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Sequent Calculi for Visser's Propositional Logics

Katsumasa Ishii, Ryo Kashima, and Kentaro Kikuchi

Abstract This paper introduces sequent systems for Visser's two propositional logics: Basic Propositional Logic (**BPL**) and Formal Propositional Logic (**FPL**). It is shown through semantical completeness that the cut rule is admissible in each system. The relationships with Hilbert-style axiomatizations and with other sequent formulations are discussed. The cut-elimination theorems are also demonstrated by syntactical methods.

1 Introduction

By interpreting implication as formal provability, Formal Propositional Logic (**FPL**) was introduced in Visser [11] together with Basic Propositional Logic (**BPL**), a preliminary one for the development of **FPL**. He described **BPL** and **FPL** in the form of natural deduction systems and showed their completeness with respect to transitive models and finite irreflexive transitive models, respectively. A decade later, Ruitenburg [3] reintroduced **BPL** with a philosophical motivation and extended **BPL** to the first order logic, **BQC** (Ruitenburg [4]). Ardeshir and Ruitenburg [2], and Suzuki, Wolter, and Zakharyaschev [8] explored the structure of propositional logics over **BPL** by model theoretic and algebraic methods.

A cut-free sequent calculus for **BPL** can be found in Ardeshir [1], but it is not satisfactory because it does not enjoy the subformula property. The present paper introduces another cut-free sequent calculus for **BPL**. This system enjoys the subformula property and is shown to be complete with respect to transitive models. In the proof of the completeness theorem, we construct a canonical model which is different from those of [11] and [2] in that the underlying set is a finite set of sequents rather than an infinite set of prime theories. By virtue of this model construction, we obtain the finite model property directly without the filtration method. It seems hard to construct directly a finite model on the lines of [11], [2]. A cut-free sequent calculus for **FPL** is introduced by extending the system for **BPL**. The above kind

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of model construction is still effective in proving the completeness of the system for **FPL** since we are expected to construct a finite irreflexive model. It follows from the completeness theorems that the cut rule is admissible in each system for **BPL** and **FPL**. The cut-elimination theorems for them are proved also by syntactical methods in Section 6.

Sections 4 and 5 are devoted to investigations into the relationships with other systems for **BPL** and **FPL**. Section 4 discusses the relationships between our sequent calculi and Hilbert-style axiom systems. The correspondence between them is shown in a syntactical way, but it also yields the completeness theorems of Hilbert-style systems by the medium of our sequent calculi. Section 5 discusses the relationships with other formulations of sequent calculi for **BPL** and **FPL** which have been recently introduced in Sasaki [5]. His systems involve an ad hoc expression $(A \to B)^+$ and have a kind of subformula property. We explain the difference between his systems and ours in detail.

One may view **BPL** and **FPL** in the light of substructural logics (Schroeder-Heister and Došen [7]), since formulas $(A \to (A \to B)) \to (A \to B)$ and $(A \to (B \to C)) \to (B \to (A \to C))$, which respectively correspond to the contraction rule and to the exchange rule, are not always true in transitive models. It would be worth developing resource sensitive aspects of them.

2 A Sequent Calculus for BPL

Our propositional language has a denumerably infinite set of propositional variables, the propositional constant $\bot$, and the binary connectives $\wedge$, $\vee$, and $\to$. Formulas are constructed from these in the usual way. We will denote propositional variables by $p, q, \ldots$, and formulas by $A, B, \ldots$, possibly with subscripts or superscripts. Capital Greek letters $\Gamma, \Delta, \ldots$ are used for finite sets of formulas. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$. Γ and Δ are called, respectively, the *antecedent* and the *succedent* of a sequent $\Gamma \Rightarrow \Delta$. As far as sequents are concerned, we usually write $A_1, \ldots, A_n$ for $\{A_1, \ldots, A_n\}$, and A, Γ for $\{A\} \cup \Gamma$, and so on.

Before the definition of our sequent calculus, we will first explain semantics for **BPL**, which is similar to Kripke semantics for intuitionistic propositional logic except that the accessibility relation is not necessarily reflexive.

Definition 2.1 A *(transitive) model* is a triple $\langle W, R, V \rangle$ where W is a nonempty set, R is a transitive relation on W, and V is a mapping from the set of propositional variables to the power set of W such that

$$x \in V(p) \text{ and } xRy \text{ imply } y \in V(p).$$

If W is a finite set, we say that a model $\langle W, R, V \rangle$ is *finite*.

Given a model $M = \langle W, R, V \rangle$, the truth-relation $\Vdash$ is defined inductively as follows:

$$\begin{array}{lll}
(M, x) \Vdash p & \text{iff} & x \in V(p) \quad \text{for each propositional variable } p, \\
(M, x) \nVdash \bot, & & \\
(M, x) \Vdash A \wedge B & \text{iff} & (M, x) \Vdash A \text{ and } (M, x) \Vdash B, \\
(M, x) \Vdash A \vee B & \text{iff} & (M, x) \Vdash A \text{ or } (M, x) \Vdash B, \\
(M, x) \Vdash A \to B & \text{iff} & \forall y \in W[xRy \text{ and } (M, y) \Vdash A \text{ imply } (M, y) \Vdash B].
\end{array}$$

If M is understood, we write simply $x \Vdash A$ instead of $(M, x) \Vdash A$. We say that a formula A is *true* in a model $\langle W, R, V \rangle$ if $x \Vdash A$ for every $x \in W$.

Formulas which are theorems of intuitionistic propositional logic but not true in every model defined above are, for example, $(p \wedge (p \rightarrow q)) \rightarrow q$ and $(p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)$.

Lemma 2.2 *For every model $\langle W, R, V \rangle$, every formula A and all $x, y \in W$,*

$$x \Vdash A \text{ and } xRy \text{ imply } y \Vdash A.$$

Proof We prove the lemma by an easy induction on A. □

Definition 2.3 For a given model M, the truth-relation for sequents is defined as follows:

$$(M, x) \Vdash \Gamma \Rightarrow \Delta \quad \text{iff} \quad \forall A \in \Gamma[(M, x) \Vdash A] \text{ implies } \exists A \in \Delta[(M, x) \Vdash A].$$

We write simply $x \Vdash \Gamma \Rightarrow \Delta$ for $(M, x) \Vdash \Gamma \Rightarrow \Delta$, if M is understood. We say that a sequent $\Gamma \Rightarrow \Delta$ is *true* in a model $\langle W, R, V \rangle$ if $x \Vdash \Gamma \Rightarrow \Delta$ for every $x \in W$.

In [11], Visser introduced the following relation $\Vdash_M$ for a model $M = \langle W, R, V \rangle$:

$$\Gamma \Vdash_M A \quad \text{iff} \quad \forall x \in W[\forall B \in \Gamma[(M, x) \Vdash B] \text{ implies } (M, x) \Vdash A].$$

He showed that $\Gamma \Vdash_M A$ for every model M if and only if A is derivable from Γ in his natural deduction system. Our notion that a sequent $\Gamma \Rightarrow \Delta$ is true in a model M almost coincides with the relation $\Gamma \Vdash_M D$ provided D is the disjunction of all formulas in Δ.

Now we will introduce a sequent calculus which we call **LBP**. Initial sequents of **LBP** are of the following forms:

$$A \Rightarrow A,$$

$$\bot \Rightarrow \ .$$

Rules of inference of **LBP** consist of the following:

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \text{ (weakening left)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \text{ (weakening right)}$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \ (\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \ (\wedge\text{right})$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \ (\vee\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \ (\vee\text{right})$$

$$\frac{\Delta_1, \Sigma, A \Rightarrow B, \Gamma_1 \quad \Delta_2, \Sigma, A \Rightarrow B, \Gamma_2 \quad \cdots \quad \Delta_{2^n}, \Sigma, A \Rightarrow B, \Gamma_{2^n}}{\Sigma, C_1 \rightarrow D_1, \ldots, C_n \rightarrow D_n \Rightarrow A \rightarrow B} \ (\rightarrow)$$

where $n \geq 0$, $\Gamma_i = \{C_j \mid j \in \gamma(i)\}$, $\Delta_i = \{D_j \mid j \in \delta(i)\}$, and the sets $\gamma(i)$ and $\delta(i)$ of natural numbers are defined as follows: $\delta(i)$ runs through the subsets of $\{1, \ldots, n\}$ ordered according to size and $\gamma(i) = \{1, \ldots, n\} \setminus \delta(i)$.

For example, when $n = 0, 1, 2$, the rule $(\rightarrow)$ is of the forms

$$\frac{\Sigma, A \Rightarrow B}{\Sigma \Rightarrow A \rightarrow B} \ (\rightarrow),$$

$$\frac{\Sigma, A \Rightarrow B, C_1 \quad D_1, \Sigma, A \Rightarrow B}{\Sigma, C_1 \to D_1 \Rightarrow A \to B} (\to), \quad \text{and}$$

$$\frac{\Sigma, A \Rightarrow B, C_1, C_2 \quad D_1, \Sigma, A \Rightarrow B, C_2 \quad D_2, \Sigma, A \Rightarrow B, C_1 \quad D_1, D_2, \Sigma, A \Rightarrow B}{\Sigma, C_1 \to D_1, C_2 \to D_2 \Rightarrow A \to B} (\to),$$

respectively.

The formulas $C_1 \to D_1, \ldots, C_n \to D_n$ and $A \to B$ in the rule $(\to)$ are called the *principal formulas* of this rule. As for the other rules, the notion is defined in the usual way.

Here are some additional remarks on the above sequent calculus **LBP**. First it has no cut rule which turns out to be admissible (Corollary 2.9). It also dispenses with the contraction and the exchange rules, since a sequent consists of finite sets of formulas. Note also that it allows more than one formula in the succedent of a sequent although **BPL** is a logic weaker than intuitionistic propositional logic for which the Gentzen **LJ** does not allow them.

