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A New Solution to a Problem of Hosoi and Ono

MICHAEL ZAKHARYASCHEV

Abstract This paper gives a new, purely semantic proof of the following theorem: if an intermediate propositional logic L has the disjunction property then a disjunction free formula is provable in L iff it is provable in intuitionistic logic. The main idea of the proof is to use the well-known semantic criterion of the disjunction property for "simulating" finite binary trees (which characterize the disjunction free fragment of intuitionistic logic) by general frames.

Hosoi and Ono [8] raised the following problem: is it true that if a consistent extension L of intuitionistic propositional logic **Int** has the disjunction property (i.e., $L \vdash \varphi \vee \psi$ implies $L \vdash \varphi$ or $L \vdash \psi$) then the disjunction free fragment of L is the same as the disjunction free fragment of **Int**? Minari [10] and Zakharyaschev [16] and [17] independently gave positive solutions to this problem. In fact, both of them constructed special sequences of disjunctions and showed that if a disjunction free nonthesis of **Int** is provable in an intermediate logic L then some disjunction in the sequence is also a thesis of L while its disjuncts are not. (An anonymous referee has kindly informed me that Minari [11] found another proof which uses a sequence of characteristic formulas.) Here I present a new, purely semantic solution to the problem of Hosoi and Ono, which is simpler and (I hope) more elegant. The idea of my approach is to use a semantic equivalent of the disjunction property for "simulating" finite binary trees, the class of which characterizes the disjunction free fragment of **Int**.

We shall use three well-known results. The first one is that an intermediate logic

$$L = \mathbf{Int} + \{\varphi_i : i \in I\}$$

has the disjunction property if and only if its greatest normal modal "companion"

$$\sigma L = \mathbf{Grz} + \{T(\varphi_i) : i \in I\}$$

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has the modal disjunction property, i.e., $\sigma L \vdash \Box\varphi \vee \Box\psi$ implies $\sigma L \vdash \Box\varphi$ or $\sigma L \vdash \Box\psi$. Here T is the Gödel translation prefixing $\Box$ to every subformula of a given intuitionistic formula and **Grz**, the Grzegorczyk logic, is obtained by adding $\Box(\Box(p \to \Box p) \to p) \to p$ to **S4** as a new axiom. This fact was first noted by Gudovschikov and Rybakov [7]; for a proof see Zakharyaschev [18].

I prefer to deal with modal logics rather than intermediate ones for purely technical (or, more exactly, aesthetic) reasons. Of course, this is a matter of taste, but it seems to me that semantic constructions based on the classical Boolean operations look clearer than constructions with the pseudo-Boolean ones. Anyway, it is not difficult to realize the idea of the proof below using only intuitionistic means.

The second result we need is the algebraic criterion for the disjunction property of intermediate logics which was found by Maksimova [9]. I will reformulate it for normal modal logics containing **S4** in terms of descriptive general frames (for definitions consult Goldblatt [6]).

Theorem 1 (Semantic criterion for the modal disjunction property) *A normal modal logic $M \supseteq$ **S4** has the modal disjunction property iff, for every two descriptive general frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$ validating M, there is a rooted descriptive general frame $\mathfrak{F}$ for M such that the disjoint union $\mathfrak{F}_1 + \mathfrak{F}_2$ is (isomorphic to) a generated subframe of $\mathfrak{F}$.*

I remind the reader that in every general frame $\mathfrak{F} = \langle W, R, P\rangle$ for **S4** the pair $\langle W, R\rangle$ is an ordinary quasi-ordered Kripke frame and P is a non-empty collection of subsets of W, which contains $\varnothing$ and is closed under the set-theoretic operations $\cap$, $\cup$, $-$ and the following (closure) operation $\downarrow$: for every $V \subseteq W$,

$$V{\downarrow} = \{a \in W : aRb \text{ for some } b \in V\}.$$

A frame $\mathfrak{F} = \left\langle W, R, 2^W\right\rangle$, called a *full* or *Kripke frame*, is denoted by $\mathfrak{F} = \langle W, R\rangle$. Without loss of generality we will assume that every finite general frame is a Kripke one. $\mathfrak{F} = \langle W, R.P\rangle$ is called *rooted* if $a{\uparrow} = W$, for some $a \in W$, where, for $V \subseteq W$,

$$V{\uparrow} = \{a \in W : bRa \text{ for some } b \in V\} \text{ and } a{\uparrow} = \{a\}{\uparrow}.$$

