function symbol g of the functional form of G_0. Then there exists $0 \leq i \leq 6$ such that $H_G|_{\{g^j(c): j \geq i\}}$ is (isomorphic to) (A, R') where $R_0 \subseteq R'$.

We define a Herbrand model H_F of F by letting

$$P(f^m(c), f^n(c)) \text{ iff } H_G \models g^{i+7m+1}(c) \in g^{i+7n+4}(c) = P'(\underline{g}_m, \underline{g}_n),$$

where with $\underline{g}_j$ we denote the sequence $(g^{i+7j+1}(c), \ldots g^{i+7j+7}(c))$.

Let $F^M(x_1/f^{n_1}(c), \ldots, x_{k+2}/f^{n_{k+2}}(c))$ where $n_2 = n_1 + 1$ be an instance of F. We have that $H_G \models G_1(\underline{x}_1/\underline{g}_{n_1}, \ldots, \underline{x}_{k+2}/\underline{g}_{n_{k+2}})$ and

$$H_G \models (E(\underline{x}_1) \wedge E(\underline{x}_2) \wedge S(\underline{x}_1, \underline{x}_2))[\underline{x}_1/\underline{g}_{n_1}, \ldots, \underline{x}_{k+2}/\underline{g}_{n_{k+2}}];$$

hence $H_G \models F^M\{Px_ix_j/P'(\underline{g}_{n_i}, \underline{g}_{n_j}) : 1 \leq i, j \leq k+2\}$.

Because of our definition of H_F we have that $H_F \models F^M\{Px_ix_j/Pf^{n_i}(c)f^{n_j}(c)\}$, that is, $H_F \models F^M(x_1/f^{n_1}(c), \ldots, x_{k+2}/f^{n_{k+2}}(c))$, that is, the instance considered. We have proved that H_F satisfies every instance of the functional form of F and thus we can conclude that H_F is a model of F. □

The decidability of the satisfiability of $\forall^*\exists\exists$ with respect to the theories *NWR*, *NWER*, $NW + AFA_1$, and their extensions is still open. For the time being, we remark that, contrary to the case of $\forall^*\exists$ sentences, to establish the decidability of *NWER* with respect to $\forall^*\exists\exists$ one cannot rely on its completeness. In fact it follows from Parlamento and Policriti [7, 8] that there are $\forall\forall\exists\exists$ sentences which are undecidable in *NWER* since they hold in *HF* but not in models which satisfy the Infinity Axiom.

Appendix A

For the reader's convenience we have included in this appendix some of the omitted details.

A.1 Every acyclic (finite) graph can be embedded in *HF* Let $G = (\{1, \ldots, n\}, R)$ be an acyclic graph.

Since R is well founded on G, we can define by induction on R a map * by letting

$$i^* = \{j^* : R_0ji\} \cup \{n+i\}.$$

Note that the sets i^* are distinct from the natural numbers considered since the i^*s have at most n predecessors while the predecessors of the natural numbers greater than $n+1$ are at least $n+1$. It is then immediate to verify that * is a map from G onto $(HF, \in)$ which is an isomorphism on G^*.

When we consider an extension $\tau = (\{1, \ldots, n+m\}, R')$ of G obtained by adding some R'-predecessors of the nodes in G, in general we will require that * in $(HF, \in)$ satisfies a stronger property that constrains the predecessor of the sets corresponding to the nodes in G: for all $s \in i^*$, $1 \leq i \leq n$ exists $j \in \tau$ such that $s = j^*$. We will say that a map of this kind is faithful on G.

τ can be embedded in *HF* through a map * which is an isomorphism on τ^* and faithful on G if and only if τ is acyclic and extensional on G.

If this is the case, we can define by induction on R' a map $^* : \tau \to HF$ by letting

$$i^* = \begin{cases} \{e^* : R'ji\} & \text{for } 1 \leq i \leq n \\ \{e^* : R'ji\} \cup \{\{i, n+m\}\} & \text{for } n+1 \leq i \leq m. \end{cases}$$

That * is faithful on G follows immediately from the definition.