Theorem 2.4 (Soundness) *For every sequent $\Gamma \Rightarrow \Delta$, if $\Gamma \Rightarrow \Delta$ is provable in* **LBP**, *then $\Gamma \Rightarrow \Delta$ is true in any model.*

Proof We prove the lemma by induction on the proof of $\Gamma \Rightarrow \Delta$ in **LBP**. Here we will consider only the rule $(\to)$. Let $\langle W, R, V \rangle$ be any model. By the induction hypothesis, each upper sequent $\Delta_i, \Sigma, A \Rightarrow B, \Gamma_i$ of the rule $(\to)$ is true in the model $\langle W, R, V \rangle$. Our aim is to show $x \Vdash \Sigma, C_1 \to D_1, \ldots, C_n \to D_n \Rightarrow A \to B$ for any $x \in W$. Suppose that $x \Vdash E$ for any $E \in \Sigma$ and that $x \Vdash C_k \to D_k$ for any k with $1 \leq k \leq n$. To show $x \Vdash A \to B$, suppose further that xRy and $y \Vdash A$. Then by Lemma 2.2, $y \Vdash E$ for any $E \in \Sigma$. Here, take the set $\Delta_i = \{D_j \mid y \Vdash D_j, 1 \leq j \leq n\}$. Since $y \Vdash \Delta_i, \Sigma, A \Rightarrow B, \Gamma_i$, we have $y \Vdash B$ or $y \Vdash C_k$ for some $C_k \in \Gamma_i$ (and so $D_k \notin \Delta_i$). However, $y \Vdash C_k$ is impossible because it together with the supposition $x \Vdash C_k \to D_k$ implies $y \Vdash D_k$, which contradicts the definition of Δ_i. Therefore we have $y \Vdash B$. □

To prove the completeness theorem, we need the following notion.

Definition 2.5 A sequent $\Gamma \Rightarrow \Delta$ is said to be *saturated* if, for all formulas A, B,

$$\begin{array}{lll} A \wedge B \in \Gamma & \text{implies} & A \in \Gamma \text{ and } B \in \Gamma, \\ A \wedge B \in \Delta & \text{implies} & A \in \Delta \text{ or } B \in \Delta, \\ A \vee B \in \Gamma & \text{implies} & A \in \Gamma \text{ or } B \in \Gamma, \\ A \vee B \in \Delta & \text{implies} & A \in \Delta \text{ and } B \in \Delta. \end{array}$$

Given a finite set Λ of formulas, $\mathcal{S}(\Lambda)$ is defined to be the set of all sequents $\Gamma \Rightarrow \Delta$ satisfying the following conditions:

1. $\Gamma \Rightarrow \Delta$ is saturated,
2. $\Gamma \Rightarrow \Delta$ is not provable in **LBP**,
3. $\Gamma \cup \Delta \subseteq \mathrm{Sub}(\Lambda)$, where $\mathrm{Sub}(\Lambda)$ is the set of all the subformulas in Λ.

Lemma 2.6 *If a sequent $\Gamma \Rightarrow \Delta$ is not provable in* **LBP**, *then there exists a sequent $\Gamma' \Rightarrow \Delta'$ in $\mathcal{S}(\Gamma \cup \Delta)$ such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$.*

Proof Let $A_1, \ldots, A_n$ be a list of all formulas in $\mathrm{Sub}(\Gamma \cup \Delta)$ such that $|A_i| \geq |A_{i+1}|$ ($|A|$ is the length of A). We define a sequence $\{\Gamma_i \Rightarrow \Delta_i\}$ $(i = 0, \ldots, n)$ of unprovable sequents as follows:

1. $(\Gamma_0 \Rightarrow \Delta_0) = (\Gamma \Rightarrow \Delta)$;
2. If $A_i = A_i' \wedge A_i''$ or $A_i' \vee A_i''$ and if $A_i \in \Gamma_{i-1}$ or $A_i \in \Delta_{i-1}$, then the sequent $\Gamma_i \Rightarrow \Delta_i$ is obtained from $\Gamma_{i-1} \Rightarrow \Delta_{i-1}$ by adding A_i' and/or A_i'' to $\Gamma_{i-1}/\Delta_{i-1}$ appropriately to the saturated condition and unprovability. Otherwise, $(\Gamma_i \Rightarrow \Delta_i) = (\Gamma_{i-1} \Rightarrow \Delta_{i-1})$.

Then we obtain $\Gamma' \Rightarrow \Delta'$ as the sequent $\Gamma_n \Rightarrow \Delta_n$. □

Theorem 2.7 (Completeness) *For every sequent $\Gamma \Rightarrow \Delta$, we have that $\Gamma \Rightarrow \Delta$ is provable in* **LBP** *if and only if $\Gamma \Rightarrow \Delta$ is true in any model.*

Proof From left to right, we have Theorem 2.4. For the other direction, suppose that $\Gamma \Rightarrow \Delta$ is not provable in **LBP**. Then by Lemma 2.6, there exists a sequent $\Gamma' \Rightarrow \Delta'$ in $\mathcal{S}(\Gamma \cup \Delta)$ such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. Now define a model $M = \langle W, R, V \rangle$ as follows:

1. $W = \mathcal{S}(\Gamma \cup \Delta)$,
2. $(\Sigma \Rightarrow \Theta)R(\Phi \Rightarrow \Psi)$ iff $\Sigma \subseteq \Phi$ and $\forall (A \to B) \in \Sigma[A \in \Psi$ or $B \in \Phi]$,
3. $V(p) = \{\Sigma \Rightarrow \Theta \in W \mid p \in \Sigma\}$.

It is easy to verify that R is transitive and that

$$\Sigma \Rightarrow \Theta \in V(p) \text{ and } (\Sigma \Rightarrow \Theta)R(\Phi \Rightarrow \Psi) \text{ imply } \Phi \Rightarrow \Psi \in V(p).$$

Thus M does indeed define a model.

To see $(M, \Gamma' \Rightarrow \Delta') \nVdash \Gamma \Rightarrow \Delta$ which means that $\Gamma \Rightarrow \Delta$ is not true in M, we show that, for any sequent $\Sigma \Rightarrow \Theta \in W$,

$$\begin{array}{lll} A \in \Sigma & \text{implies} & (M, \Sigma \Rightarrow \Theta) \Vdash A, \text{ and} \\ A \in \Theta & \text{implies} & (M, \Sigma \Rightarrow \Theta) \nVdash A, \end{array}$$

by induction on A. The only problematic case is where $A = E \to F \in \Theta$. Let $\{C_1 \to D_1, \ldots, C_n \to D_n\}$ be the set $\{C \to D \mid C \to D \in \Sigma\}$. Then, at least one of $\Delta_i, \Sigma, E \Rightarrow F, \Gamma_i$ $(i \in \{1, \ldots, 2^n\})$ is unprovable where Δ_i and Γ_i are the sets described in the rule $(\to)$. Indeed, otherwise $\Sigma \Rightarrow \Theta$ is provable by $(\to)$ and (weakening right). We apply Lemma 2.6 to such a $\Delta_k, \Sigma, E \Rightarrow F, \Gamma_k$, and we get a sequent $\Phi \Rightarrow \Psi \in \mathcal{S}(\Delta_k, \Sigma, E, F, \Gamma_k) \subseteq W$ such that $\Delta_k, \Sigma, E \subseteq \Phi$ and $F, \Gamma_k \subseteq \Psi$. Then, by the definition of R and the induction hypotheses, we have $(\Sigma \Rightarrow \Theta)R(\Phi \Rightarrow \Psi)$, $(M, \Phi \Rightarrow \Psi) \Vdash E$, and $(M, \Phi \Rightarrow \Psi) \nVdash F$. This means $(M, \Sigma \Rightarrow \Theta) \nVdash E \to F$. □

The above proof provides the model M based on the finite set $W = \mathcal{S}(\Gamma \cup \Delta)$. So we have the following theorem.

Theorem 2.8 (Finite Model Property) *For every sequent $\Gamma \Rightarrow \Delta$, if $\Gamma \Rightarrow \Delta$ is not provable in* **LBP***, then there exists a finite model in which $\Gamma \Rightarrow \Delta$ is not true.*

As a consequence of the completeness theorem, we obtain the following corollary.

Corollary 2.9 *The rule*

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{ (cut)}$$

is admissible in **LBP**.

Proof Suppose that both $\Gamma \Rightarrow \Delta, A$ and $A, \Gamma \Rightarrow \Delta$ are provable in **LBP**. Then for any model $\langle W, R, V\rangle$ and any $x \in W$, we have $x \Vdash \Gamma \Rightarrow \Delta, A$ and $x \Vdash A, \Gamma \Rightarrow \Delta$, and hence $x \Vdash \Gamma \Rightarrow \Delta$. So by the completeness theorem, $\Gamma \Rightarrow \Delta$ is provable in **LBP**. □

A syntactical proof of the cut-elimination theorem for **LBP** is also possible. See Section 6.

3 A Sequent Calculus for FPL

Next we will be concerned with **FPL**, semantics for which is based on restricted models.

Definition 3.1 An *irreflexive model* is a model $\langle W, R, V\rangle$ in which R is irreflexive, that is, there is no $x \in W$ such that xRx.

The sequent calculus **LFP** is obtained from **LBP** by providing the antecedent of each upper sequent in the rule $(\to)$ with the formula $A \to B$. Then the rule $(\to)$ of **LFP** is of the form

$$\frac{\Delta_1, \Sigma, A\to B, A \Rightarrow B, \Gamma_1 \quad \Delta_2, \Sigma, A\to B, A \Rightarrow B, \Gamma_2 \quad \cdots \quad \Delta_{2^n}, \Sigma, A\to B, A \Rightarrow B, \Gamma_{2^n}}{\Sigma, C_1\to D_1, \ldots, C_n \to D_n \Rightarrow A \to B}\ (\to)$$

where $n \geq 0$, and Γ_i and Δ_i are as in the rule $(\to)$ of **LBP**. This rule extends the rule $(\to)$ of **LBP** which is derivable from this rule and (weakening left).