A frame $\mathfrak{F} = \langle W, R, P\rangle$ is a *generated subframe* of $\mathfrak{F}_1 = \langle W_1, R_1, P_1\rangle$ if W is an upward closed subset of W_1 (i.e., $W = W{\uparrow}$) in $\mathfrak{F}_1$, R is the restriction of R_1 to W and $P = \{V \cap W : V \in P_1\}$. Finally, a frame $\mathfrak{F}_1 + \mathfrak{F}_2 = \langle W, R, P\rangle$ is called the *disjoint union* of disjoint frames $\mathfrak{F}_1 = \langle W_1, R_1, P_1\rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2, P_2\rangle$ if $W = W_1 \cup W_2$, $R = R_1 \cup R_2$ and $P = \{V_1 \cup V_2 : V_1 \in P_1,\ V_2 \in P_2\}$. The definition of descriptiveness is not vital for us here; it can be found in Goldblatt [6].

Our third auxiliary result is actually a consequence of Zakharyaschev [19] but I can give a simple straightforward inductive proof here. To formulate this result we need the following notion of cofinal subreduction.

Definition 2 A partial (i.e., not completely defined, in general) map f from W_1 onto W is called a *subreduction* (as in Fine [5] or a *partial p-morphism*, as in Zakharyaschev [19]) of a frame $\mathfrak{F}_1 = \langle W_1, R_1, P_1\rangle$ to a frame $\mathfrak{F} = \langle W, R, P\rangle$ if the following conditions are satisfied:

- aR_1b implies $f(a)Rf(b)$, for every $a, b \in \mathrm{dom}\, f$;

- cRd implies $\forall a \in f^{-1}(c) \exists b \in f^{-1}(d)\ aRb$, for every $c, d \in W$;
- $f^{-1}(V) \in P_1$, for every $V \in P$.

We say a subreduction f of $\mathfrak{F}_1$ to $\mathfrak{F}$ is *cofinal* if $a \in \mathrm{dom} f\uparrow$ implies $a \in \mathrm{dom} f\downarrow$, for every $a \in W_1$ (i.e., if a is "seen" from the domain of f then a itself "sees" at least one point in $\mathrm{dom} f$).

Lemma 3 *Let φ be an intuitionistic disjunction free formula and $\mathfrak{F}_1 \models T(\varphi)$. Then $\mathfrak{F} \models T(\varphi)$, for every cofinal subreduct $\mathfrak{F}$ of $\mathfrak{F}_1$.*

Proof: Suppose that f is a cofinal subreduction of $\mathfrak{F}_1 = \langle W_1, R_1, P_1 \rangle$ to $\mathfrak{F} = \langle W, R, P \rangle$ and $\mathfrak{F} \not\models T(\varphi)$. Using the Generation Theorem of Segerberg [12], we may assume that $W_1 = \mathrm{dom} f\uparrow$ (in any case, we can always take the subframe of $\mathfrak{F}_1$ generated by $\mathrm{dom} f$ and show that it refutes $T(\varphi)$). Given a valuation $\mathfrak{V}$ refuting $T(\varphi)$ in $\mathfrak{F}$, we define a valuation $\mathfrak{V}_1$ in $\mathfrak{F}_1$ by taking

$$\mathfrak{V}_1(p, a) = \begin{cases} \mathrm{F} & \text{if } \exists b \in a\uparrow\ \mathfrak{V}(p, f(b)) = \mathrm{F}; \\ \mathrm{T} & \text{otherwise,} \end{cases}$$

for every variable p and every $a \in W_1$. $\mathfrak{V}_1$ is a valuation, since

$$\{a \in W_1 : \mathfrak{V}_1(p, a) = \mathrm{F}\} = f^{-1}(\{b \in W : \mathfrak{V}(p, b) = \mathrm{F}\})\downarrow \in P_1.$$