Since $\text{rank}(\{i, n+m\}) = n+m$, it is immediate that $\text{rank}(i^*) > n+m$ for $n+1 \leq i \leq n+m$.

Furthermore, if in the R'-transitive closure of j there is no element in $\tau \setminus G$, then $\text{rank}(j^*) \leq n-1 < n+m$; if, on the other hand, the R'-transitive closure of j

contains some element in $\{n+1,\ldots,n+m\}$ then $\mathrm{rank}(j^*) > n+m$. Thus, for every $i \in \tau$, $\mathrm{rank}(i^*) \neq n+m$ and so $i^* \neq \{i, n+m\}$ for all $n+1 \leq i \leq n+m$.

From this it follows very easily that * is a 1-1 map and in turn that it is an isomorphism on τ^* since

$$\text{for all } i, j \in \tau, i R' j \text{ iff } i^* \in j^*. \tag{1}$$

Observe that in the extreme case in which $\tau = G$, namely, if we consider an acyclic and extensional graph G, * is simply a Mostowski collapsing.

A.2 The antifoundation axiom AFA' The Axiom of Regularity together with the Axiom of Extensionality readily entails the nonexistence of bisimulations relating different sets. In particular no such bisimulations can exist on HF. On the other hand, in ZF-Axiom of Regularity, the Axiom of Extensionality immediately follows from the nonexistence of proper bisimulations. However, that is not the case in NW as is shown by the following model.

Let M be the closure under $\mathbf{w}$ of $HF \cup \{\upsilon_1, \upsilon_2\}$ where υ_1, υ_2 are distinct objects not belonging to HF; let $\in'$ be the expansion of $\in$ over M obtained by letting $e \in' \upsilon_1$ and $e \in' \upsilon_2$ for all $e \in HF \cup \{\upsilon_1, \upsilon_2\}$ and let $\in^M$ be the least expansion of $\in'$ such that for all a, b, c in M, $b \in_M a\mathbf{w}b$ and if $c \in^M a$ then $c \in^M a\mathbf{w}b$.

Furthermore, let $\varnothing^M$ be $\varnothing \in HF$ and

$$\mathbf{w}^M(a,b) = \begin{cases} a \cup \{b\} & \text{if } a, b \in HF, \\ a\mathbf{w}b & \text{otherwise.} \end{cases}$$

$\mathcal{M}=(M, \in^M, \varnothing^M, \mathbf{w}^M)$ is a model of NW.

υ_1 and υ_2, as well as all those elements of M which are obtained by starting with υ_1 or υ_2 and applying $\mathbf{w}^M$, have all the elements in HF among their $\in^M$-predecessors. Furthermore they are the only elements of M which have infinitely many $\in^M$-predecessors. Since υ_1 and υ_2 have the same $\in^M$-predecessors and they are distinct, the Axiom of Extensionality fails in $\mathcal{M}$. Nevertheless,

$$\mathcal{M} \models \forall ab \neg \exists R\,(R \text{ is a bisimulation } \wedge (a,b) \in R \wedge a \neq b).$$

For, suppose $a, b, R \in M$ are such that $\mathcal{M} \models \forall ab \neg \exists R\,(R \text{ is a bisimulation } \wedge (a,b) \in R \wedge a \neq b)$. If $a, b \in HF$ then from R we could easily obtain a bisimulation on $(HF, \in)$ relating a and b but, as we noticed, no such bisimulation on HF can exist. On the other hand, if $a \in M \setminus HF$ then either υ_1 or υ_2, say υ_1, is in the $\in^M$-transitive closure of a. To every $\in^M$-predecessor of υ_1 must correspond an M-ordered pair $(x,y)^M$ such that $(x,y)^M \in^M R$, and to different $\in^M$-predecessors of υ_1 correspond different M-ordered pairs $\in^M$-related to R. Since υ_1 has infinitely many $\in^M$-predecessors, every element in HF is $\in^M$-related to R; in particular, there are elements $\in^M$-related to R which are not M-ordered pairs, so the assumption that $\mathcal{M} \models R$ *is a bisimulation* is contradicted.