Theorem 3.2 (Soundness) *For every sequent $\Gamma \Rightarrow \Delta$, if $\Gamma \Rightarrow \Delta$ is provable in* **LFP**, *then $\Gamma \Rightarrow \Delta$ is true in any finite irreflexive model.*

Proof We prove the lemma by induction on the proof of $\Gamma \Rightarrow \Delta$ in **LFP**. We will consider the case $(\to)$. Let $\langle W, R, V\rangle$ be a finite irreflexive model. If an element x of W is a dead end, that is, there is no $y \in W$ such that xRy, then $x \Vdash A \to B$ by the irreflexivity of R, and so $x \Vdash \Sigma, C_1 \to D_1, \ldots, C_n \to D_n \Rightarrow A \to B$. If x is not a dead end, suppose that $y \Vdash \Sigma, C_1 \to D_1, \ldots, C_n \to D_n \Rightarrow A \to B$ for any $y \in W$ such that xRy. Then, by an argument similar to that in the proof of Theorem 2.4, we have $x \Vdash \Sigma, C_1 \to D_1, \ldots, C_n \to D_n \Rightarrow A \to B$. By Noetherian induction, $x \Vdash \Sigma, C_1 \to D_1, \ldots, C_n \to D_n \Rightarrow A \to B$ for every $x \in W$, which means that $\Sigma, C_1 \to D_1, \ldots, C_n \to D_n \Rightarrow A \to B$ is true in $\langle W, R, V\rangle$. □

Theorem 3.3 (Completeness) *For every sequent $\Gamma \Rightarrow \Delta$, we have that $\Gamma \Rightarrow \Delta$ is provable in* **LFP** *if and only if $\Gamma \Rightarrow \Delta$ is true in any finite irreflexive model.*

Proof From left to right, we have Theorem 3.2. For the other direction, we construct a model $M = \langle W, R, V\rangle$ similar to that in the proof of Theorem 2.7 except that R is defined by

$$(\Sigma \Rightarrow \Theta)R(\Phi \Rightarrow \Psi) \text{ iff } \Sigma \subsetneq \Phi \text{ and } \forall(A \to B) \in \Sigma[A \in \Psi \text{ or } B \in \Phi].$$

It is easy to see that M does define a finite irreflexive model. The rest of the proof is similar to that of Theorem 2.7. Note that in the case $A = E \to F \in \Theta$ of the induction, Φ of the extended sequent $\Phi \Rightarrow \Psi$ contains $E \to F$ while Σ does not, for otherwise $\Sigma \Rightarrow \Theta$ is provable. So $\Sigma \subsetneq \Phi$ holds and we have $(\Sigma \Rightarrow \Theta)R(\Phi \Rightarrow \Psi)$. □

From the completeness theorem, we obtain the following.

Corollary 3.4 *The rule*

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{ (cut)}$$

is admissible in **LFP**.

Proof The proof is similar to the proof of Corollary 2.9. □

For syntactical proof of the cut-elimination theorem for **LFP** see Section 6.

The next corollary is often useful in searching for a proof of a given sequent.

Corollary 3.5 *In* **LFP**, $A \Rightarrow B$ *is provable if and only if* $\Rightarrow A \to B$ *is provable.*

Proof From left to right, we use (weakening left) and ($\to$). For the other direction, suppose that $\Rightarrow A \to B$ is provable in **LFP**. Then the last applied rule of the proof of $\Rightarrow A \to B$ must be ($\to$), and hence $A \to B, A \Rightarrow B$ is provable in **LFP**. Besides, $A \Rightarrow B, A \to B$ is provable in **LFP** by applying (weakening) to $\Rightarrow A \to B$. From these, we obtain $A \Rightarrow B$ by the rule (cut) that is admissible in **LFP**. □

4 Correspondence to Hilbert-style Systems

In this section, we will show the correspondence between our sequent calculi and Hilbert-style axiom systems. The completeness of Hilbert-style systems follows from this correspondence since our sequent calculi mediate between models and these systems. The Hilbert-style system for **BPL** we consider here is found in Sasaki [6]. In the following, we assume that $\wedge$ and $\vee$ bind more strongly than $\to$. Also, we consider $\top$ as an abbreviation of $\bot \to \bot$, and $\bigwedge \Gamma$ ($\bigvee \Gamma$) as the conjunction (the disjunction) of all formulas in Γ if Γ is nonempty, as $\top$ ($\bot$, respectively) if Γ empty.

The Hilbert-style system **HB** consists of the following axiom schemes and inference rule:

(B1) $A \to A$,

(B2) $(A \to B) \wedge (B \to C) \to (A \to C)$,

(B3) $A \wedge B \to A$,

(B4) $A \wedge B \to B$,

(B5) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$,

(B6) $A \to A \vee B$,

(B7) $B \to A \vee B$,

(B8) $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$,

(B9) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$,

(B10) $\bot \to A$,

(B11) $A \to (B \to A)$,

(B12) $A \to (B \to A \wedge B)$,

$$\frac{A \quad A \to B}{B} \text{ (MP)}$$

First note that in **HB** we can derive the rules

$$\frac{A}{B \to A}\ (\mathrm{AF})\,, \qquad \frac{A \quad B}{A \wedge B}\ (\wedge\,\mathrm{I})\,,$$

by applying (MP) to (B11) and (B12), respectively. We can also derive the rules

$$\frac{A \to B \quad B \to C}{A \to C}\ (\mathrm{Tr})\,, \qquad \frac{A \to B \quad A \to C}{A \to B \wedge C}\ (\to\wedge\,\mathrm{I})\,,$$

by $(\wedge\,\mathrm{I})$, (B2), and (MP), and by $(\wedge\,\mathrm{I})$, (B5), and (MP), respectively.

Now we will consider the relation between **HB** and **LBP**.

Lemma 4.1 *For every formula A, if A is provable in* **HB***, then $\Rightarrow A$ is provable in* **LBP**.

Proof We prove the lemma by induction on the proof of A in **HB**. It is straightforward to see that $\Rightarrow A$ is provable in **LBP** for all axioms A of **HB**. For example,

$$\Rightarrow (A \to B) \wedge (B \to C) \to (A \to C)$$

is provable in **LBP** as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow C, A, B} \quad \dfrac{B \Rightarrow B}{B, A \Rightarrow C, B} \quad \dfrac{A \Rightarrow A}{C, A \Rightarrow C, A} \quad \dfrac{C \Rightarrow C}{B, C, A \Rightarrow C}}{A \to B, B \to C \Rightarrow A \to C}\ (\to)}{(A \to B) \wedge (B \to C) \Rightarrow (A \to C)}\ (\wedge\mathrm{left})}{\Rightarrow (A \to B) \wedge (B \to C) \to (A \to C)}\ (\to) \quad .$$

For (MP), suppose that both $\Rightarrow A$ and $\Rightarrow A \to B$ are provable in **LBP**. Then the last applied rule of the proof of $\Rightarrow A \to B$ must be $(\to)$, and so $A \Rightarrow B$ is provable in **LBP**. From this and $\Rightarrow A$, we obtain $\Rightarrow B$ by using the rule (cut) that is admissible in **LBP**. □

For the converse of Lemma 4.1, we prove a series of lemmas.

Lemma 4.2 *For any formulas A, B, C, if $A \to B$ is provable in* **HB***, then so is $(C \to A) \to (C \to B)$.*[1]

Proof The following proof in **HB** establishes the lemma.

$$\dfrac{\dfrac{(C \to A) \to (C \to A) \quad \dfrac{A \to B}{(C \to A) \to (A \to B)}\ (\mathrm{AF})}{(C \to A) \to (C \to A) \wedge (A \to B)}\ (\to\wedge\,\mathrm{I})}{(C \to A) \to (C \to B)}\ (*)$$

where step $(*)$ is made with the help of (B2) and (Tr). □

Lemma 4.3 *For any formulas A, B, C, if $A \wedge B \to C$ is provable in* **HB***, then so is $A \to (B \to C)$.*

Proof The following proof in **HB** establishes the lemma.

$$\dfrac{A \to (B \to A \wedge B) \quad \dfrac{A \wedge B \to C}{(B \to A \wedge B) \to (B \to C)}\ (\text{Lemma 4.2})}{A \to (B \to C)}\ (\mathrm{Tr})$$

□

Lemma 4.4 *For any formulas A, B, C, D and any Π such that $C \to D \in \Pi$, if $\bigwedge \Pi \to (A \to B \vee C)$ is provable in* **HB***, then so is $\bigwedge \Pi \to (A \to B \vee D)$.*

Proof Consider the following proof in **HB**.

$$
\dfrac{\dfrac{B \to B \vee D}{(C \to D) \to (B \to B \vee D)}\,(\mathrm{AF}) \qquad \dfrac{D \to B \vee D}{(C \to D) \to (C \to B \vee D)}\,(\text{Lemma 4.2})}{\dfrac{(C \to D) \to (B \to B \vee D) \wedge (C \to B \vee D)}{(C \to D) \to (B \vee C \to B \vee D)}\,(*)}\,(\to \wedge \mathrm{I})
$$

where step $(*)$ is made with the help of (B8) and (Tr). Since $\bigwedge \Pi \to (C \to D)$ is provable in **HB**, $\bigwedge \Pi \to (B \vee C \to B \vee D)$ is also provable by (Tr). Furthermore we have

$$
\dfrac{\dfrac{\bigwedge \Pi \to (A \to B \vee C) \quad \bigwedge \Pi \to (B \vee C \to B \vee D)}{\bigwedge \Pi \to (A \to B \vee C) \wedge (B \vee C \to B \vee D)}\,(\to \wedge \mathrm{I})}{\bigwedge \Pi \to (A \to B \vee D)}\,(**)
$$

where step $(**)$ is made with the help of (B2) and (Tr), and hence

$$
\bigwedge \Pi \to (A \to B \vee D)
$$

is provable in **HB**. □

Lemma 4.5 *For any formulas A, B, C, D, if $C \to (A \wedge D \to B)$ is provable in* **HB***, then so is $C \to (A \wedge (B \vee D) \to B)$.*

Proof Consider the following proof in **HB**.

$$
\dfrac{\dfrac{\dfrac{A \wedge B \to B}{C \to (A \wedge B \to B)}\,(\mathrm{AF}) \qquad C \to (A \wedge D \to B)}{C \to (A \wedge B \to B) \wedge (A \wedge D \to B)}\,(\to \wedge \mathrm{I})}{C \to ((A \wedge B) \vee (A \wedge D) \to B)}\,(*)
$$

where step $(*)$ is made with the help of (B8) and (Tr). By (B9) and (AF), $C \to (A \wedge (B \vee D) \to (A \wedge B) \vee (A \wedge D))$ is provable in **HB**. Hence $C \to (A \wedge (B \vee D) \to B)$ is provable by $(\to \wedge \mathrm{I})$, (B2) and (Tr). □

Lemma 4.6 *For every formula A, if $\Rightarrow A$ is provable in* **LBP***, then A is provable in* **HB***.*

Proof We show that for every sequent $\Gamma \Rightarrow \Delta$, if $\Gamma \Rightarrow \Delta$ is provable in **LBP** then $\bigwedge \Gamma \to \bigvee \Delta$ is provable in **HB**. This is enough to establish the lemma, since if $\top \to A$ is provable in **HB** then so is A. We proceed by induction on the proof of $\Gamma \Rightarrow \Delta$ in **LBP**. The only problematic case is the rule $(\to)$. By the induction hypothesis, $\bigwedge(\Delta_i, \Sigma, A) \to \bigvee(B, \Gamma_i)$ is provable in **HB** for any i with $1 \leq i \leq 2^n$. Then by Lemma 4.3, $\bigwedge \Sigma \to (\bigwedge(\Delta_i, A) \to \bigvee(B, \Gamma_i))$ is provable in **HB**, and so is $\bigwedge \Pi \to (\bigwedge(\Delta_i, A) \to \bigvee(B, \Gamma_i))$ where $\Pi = \Sigma, C_1 \to D_1, \ldots, C_n \to D_n$. Hence by Lemma 4.4,

$$
\bigwedge \Pi \to (\bigwedge(\Delta_i, A) \to \bigvee(B, \Delta_i')) \tag{1}
$$

is provable in **HB** where $\Delta_i' = \{D_j \mid C_j \in \Gamma_i\}$.