Now by induction on the construction of φ we prove that, for every $a \in W_1$,

$$\mathfrak{V}_1(T(\varphi), a) = \mathrm{F} \text{ iff } \exists b \in a\uparrow\ \mathfrak{V}(T(\varphi), f(b)) = \mathrm{F}.$$

The induction basis for variables follows immediately from the definitions of $\mathfrak{V}_1$ and subreduction. As for the constant $\perp$ (falsehood), we use the cofinality of f and our assumption above according to which $a \in \mathrm{dom} f\uparrow$, for every $a \in W_1$.

Let $\varphi = \psi \to \chi$ and so $T(\varphi) = \Box(T(\psi) \to T(\chi))$. Suppose $\mathfrak{V}_1(T(\varphi), a) = \mathrm{F}$. Then $\mathfrak{V}_1(T(\psi), b) = \mathrm{T}$ and $\mathfrak{V}_1(T(\chi), b) = \mathrm{F}$, for some $b \in a\uparrow$. Therefore, by the induction hypothesis, there is $c \in b\uparrow$ such that $\mathfrak{V}(T(\chi), f(c)) = \mathrm{F}$. Notice also that $\mathfrak{V}(T(\psi), f(c)) = \mathrm{T}$, for otherwise we have $\mathfrak{V}_1(T(\psi), c) = \mathrm{F}$, which implies $\mathfrak{V}_1(T(\psi), b) = \mathrm{F}$, since $T(\psi)$ begins with $\Box$. Thus, $\mathfrak{V}(T(\varphi), f(c)) = \mathrm{F}$ for some $c \in a\uparrow$.

Conversely, suppose $\mathfrak{V}(T(\varphi), f(b)) = \mathrm{F}$, for some $b \in a\uparrow$. Then, using the definition of subreduction, we obtain $\mathfrak{V}(T(\psi), f(c)) = \mathrm{T}$ and $\mathfrak{V}(T(\chi), f(c)) = \mathrm{F}$, for some $c \in b\uparrow$. By the induction hypothesis, $\mathfrak{V}_1(T(\chi), c) = \mathrm{F}$. Besides, we have $\mathfrak{V}_1(T(\psi), c) = \mathrm{T}$, for otherwise there is $d \in c\uparrow$ such that $\mathfrak{V}(T(\psi), f(d)) = \mathrm{F}$, which implies $\mathfrak{V}(T(\psi), f(c)) = \mathrm{F}$, since cRd. Thus, $\mathfrak{V}_1(T(\psi) \to T(\chi), c) = \mathrm{F}$ and so $\mathfrak{V}_1(T(\varphi), a) = \mathrm{F}$.

The case of $\varphi = \psi \wedge \chi$ is considered analogously.

It follows that $\mathfrak{F}_1 \not\models T(\varphi)$, which is a contradiction. Thus $\mathfrak{F} \models T(\varphi)$.

Remark 4 This proof does not go through for $\varphi = \psi \vee \chi$, since there may be a situation when $\mathfrak{V}_1(T(\varphi), a) = \mathrm{F}$ because $\mathfrak{V}(T(\psi), f(b_1)) = \mathrm{F}$, $\mathfrak{V}(T(\chi), f(b_2)) = \mathrm{F}$, for some distinct $b_1, b_2 \in a\uparrow$, but $\mathfrak{V}(T(\varphi), f(b)) = \mathrm{T}$, for all $b \in a\uparrow \cap \mathrm{dom} f$. As an example, the reader can analyze the Scott Axiom

$$((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p.$$

Remark 5 It follows from Zakharyaschev [19] that an intermediate logic L is axiomatizable by disjunction free formulas if and only if L is characterized by a class of frames closed under taking cofinal subframes (not necessarily generated) and L is axiomatizable by purely implicative formulas if and only if L is characterized by a class of frames closed under taking subframes (cf. Fine [5]).