A.3 Hereditarily finite hypersets Let us recall from [1] that an accessible pointed graph (apg) is a graph with a distinguished node called its point from which any (other) node can be reached through a finite path.

Let V_{0f} be the class of all the finite apgs. For $a, b \in V_{0f}$, let $a \in_{0f} b$ hold if and only if a is a subgraph of b generated by one of the predecessors in b of the point of b. If for $a, b \in V_{0f}$ we let $a \sim_{V_{f0}} b$ mean that there is a bisimulation R on V_{0f} such that $(a,b) \in R$, then $\sim_{V_{f0}}$ is an equivalence relation.

We let V_f be the quotient of V_{0f} with respect to $\sim_{V_{f0}}$ and $\in_f$ be the relation induced over V_f by $\in_{0f}$. $(V_f, \in_f)$ is strongly extensional in the sense that no proper bisimulation with respect to $\in_f$ exists on V_f, and it is called the strongly extensional quotient of V_f. $(V_f, \in_f)$ is a model of *NW*—actually it is a model of ZF deprived of the Foundation and the Infinity Axioms.

For every finite graph $\mathcal{G} = (G, R_0)$ there is a unique system map from G onto V_f, namely, a function $\pi_{\mathcal{G}} : G \to V_f$ such that

$$\begin{aligned} aR_0b &\Rightarrow \pi_{\mathcal{G}}(a) \in_f \pi_{\mathcal{G}}(b), \\ c \in_f \pi_{\mathcal{G}}(b) &\Rightarrow \exists a \in G\ aR_0b \wedge c = \pi_{\mathcal{G}}(a). \end{aligned}$$

$\pi_{\mathcal{G}}$ is a strongly extensional quotient of $\mathcal{G}$ in the sense that $\pi_{\mathcal{G}}$ induces on G an equivalence relation which is the maximum bisimulation on $\mathcal{G}$.

$(V_f, \in_f)$ is isomorphically embedded, as an $\in$-initial part, into every model of $NW + AFA'$.

If $\mathcal{M} = (M, \in^M, \varnothing^M, \mathbf{w}^M)$ is a model of *NW*, for $a \in M$ we let Ma denote the $\in^M$-transitive closure of a, that is,

$$\begin{aligned} Ma \quad = \quad & \{b \in M : \text{ there is a finite } \in^M\text{-chain } a_0 \in^M a_1 \in^M, \ldots, \in^M a_n, \\ & \text{such that } a_0 = b \text{ and } a_n = a\}, \end{aligned}$$

and let

$$hf(\mathcal{M}) = \{a \in M : Ma \text{ is finite}\}.$$

A simple adaptation of the results in [1] leads to the following.

Proposition A.1 *If $\mathcal{M} \models NW{+}AFA'$ then $(hf(\mathcal{M}), \in^M)$ is isomorphic to $(V_f, \in_f)$.*

Proof If $a \in hf(\mathcal{M})$ then $Ma \in V_{0f}$ and the map that assigns Ma to $a \in M$ is clearly a system map which, composed with the strongly extensional quotient $\pi_f : V_{0f} \to V_f$, yields a system map $\pi : hf(\mathcal{M}) \to V_f$. $hf(\mathcal{M})$ is strongly extensional, hence π is injective, as it follows by Theorem 2.19 in [1].