Now we prove that for all i with $1 \leq i \leq 2^n$, $\bigwedge \Pi \to (A \to \bigvee(B, \Delta_i'))$ is provable in **HB**. We show this by induction on i. For $i = 1$, $\bigwedge \Pi \to (A \to \bigvee(B, \Delta_1'))$

is just (1) since $\Delta_1 = \varnothing$. For $i > 1$, let $D_{i_1}, \ldots, D_{i_m}$ be a list of all formulas in Δ_i. Then we have that

$$\bigwedge \Pi \to (A \wedge D_{i_1} \wedge \cdots \wedge D_{i_m} \to \bigvee(B, \Delta_i')) \tag{2}$$

is provable in **HB**. So by Lemma 4.5,

$$\bigwedge \Pi \to (A \wedge \bigvee(B, \Delta_i', D_{i_1}) \wedge \cdots \wedge \bigvee(B, \Delta_i', D_{i_m}) \to \bigvee(B, \Delta_i')) \tag{3}$$

is provable in **HB**. Note here that for each k with $1 \le k \le m$, we can find $h < i$ such that $\Delta_h' = \Delta_i' \cup \{D_{i_k}\}$. Therefore by the induction hypothesis, $\bigwedge \Pi \to (A \to \bigvee(B, \Delta_i', D_{i_k}))$ is provable in **HB** for each k with $1 \le k \le m$. Hence

$$\bigwedge \Pi \to (A \to A \wedge \bigvee(B, \Delta_i', D_{i_1}) \wedge \cdots \wedge \bigvee(B, \Delta_i', D_{i_m})) \tag{4}$$

is provable in **HB**, and from (3) and (4), we obtain $\bigwedge \Pi \to (A \to \bigvee(B, \Delta_i'))$. Thus by induction on i, $\bigwedge \Pi \to (A \to \bigvee(B, \Delta_i'))$ is provable in **HB** for all i with $1 \le i \le 2^n$. In particular, $\bigwedge \Pi \to (A \to \bigvee(B, \Delta_{2^n}'))$, that is, $\bigwedge \Pi \to (A \to B)$ is provable in **HB**. $\square$

Combining Lemmas 4.1 and 4.6 with Theorem 2.7, we have the following theorem.

Theorem 4.7 (Completeness of HB) *For every formula A, A is provable in* **HB** *if and only if A is true in any model.*

In fact, it is possible to give any formula A that is not provable in **HB** the finite model in which A is not true via the construction described in the proof of Theorem 2.7.

Next we will consider a Hilbert-style axiom system for **FPL** and its correspondence to **LFP**. The Hilbert-style system **HF** is obtained from **HB** by adding the axiom scheme

(Löb) $\qquad ((\top \to A) \to A) \to (\top \to A).$

This axiom scheme can be generalized to the scheme $((A \to B) \to B) \to (A \to B)$ as easily shown.

Lemma 4.8 *For every formula A, if A is provable in* **HF**, *then $\Rightarrow A$ is provable in* **LFP**.

Proof We prove the lemma by induction on the proof of A in **HF**. To see that

$$\Rightarrow ((\top \to A) \to A) \to (\top \to A)$$

is provable in **LFP**, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\top \to A \Rightarrow \top \to A}{\top \to A, \top \Rightarrow A, \top \to A} \qquad \dfrac{A \Rightarrow A}{A, \top \to A, \top \Rightarrow A}}{(\top \to A) \to A \Rightarrow \top \to A}\ (\to)}{((\top \to A) \to A) \to (\top \to A), (\top \to A) \to A \Rightarrow \top \to A}}{\Rightarrow ((\top \to A) \to A) \to (\top \to A)}\ (\to).$$

For (MP), suppose that both $\Rightarrow A$ and $\Rightarrow A \to B$ are provable in **LFP**. Then by Corollary 3.5, $A \Rightarrow B$ is provable in **LFP** and by using (cut) we obtain $\Rightarrow B$. $\square$

Lemma 4.9 *For every formula A, if $\Rightarrow A$ is provable in* **LFP**, *then A is provable in* **HF**.

Proof We proceed in the same way as the proof of Lemma 4.6 except that in the case $(\rightarrow)$ of the induction we work with $A \wedge (A \rightarrow B)$ for A. Namely, by the induction hypothesis we have $\bigwedge(\Delta_i, \Sigma, A \wedge (A \rightarrow B)) \rightarrow \bigvee(B, \Gamma_i)$, and then finally obtain $\bigwedge \Pi \rightarrow (A \wedge (A \rightarrow B) \rightarrow B)$ instead of $\bigwedge \Pi \rightarrow (A \rightarrow B)$ where $\Pi = \Sigma, C_1 \rightarrow D_1, \ldots, C_n \rightarrow D_n$. However, $\bigwedge \Pi \rightarrow (A \rightarrow B)$ is also provable in **LFP** since $(A \wedge (A \rightarrow B) \rightarrow B) \rightarrow (A \rightarrow B)$ is provable in **LFP** as follows.

First let C be the formula $A \rightarrow B$. Then $(C \rightarrow B) \rightarrow C$ is a formula of the form generalizing the axiom (Löb). By Lemma 4.2, $(A \rightarrow (C \rightarrow B)) \rightarrow (A \rightarrow C)$ is provable in **LFP**, and so is $(A \rightarrow (C \rightarrow B)) \rightarrow (A \rightarrow A \wedge C)$. On the other hand, $(A \wedge C \rightarrow B) \rightarrow (A \rightarrow (C \rightarrow B))$ is provable in some calculations, and so by (Tr), $(A \wedge C \rightarrow B) \rightarrow (A \rightarrow A \wedge C)$ is provable in **LFP**. From this and $(A \wedge C \rightarrow B) \rightarrow (A \wedge C \rightarrow B)$, we obtain $(A \wedge C \rightarrow B) \rightarrow (A \rightarrow B)$ as required. □

The following theorem combines Lemmas 4.8 and 4.9 with Theorem 3.3.

Theorem 4.10 (Completeness of HF) *For every formula A, A is provable in* **HF** *if and only if A is true in any finite irreflexive model.*

5 Relationships with Other Sequent Calculi

In this section, we will investigate the relation between **LBP** and another sequent calculus **GVPL$^+$** which has been recently introduced in [5]. The system **GVPL$^+$** involves an ad hoc expression $(A \rightarrow B)^+$ which departs from ordinary formulations of sequent calculus. In [5], Sasaki showed that **GVPL$^+$** and Visser's natural deduction system are equivalent in provability of formulas without this expression. He also showed the cut-elimination theorem for **GVPL$^+$**, but it only led to a weak form of subformula property in the sense that even $(A \rightarrow B)^+$ is included in the subformulas of $A \rightarrow B$.

Here we give a precise definition of the system **GVPL$^+$**. A sequent of **GVPL$^+$** is an expression of the form $\Gamma \Rightarrow A$ where Γ is a finite *sequence* of formulas, A may be empty, and each formula in Γ, A is possibly of the form $(B \rightarrow C)^+$ in which B and C are ordinary ones. For a sequence Γ, we define Γ^+ as the sequence obtained from Γ by replacing every $A \rightarrow B$ with $(A \rightarrow B)^+$. Then, the system **GVPL$^+$** is defined from the following initial sequents:

$$A \Rightarrow A,$$

$$\perp \Rightarrow,$$

and the following rules of inference:

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ (\Rightarrow \wedge) \qquad \frac{A, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ (\wedge \Rightarrow_1)$$

$$\frac{B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ (\wedge \Rightarrow_2) \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ (\Rightarrow \vee_1)$$

$$\frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ (\Rightarrow \vee_2) \qquad \frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ (\vee \Rightarrow)$$

$$\frac{A, \Gamma^+ \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}\ (\Rightarrow \rightarrow) \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow (A \rightarrow B)^+}\ (\Rightarrow \rightarrow^+)$$

$$\frac{\Gamma \Rightarrow A \quad B, \Pi \Rightarrow C}{(A \to B)^+, \Gamma, \Pi \Rightarrow C}\ (\to^+\Rightarrow) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\ (\Rightarrow T)$$

$$\frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ (T \Rightarrow) \qquad \frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ (C \Rightarrow)$$

$$\frac{\Gamma, A, B, \Pi \Rightarrow C}{\Gamma, B, A, \Pi \Rightarrow C}\ (I \Rightarrow) \qquad \frac{\Gamma \Rightarrow A \quad A, \Pi \Rightarrow C}{\Gamma, \Pi \Rightarrow C}\ (\mathrm{cut})\,.$$

Note that neither A nor B in the rules for logical connectives is of the form $(D \to E)^+$ by the definition of sequents of $\mathbf{GVPL^+}$.