It is not difficult to see that, for each finite rooted partially ordered frame $\mathfrak{F}$, there is a finite frame $\mathfrak{F}_1$ which has the form of binary tree and is cofinally subreducible to $\mathfrak{F}$ (one can prove this by induction on the number of points in $\mathfrak{F}$). Thus, for each disjunction free nonthesis φ of **Int**, there is a finite binary tree $\mathfrak{F}$ such that $\mathfrak{F} \not\models T(\varphi)$. This is just another form of the well-known result of Segerberg [13].

Now we are in a position to prove our main theorem.

Theorem 6 *If an intermediate logic L has the disjunction property then the disjunction free fragments of L and* **Int** *are the same.*

Proof: Suppose that L has the disjunction property and φ is a disjunction free formula such that $\mathbf{Int} \nvdash \varphi$. Then σL has the modal disjunction property and $\mathbf{Grz} \nvdash T(\varphi)$.

Let $\mathfrak{F}_0 = \langle W_0, R_0 \rangle$ be a finite frame for **Grz** having the form of binary tree and refuting $T(\varphi)$. Using the Criterion for the modal disjunction property, we shall construct a descriptive frame $\mathfrak{F} = \langle W, R, P \rangle$ for σL which is cofinally subreducible to $\mathfrak{F}_0$. Then, by Lemma 3, we shall have: $\mathfrak{F} \models \sigma L$, $\mathfrak{F} \not\models T(\varphi)$ and so $\sigma L \nvdash T(\varphi)$, which implies $L \nvdash \varphi$.

We construct $\mathfrak{F}$ by induction on the number of points in $\mathfrak{F}_0$. The case when $\mathfrak{F}_0$ consists of a single point is trivial: we can simply take $\mathfrak{F} = \mathfrak{F}_0$. Suppose now that a_0 is the least point (i.e., the root) in $\mathfrak{F}_0$ and $\mathfrak{F}_1 = \langle W_1, R_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2 \rangle$ are the two maximal subtrees of $\mathfrak{F}_0$ which do not contain a_0. By the induction hypothesis, we have two descriptive general frames $\mathfrak{F}^1 = \langle W^1, R^1, P^1 \rangle$ and $\mathfrak{F}^2 = \langle W^2, R^2, P^2 \rangle$ for σL such that there are cofinal subreductions f_1 of $\mathfrak{F}^1$ to $\mathfrak{F}_1$ and f_2 of $\mathfrak{F}^2$ to $\mathfrak{F}_2$. According to the criterion for the modal disjunction property, there is a rooted descriptive general frame $\mathfrak{F} = \langle W, R, P \rangle$ for σL, containing $\mathfrak{F}^1 + \mathfrak{F}^2$ as its generated subframe.

To define a cofinal subreduction f of $\mathfrak{F}$ to $\mathfrak{F}_0$, with each $a \in W_0$ we associate a set G_a which then will be mapped onto a by f. It should be clear that f will be a cofinal subreduction if the sets G_a, for all $a \in W_0$, satisfy the following conditions:

(i) $G_a \neq \varnothing$;

(ii) if $a, b \in W_0$ and not aR_0b then $G_b\!\downarrow \cap\, G_a = \varnothing$;

(iii) if aR_0b then $G_a \subseteq G_b\!\downarrow$;

(iv) $G_a \in P$;

(v) $W = \bigcup_{a \in W_0} G_a\!\downarrow$.

(The first four conditions guarantee that f will be a subreduction and the last one ensures its cofinality.)