If $a \in V_f$ then $V_f a$ is an apg and, since it is finite and $\mathcal{M} \models NW$, there exists in $\mathcal{M}$ the corresponding graph g. Furthermore since $\mathcal{M} \models AFA'$ there exists in $\mathcal{M}$ a decoration $d^{\mathcal{M}}$ of g from which we obtain an $\mathcal{M}$-decoration d of $V_f a$. Then $\pi \circ d : V_f a \to V_f$ is a system map. As V_f is strongly extensional $\pi \circ d$ must be the identity map on $V_f a$ since the identity is a system map on $V_f a$ and there is only one system map on $V_f a$ ([1], Theorem 2.19). In particular, $a = \pi(da)$. Thus π is surjective as well as injective; hence it is an isomorphism. □

A.4 Every (finite) graph can be embedded in V_f Let $G = (\{1, \ldots, n\}, R)$ be a graph and * a map from G to V_f that satisfies the equations

$$i^* = \{j^* : Rji\} \cup \{n + i\}.$$

A map that satisfies this equation is the restriction to G of the decoration of the graph obtained extending G by adding in a suitable way the pictures of the natural numbers considered.

Note that the sets i^* are distinct from the natural numbers $n + 1, \ldots, 2n$ since the i^*s have at most n predecessors while the predecessors of the natural numbers greater than $n + 1$ are at least $n + 1$. It is then immediate to verify that * is a map from G onto $(V_f, \in)$ which is an isomorphism on G^*.

As in the case of HF, when we consider an extension $\tau = (\{1, \ldots, n+m\}, R')$ of G obtained by adding some R'-predecessors of the nodes in G, in general we will require that * is faithful on G.

τ can be embedded in V_f through a map * which is iso on τ^* and faithful (according to the definition given in Appendix A.1 on G if and only if τ is strongly extensional on G.

Let H be the Herbrand universe over $c_0, \ldots, c_{n+m}, \mathbf{w}$ where $c_0, \ldots, c_{n+m}$ are constant symbols and $\in^H$ is defined as follows.

Let

$$\begin{aligned} \in' \quad = \quad & \{(c_i, c_j) : 1 \leq i, j \leq n+m(i, j) \in R\} \cup \\ & \{(j, c_i) : n+1 \leq i \leq n+m, 2n+1+i \leq j \leq 3n+2+i\} \end{aligned}$$

and $\in^H$ be the closure over H of $\in'$ with respect to the axiom W, namely, the least binary relation over H such that

1. $\in' \subseteq \in_H$,
2. $b \in_H a\mathbf{w}b$,
3. if $c \in^H a$ then $c \in^H a\mathbf{w}b$.

It is straightforward to check that the map $i \rightarrow c_i$ is an isomorphism from τ to $(H, \in_H)|_{\{c_0, \ldots, c_n+m\}}$.

Let π be the strongly extensional quotient of $(H, \in^H)$ in $(V_f, \in_f)$, namely, the unique system map from $(H, \in^H)$ onto $(V_f, \in_f)$. Since π is a system map, for every c_i, $1 \leq i \leq n+m |\pi(c_i)| \leq |\{c_j : i R' j\}|$. Furthermore, since there is no bisimulation relating two distinct natural numbers, for $i, j \in \mathbb{N}$ if $i \neq j$ then $\pi(i) \neq \pi(j)$ and $|\pi(i)| = i$. It follows that

$$(*) \qquad \pi \text{ is } 1-1 \text{ on } \{c_{n+1}, \ldots, c_{n+m}\} \text{ and}$$

$$(**) \qquad \pi(c_j) \neq \pi(c_h) \text{ for } n+1 \leq j \leq n+m \text{ and } 1 \leq h \leq n.$$

From $(*)$ and $(**)$ it follows that if $\pi(c_h) = \pi(c_\ell)$, $1 \leq h, \ell \leq n$ then h and ℓ have the same R'-predecessors in $\{n+1, \ldots, n+m\}$. Since $\in^H$ is strongly extensional over $c_1, \ldots, c_n$ it follows that $c_h = c_\ell$ and then that h and ℓ are the same element of τ.