Now we will consider the relation between **LBP** and $\mathbf{GVPL^+}$. To facilitate a comparison between them, we use a version of **LBP** whose sequents consist of finite sequences of formulas rather than finite sets. According to this, we add the contraction and the exchange rules to the system and modify the rules ($\wedge$left) and ($\vee$right) as follows:

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\mathrm{left1}) \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\mathrm{left2})$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B}\ (\vee\mathrm{right1}) \qquad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}\ (\vee\mathrm{right2})\,.$$

It is easily seen that the resulting system is equivalent to the original one. Hence we call this also **LBP**.

Lemma 5.1 *Let Γ, Δ contain no formula of the form $(B \to C)^+$. If $\Gamma \Rightarrow \Delta$ is provable in* **LBP***, then $\Gamma \Rightarrow \bigvee \Delta$ is provable in* $\mathbf{GVPL^+}$.

Proof We prove the lemma by induction on the proof of $\Gamma \Rightarrow \Delta$ in **LBP**. The only problematic case is the rule ($\to$). For this case, we first observe that if $\Gamma \Rightarrow B \vee C$ is provable in $\mathbf{GVPL^+}$ then so is $\Gamma, (C \to D)^+ \Rightarrow B \vee D$ as follows:

$$\dfrac{\Gamma \Rightarrow B \vee C \qquad \dfrac{\dfrac{\dfrac{B \Rightarrow B}{B, (C \to D)^+ \Rightarrow B}\ (T \Rightarrow)}{B, (C \to D)^+ \Rightarrow B \vee D}\ (\Rightarrow \vee) \qquad \dfrac{\dfrac{C \Rightarrow C \quad D \Rightarrow D}{C, (C \to D)^+ \Rightarrow D}\ (\to^+\Rightarrow)}{C, (C \to D)^+ \Rightarrow B \vee D}\ (\Rightarrow \vee)}{B \vee C, (C \to D)^+ \Rightarrow B \vee D}\ (\vee \Rightarrow)}{\Gamma, (C \to D)^+ \Rightarrow B \vee D}\ (\mathrm{cut})\,.$$

Moreover, if $A \to B, \Gamma \Rightarrow C$ is provable in $\mathbf{GVPL^+}$ then so is $(A \to B)^+, \Gamma \Rightarrow C$ by using (cut) and $(A \to B)^+ \Rightarrow A \to B$ which is provable in $\mathbf{GVPL^+}$. Now by the induction hypothesis, we may suppose that $\Delta_i, \Sigma, A \Rightarrow \bigvee(B, \Gamma_i)$ is provable in $\mathbf{GVPL^+}$ for any i with $1 \le i \le 2^n$. Let $\Pi = \Sigma, C_1 \to D_1, \ldots, C_n \to D_n$ and let Δ'_i be the sequence obtained from Γ_i by replacing every C_j with D_j. Then by the observations above,

$$\Delta_i, \Pi^+, A \Rightarrow \bigvee(B, \Delta'_i) \tag{5}$$

is provable in $\mathbf{GVPL^+}$. For $i > 1$, let $\Delta_i = D_{i_1}, \ldots, D_{i_m}$. Then we have that

$$D_{i_1}, \ldots, D_{i_m}, \Pi^+, A \Rightarrow \bigvee(B, \Delta'_i) \tag{6}$$

is provable in $\mathbf{GVPL^+}$. So by ($\vee \Rightarrow$),

$$\bigvee(B, \Delta'_i, D_{i_1}), \ldots, \bigvee(B, \Delta'_i, D_{i_m}), \Pi^+, A \Rightarrow \bigvee(B, \Delta'_i) \tag{7}$$

is provable in $\mathbf{GVPL^+}$. Here we apply (cut) to each two sequents of the form (7) such that $1 \leq h < i \leq 2^n$ and $\bigvee(B, \Delta'_h) = \bigvee(B, \Delta'_i, D_{i_k})$ for some k. Then we obtain $\Pi^+, A \Rightarrow \bigvee(B, \Delta'_{2^n})$, that is, $\Pi^+, A \Rightarrow B$. Finally we apply $(\Rightarrow\rightarrow)$ and obtain $\Pi \Rightarrow A \rightarrow B$. □

Lemma 5.2 *Let Γ, A contain no formula of the form $(B \rightarrow C)^+$ and let Σ^- be the sequence obtained from Σ by replacing every $(B \rightarrow C)^+$ with $B \rightarrow C$. If $\Gamma, \Sigma \Rightarrow A$ is provable in $\mathbf{GVPL^+}$, then $\Sigma^- \Rightarrow \bigwedge\Gamma \rightarrow A$ is provable in* **LBP**.

Proof We prove the lemma by induction on the proof of $\Gamma, \Sigma \Rightarrow A$ in $\mathbf{GVPL^+}$. We treat here some cases.

Case 1 $(\Rightarrow \wedge)$ Suppose that the last inference is of the form

$$\frac{\Gamma, \Sigma \Rightarrow A \quad \Gamma, \Sigma \Rightarrow B}{\Gamma, \Sigma \Rightarrow A \wedge B}\ (\Rightarrow \wedge)\,.$$

By the induction hypothesis, $\Sigma^- \Rightarrow \bigwedge\Gamma \rightarrow A$ and $\Sigma^- \Rightarrow \bigwedge\Gamma \rightarrow B$ are provable in **LBP**. Since $\bigwedge\Gamma \rightarrow A, \bigwedge\Gamma \rightarrow B \Rightarrow \bigwedge\Gamma \rightarrow A \wedge B$ is provable in **LBP**, we obtain $\Sigma^- \Rightarrow \bigwedge\Gamma \rightarrow A \wedge B$ by applying (cut) twice.

Case 2 $(\vee \Rightarrow)$ There are two possibilities. One is the case where $A \vee B$ is in Γ. Suppose that the last inference is of the form

$$\frac{(A, \Gamma'), \Sigma \Rightarrow C \quad (B, \Gamma'), \Sigma \Rightarrow C}{(A \vee B, \Gamma'), \Sigma \Rightarrow C}\ (\vee \Rightarrow)\,.$$

By the induction hypothesis, $\Sigma^- \Rightarrow \bigwedge(A, \Gamma') \rightarrow C$ and $\Sigma^- \Rightarrow \bigwedge(B, \Gamma') \rightarrow C$ are provable in **LBP**. Since $\bigwedge(A, \Gamma') \rightarrow C, \bigwedge(B, \Gamma') \rightarrow C \Rightarrow \bigwedge(A \vee B, \Gamma') \rightarrow C$ is provable in **LBP**, we obtain $\Sigma^- \Rightarrow \bigwedge(A \vee B, \Gamma') \rightarrow C$ by (cut).

The other is the case where $A \vee B$ is in Σ. Suppose that the last inference is of the form

$$\frac{\Gamma, (A, \Sigma') \Rightarrow C \quad \Gamma, (B, \Sigma') \Rightarrow C}{\Gamma, (A \vee B, \Sigma') \Rightarrow C}\ (\vee \Rightarrow)\,.$$

By the induction hypothesis, $(A, \Sigma')^- \Rightarrow \bigwedge\Gamma \rightarrow C$ and $(B, \Sigma')^- \Rightarrow \bigwedge\Gamma \rightarrow C$ are provable in **LBP**, and by ($\vee$left), so is $(A \vee B, \Sigma')^- \Rightarrow \bigwedge\Gamma \rightarrow C$.

Case 3 $(\Rightarrow\rightarrow)$ Suppose that the last inference is of the form

$$\frac{A, \Gamma^+, \Sigma^+ \Rightarrow B}{\Gamma, \Sigma \Rightarrow A \rightarrow B}\ (\Rightarrow\rightarrow)\,.$$

By the induction hypothesis, $(\Gamma^+, \Sigma^+)^- \Rightarrow A \rightarrow B$ is provable in **LBP**, that is, $\Gamma, \Sigma^- \Rightarrow A \rightarrow B$ is provable in **LBP**. By applying ($\wedge$left) and $(\rightarrow)$, we obtain $\Sigma^- \Rightarrow \bigwedge\Gamma \rightarrow (A \rightarrow B)$.

Case 4 $(\rightarrow^+\Rightarrow)$ Suppose that the last inference is of the form

$$\frac{\Gamma, \Sigma' \Rightarrow A \quad (B, \Gamma), \Sigma' \Rightarrow C}{\Gamma, ((A \rightarrow B)^+, \Sigma') \Rightarrow C}\ (\rightarrow^+\Rightarrow)\,.$$

By the induction hypothesis,

$$(\Sigma')^- \Rightarrow \bigwedge\Gamma \rightarrow A, \tag{8}$$

$$(\Sigma')^- \Rightarrow \bigwedge(B, \Gamma) \rightarrow C, \tag{9}$$

are provable in **LBP**. Since

$$\bigwedge\Gamma \to A, A \to B \Rightarrow \bigwedge\Gamma \to B \tag{10}$$

is provable in **LBP**, we obtain

$$A \to B, (\Sigma')^- \Rightarrow \bigwedge\Gamma \to B \tag{11}$$

by applying (cut) to (8) and (10), and hence we obtain

$$A \to B, (\Sigma')^- \Rightarrow \bigwedge\Gamma \to \bigwedge(B, \Gamma). \tag{12}$$

Finally, since

$$\bigwedge\Gamma \to \bigwedge(B, \Gamma), \bigwedge(B, \Gamma) \to C \Rightarrow \bigwedge\Gamma \to C \tag{13}$$

is provable in **LBP**, we obtain

$$A \to B, (\Sigma')^- \Rightarrow \bigwedge\Gamma \to C \tag{14}$$

by applying (cut) to (12), (13), and (9). □

The above two lemmas lead to the following.

Theorem 5.3 *Let Γ, A contain no formula of the form $(B \to C)^+$. Then we have that $\Gamma \Rightarrow A$ is provable in* **LBP** *if and only if $\Gamma \Rightarrow A$ is provable in* **GVPL⁺**.