We begin our construction of the sets G_a by observing that, for every $a \in W_1 \cup W_2$, there is a set $H_a \subseteq W$ such that

- $H_a \cap W^i = f_i^{-1}(a)$, for $a \in W_i$, $i = 1, 2$;
- $H_a \cap W^i = \varnothing$, for $a \notin W_i$, $i = 1, 2$;

- $H_a \in P$;
- $H_a \cap H_b = \varnothing$, for $a \neq b$.

Indeed, by the definitions of generated subframe and disjoint union and the fact that $f_i^{-1}(a) \in P^i \subseteq P$, for $i = 1, 2$, we have some sets H'_a satisfying the first three conditions. To satisfy the last one, we can take, for every $a \in W_0$,

$$H_a = H'_a - \bigcup_{b \neq a} H'_b.$$

Clearly $f_i^{-1}(a) \subseteq H_a$, since $f_i^{-1}(a) \cap f_j^{-1}(b) = \varnothing$ for $a \neq b$, and $H_a \in P$, since $\mathfrak{F}_0$ is finite.

So H_a is a good candidate to the role of G_a. However, it may contain points violating (i) – (v), and we should get rid of them.

Let us consider first only the sets H_a corresponding to the final points (i.e., the leaves) a in $\mathfrak{F}_0$. Removing from H_a those points that "see" some other set H_b (and so violate (ii), since not aR_0b), we obtain the set

$$U_a = H_a - \bigcup_{b \in W_1 \cup W_2, b \neq a} H_b{\downarrow} \in P,$$

for each final a. Notice that $f_i^{-1}(a) \subseteq U_a$ if $a \in W_i$, since otherwise xR^iy, for some $x \in f_i^{-1}(a)$ and $y \in H_b$ with $b \neq a$; but then $y \in W^i$ and so $y \in f_i^{-1}(b)$, which leads to a contradiction between aR_ib and a being final in $\mathfrak{F}_0$.

Next, removing from U_a those points x that "see" some $y \notin U_a{\downarrow}$, we obtain the set

$$V_a = U_a - (-(U_a{\downarrow})){\downarrow} \in P,$$

for each final a. Again we have $f_i^{-1}(a) \subseteq V_a$, for every final $a \in W_i$, since the subreduction f_i is cofinal and $\mathfrak{F}^1 + \mathfrak{F}^2$ is a generated subframe of $\mathfrak{F}$.

Now, in order to satisfy (v), we pick some final a in $\mathfrak{F}_0$ and add to V_a all those points that "see" no V_c, for all final c in $\mathfrak{F}_0$. Thus we arrive at

$$G_a = V_a \cup -\bigcup_{c \text{ is final in } \mathfrak{F}_0} V_c{\downarrow} \in P,$$

and we let

$$G_b = V_b \in P,$$

for the other final $b \neq a$. So we have:

- $f_i^{-1}(a) \subseteq G_a$, for every final $a \in W_i$, $i = 1, 2$;
- $G_b{\downarrow} \cap G_a = \varnothing$, for all the distinct final a and b;
- $W = \bigcup_{a \text{ is final in } \mathfrak{F}_0} G_a{\downarrow}$.

All technical difficulties are now behind. Suppose that $a \in W_1 \cup W_2$ and that we have already defined the sets G_b for all successors b of a. Then we let

$$G_a = H_a \cap \bigcap_{aR_0b, a \neq b} G_b{\downarrow} \cap - \bigcap_{c \text{ is final in } \mathfrak{F}_0,\ \neg aR_0c} G_c{\downarrow} \in P.$$

Notice that $f_i^{-1}(a) \subseteq G_a$, for $a \in W_i$, since aR_ib implies $f_i^{-1}(a) \subseteq f_i^{-1}(b){\downarrow}$ by the definition of subreduction.