We have thus proved that π is 1-1 over $\{c_0, \ldots, c_{n+m}\}$. It is then immediate to verify that $^* : \tau \rightarrow \pi(\{c_1, \ldots, c_n + m\}), i \rightarrow \pi(c_i)$ is an isomorphism faithful on G.

Observe that in the extreme case in which $\tau = G$, namely, if we consider a strongly extensional graph G, * is simply a strongly extensional quotient.

A.5 Strong extensionality of (A, R)

Lemma A.2 *Let A be a countable set $\{a_i\}$ and*

$$\begin{aligned} R_0 \quad = \quad & \{(a_i, a_i) : i \in N \setminus \{0\}\} \cup \\ & \{(a_i, a_{i+1}) : i \in N\} \cup \\ & \{(a_{7n+1}, a_{7n+7}) : i \in N\} \cup \\ & \{(a_{7n+7}, a_{7n+1}) : i \in N\}. \end{aligned}$$

For any subset M of N^2, the structure (A, R), where

$$\begin{aligned} R \quad = \quad & R_0 \cup \{(a_{7m+1}, a_{7n+4}) : (m, n) \in M\} \cup \\ & \{(a_{7n+4}, a_{7m+1}) : (m, n) \in M\}, \end{aligned}$$

is strongly extensional.

Proof By induction on n we show that, for all n, a_n cannot be bisimulated with any other element of A, that is, given an $a \in A$ distinct from a_n there does not exist any bisimulation B on (A, R) such that $(a, a_n) \in B$. a_0 cannot be bisimulated with any other element of A since they all have some R_0 predecessor while a_0 has none.

Assuming a_n cannot be bisimulated with any other element of A let us show that this is the case for a_{n+1} as well. Since $a_n R a_{n+1}$ and a_n cannot be bisimulated with any other element of A, a_{n+1} could possibly be bisimulated only with elements of A having a_n as an R-predecessor.

If $n + 1 = 7k + 1$ for some k, the only elements, besides a_{n+1}, having a_n as an R-predecessor are a_n and $a_{7(k-1)+1}$ none of which can be bisimulated with a_{n+1} by induction hypothesis.

If $n + 1 = 7k + 3$, $n + 1 = 7k + 4$, $n + 1 = 7k + 6$, or $n + 1 = 7k + 7$ for some k the only element, besides a_{n+1}, having a_n as an R-predecessor is a_n itself and the induction hypothesis rules out the possibility that a_{n+1} could be bisimulated with a_n.

If $n+1 = 7k+2$, the only elements, besides a_{n+1}, having a_n as an R-predecessor are a_n, a_{n+7}, and possibly an element of the form a_{7h+4} for some h. a_{n+1} cannot be bisimulated with a_{n+7} since a_{n+6} is an R-predecessor of a_{n+7} which cannot be bisimulated with any of the R-predecessors of a_{n+1}, neither with a_{n+1} since a_n is not an R-predecessor of a_{n+6} nor with a_n by induction hypothesis. The same argument shows that a_{n+1} cannot be bisimulated with a_{7h+4} either.

If $n+1 = 7k+2$, the only elements, besides a_{n+1}, having a_n as an R-predecessor are a_n and possibly an element of the form a_{7h+1} for some h. The last possibility is ruled out since $a_{7(h-1)+7}$ is an R-predecessor of a_{7h+1} which cannot be bisimulated with any of the R-predecessors of a_{n+1}, neither with a_{n+1} since a_n is not an R-predecessor of $a_{7(h-1)+7}$ nor with a_n by induction hypothesis. □

Remark A.3 Were we concerned only with the construction of an extensional but not necessarily strongly extensional model, the use of 5-tuple instead of 7-tuple in the proof of Proposition 4.1 would be equally appropriate. Furthermore, if the requirement of the extensionality were to be dropped, even the use of 4-tuple of elements would be sufficient.

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