Proof From left to right, it is just when Δ consists of one formula in Lemma 5.1. For the other direction, suppose that $\Gamma \Rightarrow A$ is provable in **GVPL⁺**. Then by Lemma 5.2 with Σ empty, $\Rightarrow \bigwedge\Gamma \to A$ is provable in **LBP**. The last inference of the proof of $\Rightarrow \bigwedge\Gamma \to A$ is $(\to)$, and so $\bigwedge\Gamma \Rightarrow A$ is provable in **LBP**. Hence $\Gamma \Rightarrow A$ is provable in **LBP** by using (cut) and $\Gamma \Rightarrow \bigwedge\Gamma$. □

This theorem states that **LBP** and **GVPL⁺** are equivalent in provability as far as ordinary formulas are concerned. Let us survey a further difference between the two from the point of view of proof-search for a given sequent. When we try to construct a possible proof from the end-sequent upward and decompose implicational formulas by applying upside down the rule for implication, we have no choice in **LBP** because it has the only rule for implication, and the implicational formulas in the antecedent and the succedent are to be decomposed at the same time. On the other hand, this process is simulated in **GVPL⁺** as follows. First decompose the implicational formula in the succedent by $(\Rightarrow\to)$, marking all implicational formulas in the antecedent with $^+$. Then by $(\to^+\Rightarrow)$ decompose each marked formula one by one. This is all the role that $^+$ plays in proof-search for a sequent consisting of ordinary formulas, and there we need no $(\Rightarrow\to^+)$ which is used only for proving the cut-elimination theorem for **GVPL⁺**. Finally, we mention that all the observations in this section hold between **LFP** and **GFPL⁺** [5] which differs from **GVPL⁺** in the rule $(\Rightarrow\to)$ where the formula $A \to B$ is placed next to Γ^+.

6 Proof of the Cut-Elimination Theorems: Syntactical Method

In this section, we will prove the cut-elimination theorems for **LBP**+(cut) and **LFP**+(cut) following Gentzen's method (see, e.g., Takeuti [9]). It requires us to make some devices in addition to the usual technique of cut-elimination. Here we consider **LBP** and **LFP** as the systems whose sequents are based on finite sequences of formulas as in the previous section. Furthermore we assume that the cut rule is built into their systems. Then our aim is to prove the following theorems.

Theorem 6.1 *If a sequent is provable in* **LBP**, *then it is provable in* **LBP** *without using the cut rule.*

Theorem 6.2 *If a sequent is provable in* **LFP**, *then it is provable in* **LFP** *without using the cut rule.*

In order to prove these, we introduce as usual the mix rule as follows:

$$\frac{\Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Lambda}{\Gamma, \Pi^A \Rightarrow \Delta^A, \Lambda}$$

where both Δ and Π contain the formula A, and Δ^A and Π^A are obtained from Δ and Π, respectively, by deleting all the occurrences of A in them. A is called the *mix formula* of this inference. The systems **LBP*** and **LFP*** are obtained from **LBP** and **LFP**, respectively, by replacing the cut rule by the mix rule.

Then in order to prove Theorems 6.1 and 6.2, it suffices to show the following lemmas.

Lemma 6.3 *If P is a proof of a sequent S in* **LBP*** *which contains only one mix occurring as the last inference, then we can transform P into a proof of S in* **LBP*** *in which no mix occurs.*

Lemma 6.4 *If P is a proof of a sequent S in* **LFP*** *which contains only one mix occurring as the last inference, then we can transform P into a proof of S in* **LFP*** *in which no mix occurs.*

A proof which contains no mix is called a *mix-free* proof. Before proving the above lemmas, we need some auxiliary definitions.

Definition 6.5 Let P be a proof in **LBP*** or **LFP*** and let E be an occurrence of a formula in P. The *direct ancestors* of E are defined inductively as follows.

1. E is a direct ancestor of itself.
2. If a direct ancestor E_1 of E is in the lower sequent of a contraction rule in P as the principal formula, for example,
$$\frac{E_1, E_1, \Gamma \Rightarrow \Delta}{E_1, \Gamma \Rightarrow \Delta} \ ,$$
then the two E_1s in the upper sequent are direct ancestors of E.
3. If a direct ancestor E_1 of E is in the lower sequent of an exchange rule in P as one of the principal formulas, for example,
$$\frac{\Gamma, E_1, E_2, \Sigma \Rightarrow \Delta}{\Gamma, E_2, E_1, \Sigma \Rightarrow \Delta} \ ,$$
then the E_1 in the upper sequent is a direct ancestor of E.
4. If a direct ancestor E_1 of E is in the lower sequent of an inference in P not as one of the principal formulas, for example, if E_1 is the kth occurrence of Σ in the lower sequent of a rule $(\rightarrow)$
$$\frac{\Delta_1, \Sigma, A \rightarrow B, A \Rightarrow B, \Gamma_1 \quad \Delta_2, \Sigma, A \rightarrow B, A \Rightarrow B, \Gamma_2 \quad \cdots \quad \Delta_{2^n}, \Sigma, A \rightarrow B, A \Rightarrow B, \Gamma_{2^n}}{\Sigma, C_1 \rightarrow D_1, \ldots, C_n \rightarrow D_n \Rightarrow A \rightarrow B} \ ,$$
then the kth occurrence of Σ in each upper sequent is a direct ancestor of E.

Definition 6.6 The *grade* of a formula A, denoted by $g(A)$, is defined inductively as follows.

1. $g(p) = 0$ for each propositional variable p,
2. $g(\bot) = 0$,
3. $g(A \to B) = g(A \wedge B) = g(A \vee B) = g(A) + g(B) + 1$.

The grade of a mix is the grade of the mix formula. When a proof P has a mix only as the last inference, we define the grade of P, denoted by $g(P)$, to be the grade of this mix.

Definition 6.7 Let P be a proof in **LBP*** or **LFP*** which has a mix only as the last inference and P_1 (P_2) be the subproof of P whose end-sequent is the left (the right) upper sequent of the mix. We define the *rank* of a sequent S contained in P, denoted by $r(S)$, as follows.

1. S is contained in P_1.
   - (a) If the succedent of S contains no direct ancestor of the occurrences of the mix formula, then $r(S) = 0$.
   - (b) Otherwise, if S is an initial sequent, then $r(S) = 1$, if S is the lower sequent of an inference whose upper sequents are $S_1, \ldots, S_n$, then $r(S) = \max\{r(S_1), \ldots, r(S_n)\} + 1$.
2. S is contained in P_2.
   - (a) If the antecedent of S contains no direct ancestor of the occurrences of the mix formula, then $r(S) = 0$.
   - (b) Otherwise, similar to (1b) above.

Definition 6.8 Let P be a proof in **LBP*** or **LFP*** which has a mix only as the last inference and S_1 (S_2) be the left (the right) upper sequent of this mix. We define $r_l(P) = r(S_1)$ and $r_r(P) = r(S_2)$. The *rank* of P, denoted by $r(P)$, is defined as $r_l(P) + r_r(P)$.

Now we will prove Lemma 6.3. One of the difficulties in proving this lemma by means of the usual technique of cut-elimination is caused by the rule $(\to)$ which only restricts the succedent of the lower sequent to one formula. The same difficulty arises in proof of cut-elimination even for a version of sequent calculus for intuitionistic logic with similar restriction. To overcome this difficulty, we first consider a special case which is proved in the same way as the cut-elimination theorem for **LK** (and so we omit the proof).

Lemma 6.9 *Let P be a proof of a sequent S in* **LBP*** *which contains only one mix occurring as the last inference and let P_1 be the subproof of P whose end-sequent is the left upper sequent of this mix. If P_1 contains no $(\to)$ and if the succedent of each sequent occurring in P_1 consists of at most one formula, then S is provable in* **LBP*** *with no mix.*

This lemma is used in Subcase 2.2.1(c) of the following proof.

Proof of Lemma 6.3 We prove the lemma by transfinite induction on $\omega \cdot g(P) + r(P)$.

Case 1 $r(P) = 2$. We treat the case where each upper sequent of the mix is the lower sequent of $(\to)$. For simplicity, we write

$$\dfrac{\begin{array}{c} \vdots\, Q_k \\ [\Delta_k, \Sigma, A \Rightarrow B, \Gamma_k]_{1 \le k \le 2^n} \end{array}}{\Sigma, C_1 \to D_1, \ldots, C_n \to D_n \Rightarrow A \to B}$$

instead of

$$\dfrac{\begin{array}{c}\vdots\ Q_1\\ \Delta_1, \Sigma, A \Rightarrow B, \Gamma_1\end{array} \quad \begin{array}{c}\vdots\ Q_2\\ \Delta_2, \Sigma, A \Rightarrow B, \Gamma_2\end{array} \quad \cdots \quad \begin{array}{c}\vdots\ Q_{2^n}\\ \Delta_{2^n}, \Sigma, A \Rightarrow B, \Gamma_{2^n}\end{array}}{\Sigma, C_1 \to D_1, \ldots, C_n \to D_n \Rightarrow A \to B} \ .$$

We also remark that if $C_i \to D_i$ is identical with $C_j \to D_j$ for $i \neq j$ then the lower sequent (without $C_j \to D_j$) is derivable only from the upper sequents $\Delta_k, \Sigma, A \Rightarrow B, \Gamma_k$ such that either $C_i, C_j \in \Gamma_k$ or $D_i, D_j \in \Delta_k$. Modified so that the mix formula may appear just once in the antecedent of the right upper sequent of the mix, the last part of P is as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ Q'_l\\ [\Delta'_l, \Sigma', C_i \Rightarrow D_i, \Gamma'_l]_{1 \leq l \leq 2^m}\end{array}}{\Sigma', E_1 \to F_1, \ldots, E_m \to F_m \Rightarrow C_i \to D_i} \quad \dfrac{\begin{array}{c}\vdots\ Q_k\\ [\Delta_k, \Sigma, A \Rightarrow B, \Gamma_k]_{1 \leq k \leq 2^n}\end{array}}{\Sigma, C_1 \to D_1, \ldots, C_n \to D_n \Rightarrow A \to B}}{\Sigma', E_1 \to F_1, \ldots, E_m \to F_m, \Sigma, C_1 \to D_1, \ldots, C_{i-1} \to D_{i-1}, C_{i+1} \to D_{i+1}, \ldots, C_n \to D_n \Rightarrow A \to B} \ .$$

Now take any l and k such that $1 \leq l \leq 2^m$, $1 \leq k \leq 2^n$ and C_i is contained in Γ_k. We consider the following proof P_1:

$$\dfrac{\begin{array}{c}\vdots\ Q_k\\ \Delta_k, \Sigma, A \Rightarrow B, \Gamma_k\end{array} \quad \begin{array}{c}\vdots\ Q'_l\\ \Delta'_l, \Sigma', C_i \Rightarrow D_i, \Gamma'_l\end{array}}{\Delta_k, \Sigma, A, (\Delta'_l)^{C_i}, (\Sigma')^{C_i} \Rightarrow B^{C_i}, (\Gamma_k)^{C_i}, D_i, \Gamma'_l} \ (\text{mix}) \ .$$

Since $g(P_1) < g(P)$, we can obtain a mix-free proof P_2 of

$$\Delta_k, \Sigma, A, (\Delta'_l)^{C_i}, (\Sigma')^{C_i} \Rightarrow B^{C_i}, (\Gamma_k)^{C_i}, D_i, \Gamma'_l$$

by the induction hypothesis.