And, finally, the last step: for the root a_0 in $\mathfrak{F}_0$ we let

$$G_{a_0} = \bigcap_{a_0 R_0 a} G_a{\downarrow} \in P.$$

$G_{a_0} \neq \varnothing$, since $\mathfrak{F}$ is a rooted frame.

It is clear that the sets G_a we have just constructed satisfy (i) – (v), and so the partial map f from W onto W_0, defined by

$$f(x) = a \text{ iff } x \in G_a,$$

is a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{F}_0$.

I conclude this paper with a few remarks concerning related results.

Remark 7 Theorem 6 proved above may be regarded as a syntactical necessary condition for the disjunction property of intermediate logics. Unfortunately, there are no syntactical *sufficient* conditions of equal generality and simplicity. A pair of rather general sufficient conditions was proved in Chagrov and Zakharyaschev [1], [4]. These conditions are imposed on the canonical formulas of Zakharyaschev [19], using which one can axiomatize all intermediate logics. Both of them cover a continuum of intermediate logics.

Remark 8 Difficulties in finding general effective necessary and sufficient conditions for the disjunction property of intermediate and modal logics turned out to be of principle nature: Chagrov and Zakharyaschev [1], [4] proved that there is no algorithm which is capable of deciding, given an intuitionistic or modal formula φ, whether the logic $\mathbf{Int} + \varphi$ or $\mathbf{Grz} + \varphi$ has the disjunction property.

Remark 9 The proof above would be much easier if every intermediate logic with the disjunction property was complete with respect to Kripke frames. (However, in this case we ought to adapt our proof either to the least modal “companion” of L, viz., $\tau L = \mathbf{S4} + \{T(\varphi) : L \vdash \varphi\}$, or to L itself, since, as was shown by Shehtman [15], σL may be incomplete even though L is complete. As for τL, it was proved by Zakharyaschev [18] that it is complete and/or has the modal disjunction property iff L is complete and/or has the disjunction property.) Minari [10] mentioned the question concerning completeness of intermediate logics with the disjunction property as an open problem. Unfortunately, the difficulties we have encountered in the proof above were unavoidable: in Appendix to Chagrov and Zakharyaschev [1] I gave a method for constructing an incomplete logic (calculus) with the disjunction property beginning with any incomplete intermediate logic (calculus) which can be axiomatized by negation free formulas. (Such a calculus can be found in Shehtman [14].) This method uses one of the sufficient conditions mentioned in (Remark 7) and the apparatus of the canonical formulas. In exactly the same way undecidable intermediate logics and calculi with the disjunction property can be constructed.

Remark 10 Theorem 6 has an interesting consequence concerning the *complexity function* $f_L(n)$ for a logic L, which is defined as follows:

$$f_L(n) = \max_{L \not\vdash \varphi, l(\varphi) \leq n} \min_{\mathfrak{F} \models L, \mathfrak{F} \not\models \varphi} |\,\mathfrak{F}\,|$$

where $l(\varphi)$ is the length of φ and $|\,\mathfrak{F}\,|$ the cardinality of $\mathfrak{F}$. Answering Kuznetsov's problem on the lower bound for $f_{\mathbf{Int}}(n)$, in Zakharyaschev and Popov [20] I constructed a sequence of disjunction free formulas φ_n such that

$$\min_{\mathfrak{F} \not\models \varphi_n} |\,\mathfrak{F}\,| \geq 2^n$$

and $l(\varphi_n)$ is a linear function of n. Thus, it follows from this result and Theorem 6 that, for each intermediate logic L with the disjunction property

$$f_L(n) \geq 2^{cn},$$

where c is a positive constant. This fact was first noted by Chagrov and Zakharyaschev [2].

Remark 11 For a more detailed discussion of the disjunction property of intermediate and modal propositional logics consult a review by Chagrov and Zakharyaschev [3].

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Mathematisches Institut
Universität Berlin
Arnimallee 3, D–14195 Berlin
Germany
email: mishaz@math.fu-berlin.de