Next, let $\Gamma_{k'}$ be the sequence obtained from Γ_k by deleting C_i. Then D_i is contained in $\Delta_{k'}$ and we consider the following proof P_3:

$$\dfrac{\begin{array}{c}\vdots\ P_2\\ \Delta_k, \Sigma, A, (\Delta'_l)^{C_i}, (\Sigma')^{C_i} \Rightarrow B^{C_i}, (\Gamma_k)^{C_i}, D_i, \Gamma'_l\end{array} \quad \begin{array}{c}\vdots\ Q_{k'}\\ \Delta_{k'}, \Sigma, A \Rightarrow B, \Gamma_{k'}\end{array}}{\Delta_k, \Sigma, A, (\Delta'_l)^{C_i}, (\Sigma')^{C_i}, (\Delta_{k'})^{D_i}, \Sigma^{D_i}, A^{D_i} \Rightarrow (B^{C_i})^{D_i}, ((\Gamma_k)^{C_i})^{D_i}, (\Gamma'_l)^{D_i}, B, \Gamma_{k'}} \ (\text{mix}) \ .$$

Since $g(P_3) < g(P)$, we can eliminate the above mix by the induction hypothesis. Noticing that $(\Gamma_k)^{C_i}$ is identical with or a subsequence of $\Gamma_{k'}$ and that $(\Delta_{k'})^{D_i}$ is identical with or a subsequence of Δ_k, we obtain a mix-free proof ending with $\Delta_k, \Sigma, A, (\Delta'_l)^{C_i}, (\Sigma')^{C_i}, \Delta_k, \Sigma^{D_i}, A^{D_i} \Rightarrow (B^{C_i})^{D_i}, (\Gamma_{k'})^{D_i}, (\Gamma'_l)^{D_i}, B, \Gamma_{k'}$ and hence obtain a mix-free proof of $\Delta_k, \Delta'_l, \Sigma, \Sigma', A \Rightarrow B, \Gamma_{k'}, \Gamma'_l$. This holds for any l and k such that $1 \leq l \leq 2^m$, $1 \leq k \leq 2^n$ and C_i is contained in Γ_k.

Now we set $C_{n+h} = E_h$ and $D_{n+h} = F_h$ for all h such that $1 \leq h \leq m$. Let

$$\langle [q_1, \ldots, q_t], [q_{t+1}, \ldots, q_{n+m-1}] \rangle$$

be the qth division of

$$[1, \ldots, i-1, i+1, \ldots, n, n+1, \ldots, n+m]$$

(cf. the definition of **LBP** in Section 2). Then there exists a (k, l) considered above such that $\Delta_k, \Delta'_l = D_{q_1}, \ldots, D_{q_t} (= \Delta^*_q)$ and $\Gamma_{k'}, \Gamma'_l = C_{q_{t+1}}, \ldots, C_{q_{n+m-1}} (= \Gamma^*_q)$.

Therefore we can construct a required proof as follows:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Delta_1^*,\Sigma,\Sigma',A\Rightarrow B,\Gamma_1^*\end{matrix} \quad \begin{matrix}\vdots \\ \Delta_2^*,\Sigma,\Sigma',A\Rightarrow B,\Gamma_2^*\end{matrix} \quad \cdots \quad \begin{matrix}\vdots \\ \Delta_{2^{n+m-1}}^*,\Sigma,\Sigma',A\Rightarrow B,\Gamma_{2^{n+m-1}}^*\end{matrix}}{\Sigma,\Sigma',C_1\to D_1,\ldots,C_{i-1}\to D_{i-1},C_{i+1}\to D_{i+1},\ldots,C_n\to D_n,E_1\to F_1,\ldots,E_m\to F_m\Rightarrow A\to B}\,(\to)}{\Sigma',E_1\to F_1,\ldots,E_m\to F_m,\Sigma,C_1\to D_1,\ldots,C_{i-1}\to D_{i-1},C_{i+1}\to D_{i+1},\ldots,C_n\to D_n\Rightarrow A\to B}\,\text{(some exchanges)}\ .$$

Case 2 $r(P) > 2$.

Subcase 2.1 $r_l(P) > 1$. The last part of P is as follows:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Phi \Rightarrow \Psi\end{matrix}}{\Gamma \Rightarrow \Delta}\,I \qquad \begin{matrix}\vdots \\ \Pi \Rightarrow \Lambda\end{matrix}}{\Gamma, \Pi^A \Rightarrow \Delta^A, \Lambda}\ .$$

Since $r_l(P) > 1$, the inference I cannot be $(\to)$. Then the proof is carried out in the same way as that for **LK**.

Subcase 2.2 $r_l(P) = 1$ and $r_r(P) > 1$.

Subcase 2.2.1 The right upper sequent of the mix is the lower sequent of either a logical inference whose principal formulas contain no A or a structural inference. The last part of P is as follows:

$$\dfrac{\begin{matrix}\vdots \\ \Gamma \Rightarrow \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\vdots \\ \Phi \Rightarrow \Psi\end{matrix}}{\Pi \Rightarrow \Lambda}\,I}{\Gamma, \Pi^A \Rightarrow \Delta^A, \Lambda}\ .$$

We treat the case where I is $(\to)$.

(a) $\Gamma \Rightarrow \Delta$ is an initial sequent or the lower sequent of a weakening rule. In this case, the claim is easy to see and we omit the detail.

(b) $\Gamma \Rightarrow \Delta$ is the lower sequent of $(\to)$. Since the succedent Δ consists of one formula A, the last part of P is as follows:

$$\dfrac{\begin{matrix}\vdots\, Q \\ \Gamma \Rightarrow A\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\, R_1 \\ \Delta_1, \Sigma, B \Rightarrow C, \Gamma_1\end{matrix} \quad \begin{matrix}\vdots\, R_2 \\ \Delta_2, \Sigma, B \Rightarrow C, \Gamma_2\end{matrix} \quad \cdots \quad \begin{matrix}\vdots\, R_{2^n} \\ \Delta_{2^n}, \Sigma, B \Rightarrow C, \Gamma_{2^n}\end{matrix}}{\Pi \Rightarrow B \to C}\,I}{\Gamma, \Pi^A \Rightarrow B \to C}$$

Consider the following proof P_k:

$$\dfrac{\begin{matrix}\vdots\, Q \\ \Gamma \Rightarrow A\end{matrix} \qquad \begin{matrix}\vdots\, R_k \\ \Delta_k, \Sigma, B \Rightarrow C, \Gamma_k\end{matrix}}{\Gamma, (\Delta_k)^A, \Sigma^A, B^A \Rightarrow C, \Gamma_k}\,\text{(mix)}$$

Since $r(P_k) < r(P)$, we can eliminate the above mix by the induction hypothesis. Therefore for all k such that $1 \le k \le 2^n$, we can obtain mix-free proofs P'_k ending with $\Delta_k, \Gamma, \Sigma^A, B \Rightarrow C, \Gamma_k$. Noticing that no

A is in the principal formulas of I, we can construct a mix-free proof of $\Gamma, \Pi^A \Rightarrow B \to C$ as follows:

$$\dfrac{\begin{array}{c}\vdots\, P'_1\\ \Delta_1, \Gamma, \Sigma^A, B \Rightarrow C, \Gamma_1\end{array} \quad \begin{array}{c}\vdots\, P'_2\\ \Delta_2, \Gamma, \Sigma^A, B \Rightarrow C, \Gamma_2\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, P'_{2^n}\\ \Delta_{2^n}, \Gamma, \Sigma^A, B \Rightarrow C, \Gamma_{2^n}\end{array}}{\Gamma, \Pi^A \Rightarrow B \to C}\ (\to)\ .$$

(c) $\Gamma \Rightarrow \Delta$ is the lower sequent of ($\wedge$right) or ($\vee$right). We treat the case of ($\wedge$right). The last part of P is as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, Q_1\\ \Gamma \Rightarrow \Delta', B\end{array} \quad \begin{array}{c}\vdots\, Q_2\\ \Gamma \Rightarrow \Delta', C\end{array}}{\Gamma \Rightarrow \Delta', B \wedge C} \quad \begin{array}{c}\vdots\, R\\ \Pi \Rightarrow \Lambda\end{array}}{\Gamma, \Pi^{B\wedge C} \Rightarrow \Delta', \Lambda}\ .$$

Consider the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B}{C, B \Rightarrow B}}{B, C \Rightarrow B} \quad \dfrac{C \Rightarrow C}{B, C \Rightarrow B}}{B, C \Rightarrow B \wedge C} \quad \begin{array}{c}\vdots\, R\\ \Pi \Rightarrow \Lambda\end{array}}{B, C, \Pi^{B\wedge C} \Rightarrow \Lambda}\ .$$

By applying Lemma 6.9 to the above proof, we obtain a mix-free proof P_1 ending with $B, C, \Pi^{B\wedge C} \Rightarrow \Lambda$. Then consider the following proof P_2:

$$\dfrac{\begin{array}{c}\vdots\, Q_1\\ \Gamma \Rightarrow \Delta', B\end{array} \quad \begin{array}{c}\vdots\, P_1\\ B, C, \Pi^{B\wedge C} \Rightarrow \Lambda\end{array}}{\Gamma, C^B, (\Pi^{B\wedge C})^B \Rightarrow (\Delta')^B, \Lambda}\ (\text{mix})\ .$$

Since $g(P_2) < g(P)$, we can eliminate the above mix by the induction hypothesis, and obtain a mix-free proof P_3 ending with $\Gamma, C, \Pi^{B\wedge C} \Rightarrow \Delta', \Lambda$.

Next, consider the following proof P_4:

$$\dfrac{\begin{array}{c}\vdots\, Q_2\\ \Gamma \Rightarrow \Delta', C\end{array} \quad \begin{array}{c}\vdots\, P_3\\ \Gamma, C, \Pi^{B\wedge C} \Rightarrow \Delta', \Lambda\end{array}}{\Gamma, \Gamma^C, (\Pi^{B\wedge C})^C \Rightarrow (\Delta')^C, \Delta', \Lambda}\ (\text{mix})\ .$$

Since $g(P_4) < g(P)$, we can obtain a mix-free proof ending with

$$\Gamma, \Gamma^C, (\Pi^{B\wedge C})^C \Rightarrow (\Delta')^C, \Delta', \Lambda$$

by the induction hypothesis, and hence obtain a mix-free proof of

$$\Gamma, \Pi^{B\wedge C} \Rightarrow \Delta', \Lambda.$$

Subcase 2.2.2 The right upper sequent of the mix is the lower sequent of a logical inference whose principal formulas contain the mix formula A. This case is treated similarly to that of the cut-elimination theorem for **LK**. □

Next we will prove Lemma 6.4. We assume here that the initial sequents of the form $A \Rightarrow A$ are restricted to the form such that A is a propositional variable. It is easy to see that the resulting system is equivalent to the original one. Hence we call this also **LFP***.

Definition 6.10 Let P be a proof in **LFP*** which has a rule $(\to)$

$$\frac{\Delta_1, \Sigma, A\to B, A\Rightarrow B, \Gamma_1 \quad \Delta_2, \Sigma, A\to B, A\Rightarrow B, \Gamma_2 \quad \cdots \quad \Delta_{2^n}, \Sigma, A\to B, A\Rightarrow B, \Gamma_{2^n}}{\Sigma, C_1\to D_1, \ldots, C_n\to D_n \Rightarrow A\to B}\ .$$

The $(A \to B)$s in the upper sequents are called the *diagonal formulas* of this inference. We define the *width* of this inference I to be the number of the inferences I' whose principal formulas contain a direct ancestor of the diagonal formulas of I.

Definition 6.11 Let P be a proof in **LFP*** which has a mix only as the last inference. The *width* of P, denoted by $w(P)$, is defined as follows:

1. The case where the mix formula is of the form $A \to B$. Let P' be the subproof of P whose end-sequent is the left upper sequent of the mix. Then $w(P)$ is the sum of the width of all lowermost $(\to)$s in P'.
2. Otherwise, $w(P) = 0$.

Now we prove Lemma 6.4. The following technique is an analogue of that used in [10].

Proof of Lemma 6.4 We prove the lemma by transfinite induction on $\omega^2 \cdot g(P) + \omega \cdot w(P) + r(P)$. Here we treat only the case where $r(P) = 2$ and each upper sequent of the mix is the lower sequent of $(\to)$. With the modification remarked in the proof of Lemma 6.3, the last part of P is as follows:

$$\frac{\dfrac{\begin{array}{c}\vdots\ Q'_l\\ [\Delta'_l, \Sigma', C_i\to D_i, C_i\Rightarrow D_i, \Gamma'_l]_{1\le l\le 2^m}\end{array}}{\Sigma', E_1\to F_1, \ldots, E_m\to F_m \Rightarrow C_i\to D_i}\ I \qquad \dfrac{\begin{array}{c}\vdots\ Q_k\\ [\Delta_k, \Sigma, A\to B, A\Rightarrow B, \Gamma_k]_{1\le k\le 2^n}\end{array}}{\Sigma, C_1\to D_1, \ldots, C_n\to D_n \Rightarrow A\to B}}{\Sigma', E_1\to F_1, \ldots, E_m\to F_m, \Sigma, C_1\to D_1, \ldots, C_{i-1}\to D_{i-1}, C_{i+1}\to D_{i+1}, \ldots, C_n, \to D_n \Rightarrow A\to B}\ .$$

Case 1 $w(P) = 0$. In this case, each topmost direct ancestor of the diagonal formulas $C_i \to D_i$ of I is the principal formula of a weakening rule (note that all the initial sequents of the form $A \Rightarrow A$ of our system are of the form $p \Rightarrow p$ for propositional variable p). Hence by deleting the diagonal formulas $C_i \to D_i$ as well as all direct ancestors of them from P and some trivial modifications, we obtain proofs ending with $[\Delta'_l, \Sigma', C_i \Rightarrow D_i, \Gamma'_l]_{1\le l\le 2^m}$. Then we proceed in the same way as Case 1 in the proof of Lemma 6.3.

Case 2 $w(P) > 0$. Let P_1 be the subproof of P whose end-sequent is the left upper sequent of the mix. In this case, there exists an inference I' whose principal formulas contain a direct ancestor of the diagonal formulas $C_i \to D_i$ of I. Then P_1 looks like this:

$$\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\ R_j\\ [\Delta^*_j, \Sigma^*, G \to H, G \Rightarrow H, \Gamma^*_j]_{1\le j\le 2^r}\end{array}}{\Sigma^*, J_1 \to K_1, \ldots (C_i \to D_i) \ldots, J_r \to K_r \Rightarrow G \to H}\ I' \\
\vdots\ Q'_{l'} \\
\dfrac{\cdots \quad \Delta'_{l'}, \Sigma', C_i \to D_i, C_i \Rightarrow D_i, \Gamma'_{l'} \quad \cdots}{\Sigma', E_1 \to F_1, \ldots, E_m \to F_m \Rightarrow C_i \to D_i}\ I
\end{array}$$

where $C_i \to D_i$ appears in $J_1 \to K_1, \ldots, J_r \to K_r$, which are all distinct with the modification remarked in the proof of Lemma 6.3. We transform P_1 as follows:

1. Delete the part over $\Sigma^*, J_1 \to K_1, \ldots, J_r \to K_r \Rightarrow G \to H$.

2. Transform each remaining sequent $\Pi \Rightarrow \Lambda$ to $G \to H, \Pi \Rightarrow \Lambda$.

In the figure obtained in this way, each inference is correct, and each topmost sequent is provable using structural inferences only. Therefore from this figure, we can obtain a mix-free proof P_2 ending with

$$G \to H, \Sigma', E_1 \to F_1, \ldots, E_m \to F_m \Rightarrow C_i \to D_i.$$

Now take any l such that $1 \leq l \leq 2^m$, and consider the following proof P'_l:

$$\frac{\overset{\vdots\, P_2}{G\to H,\Sigma',E_1\to F_1,\ldots,E_m\to F_m\Rightarrow C_i\to D_i} \quad \overset{\vdots\, Q'_l}{\Delta'_l,\Sigma',C_i\to D_i,C_i\Rightarrow D_i,\Gamma'_l}}{G\to H,\Sigma',E_1\to F_1,\ldots,E_m\to F_m,(\Delta'_l)^{C_i\to D_i},(\Sigma')^{C_i\to D_i},C_i,\Rightarrow D_i,\Gamma'_l}\ (\text{mix})\ .$$

Since $w(P'_l) < w(P)$, we can obtain a mix-free proof R'_l of this end-sequent. Furthermore take any j with $1 \leq j \leq 2^r$ such that C_i is contained in Γ^*_j, and by the same argument as Case 1 in the proof of Lemma 6.3 with R_j and R'_l instead of Q_k and Q'_l there, we finally obtain a mix-free proof R ending with

$$\Sigma', E_1 \to F_1, \ldots, E_m \to F_m, \Sigma^*, (J_1 \to K_1, \ldots, J_r \to K_r)^{C_i\to D_i} \Rightarrow G \to H.$$

Next, we again transform P_1 as follows:

1. Delete the part over $\Sigma^*, J_1 \to K_1, \ldots, J_r \to K_r \Rightarrow G \to H$.
2. Transform each remaining sequent $\Pi \Rightarrow \Lambda$ to
$$\Sigma', E_1 \to F_1, \ldots, E_m \to F_m, \Pi \Rightarrow \Lambda.$$
3. Put on the deleted part the mix-free proof R followed by a weakening rule (and some exchange rules) whose principal formula is $C_i \to D_i$.

Since each topmost sequent not in R is provable by structural inferences, we can obtain a mix-free proof P_3 ending with $\Sigma', E_1 \to F_1, \ldots, E_m \to F_m \Rightarrow C_i \to D_i$.

Finally, consider the following proof P_4:

$$\frac{\overset{\vdots\, P_3}{\Sigma',E_1\to F_1,\ldots,E_m\to F_m\Rightarrow C_i\to D_i} \quad \dfrac{\overset{\vdots\, Q_k}{[\Delta_k,\Sigma,A\to B,A\Rightarrow B,\Gamma_k]_{1\leq k\leq 2^n}}}{\Sigma,C_1\to D_1,\ldots,C_n\to D_n\Rightarrow A\to B}}{\Sigma',E_1\to F_1,\ldots,E_m\to F_m,\Sigma,C_1\to D_1,\ldots,C_{i-1}\to D_{i-1},C_{i+1}\to D_{i+1},\ldots,C_n,\to D_n\Rightarrow A\to B}\ .$$

It is easy to see that $w(P_4) < w(P)$. Therefore we can eliminate the above mix by the induction hypothesis, and obtain a required mix-free proof. □

Note

1. Similarly, $(B \to C) \to (A \to C)$ is provable in **HB** whenever $A \to B$ is. They are used in an inductive proof of the equivalent replacement, which justifies such an expression as $\bigwedge \Gamma$ with the associativity and commutativity of $\wedge$ on provability in **HB**.

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School of Information Science
Japan Advanced Institute of Science and Technology
Tatsunokuchi
Ishikawa 923-1292
JAPAN
k-ishii@jaist.ac.jp

Department of Mathematical and Computing Sciences
Tokyo Institute of Technology
Ookayama Meguro
Tokyo 152-8552
JAPAN
kashima@is.titech.ac.jp

School of Information Science
Japan Advanced Institute of Science and Technology
Tatsunokuchi
Ishikawa 923-1292
JAPAN
kentaro@jaist.ac.jp