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Well Ordered Subsets of Linearly Ordered Sets

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Abstract The deductive relationships between six statements are examined in set theory without the axiom of choice. Each of these statements follows from the axiom of choice and involves linear orderings in some way.

1 Introduction We consider the following six consequences of the axiom of choice:

- CF: Every linearly ordered set has a cofinal sub-well-ordering.
- LFC: If a linear order has the fixed point property then it is complete.
- DS: If a linear order has no infinite descending sequences then it is a well ordering.
- $LDF = F$: Every linearly orderable Dedekind finite set is finite.
- PDF: $\forall X$, if $\mathcal{P}(X)$ is Dedekind finite then every subset of $\mathcal{P}(X)$ which is linearly ordered by $\subseteq$ has a maximum element.
- $DF = F$: Every Dedekind finite set is finite.

Where the relevant definitions are

1. A partially ordered set $(A, \leq)$ has the *fixed point property* (fpp) if every function $f : A \to A$ which satisfies $(x \leq y \Rightarrow f(x) \leq f(y))$ has a fixed point.
2. A partially ordered set $(A, \leq)$ is *complete* if every subset of A has a least upper bound.
3. A set A is *Dedekind finite* if it has no countably infinite subsets.
4. If $(A, \leq)$ is a linearly ordered set, then $C \subseteq A$ is a *cofinal sub-well-ordering of* A if $\leq$ well orders C and

$$(\forall a \in A)(\exists c \in C)(a \leq c)$$

The statement $DF = F$ is the best known of these weak forms of the axiom of choice. Both Cantor and Dedekind asserted that it was *"true."* Other historical details can be found in [7]. The statement DS is frequently used in set theory with the axiom

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of choice to show that a set is well ordered [5]. The statement CF was studied by Sierpinski [9] and Manka [6]. Jech showed (see Problem 6.9 on page 95 of his [4]) that PDF is not provable in set theory without the axiom of choice. PDF was also studied in [3]. Hickman in [2] considered $LDF = F$.

We show that in the theory ZFU (Zermelo-Fraenkel set theory weakened to permit the existence of atoms) that

$$CF \Rightarrow LFC \Rightarrow DS \Rightarrow LDF = F \Rightarrow PDF$$

and that $DF = F$ implies $LDF = F$. Further the following implications are not provable in the theory ZFU:

$$LFC \Rightarrow CF,\ DS \Rightarrow LFC,\ DF{=}F \Rightarrow DS,$$

$$CF \Rightarrow DF = F,\ PDF \Rightarrow LDF = F.$$

Our results are summarized in the following diagram. Numbers refer to references, lemmas and theorems where the results are proved.

$$\begin{tikzcd} CF \arrow[r, shift left, "{[1]}"] & LFC \arrow[l, shift left, "/" marking, "3.3\ \&\ 3.4"] \arrow[r, shift left, "2.1"] & DS \arrow[l, shift left, "/" marking, "3.5"] \arrow[r, "2.2"] & LDF{=}F \arrow[r, shift left, "\text{trivial}"] & PDF \arrow[l, shift left, "/" marking, "3.6\text{–}3.9"] \end{tikzcd}$$

$$CF \underset{3.1}{\not\longrightarrow} DF{=}F \underset{3.2}{\not\longrightarrow} DS$$

$$DF{=}F \underset{\text{trivial}}{\longrightarrow} LDF{=}F$$

This decides, in the theory ZFU, whether or not $A \Rightarrow B$ is provable for every A and B chosen from our six consequences of AC.

Some of our independence results transfer to Zermelo-Fraenkel set theory (ZF) using the transfer theorems of Pincus. We can show that the implications

$$DF{=}F \Rightarrow DS,\ LDF{=}F \Rightarrow DF{=}F \text{ and } PDF \Rightarrow LDF{=}F$$

are not provable in ZF using these transfer theorems since the statements $DF{=}F$, $LDF{=}F$ and PDF are injectively boundable. We refer the reader to [8] for the details. Other independence results in ZF, except those that follow directly from the three above and known implications, are open problems. For example, we conjecture that $ZF \nvdash DS \to CF$ and $ZF \nvdash DS \to DF{=}F$.

2 *The Implications* The implication $CF \Rightarrow LFC$ is due to Davis [1] who proved LFC in the theory $ZF + AC$. An examination of the proof shows that only CF is needed. The implications $LDF{=}F \Rightarrow PDF$ and $DF{=}F \Longrightarrow LDF{=}F$ are clear. We prove:

Theorem 2.1 *LFC implies DS.*

Proof: Assume LFC and that $(A, \leq)$ is a chain with no infinite descending sequences. Let C be any non-empty subset of A, let $B = C \cup \{b_0\}$ where $b_0 \notin C$ and let $\leq^*$ be the linear ordering on B defined by $s \leq^* t$ if and only if $(s, t \in B \wedge s \leq t)$ or $t = b_0$. That is, $(B, \leq^*)$ is the linear ordering obtained from C by adding b_0 as largest element. $(B, \leq^*)$ has fpp otherwise the sequence $f(b_0)$, $f^{(2)}(b_0)$, $f^{(3)}(b_0), \ldots$ is an infinite descending sequence in $(A, \leq)$. By LFC, $(B, \leq^*)$ is complete. Therefore $(B, \leq^*)$ has a least element which must be a $\leq$-least element of C.

Theorem 2.2 *DS implies LDF=F.*

Proof: Assume DS and that $(A, \leq)$ is a linear order where A is Dedekind finite. Then $(A, \leq)$ has neither infinite descending sequences nor infinite ascending sequences. By DS, $(A, \leq)$ is a well order with no infinite ascending sequences which implies that A is finite.

3 *The Models*

In this section we construct several models of the theory ZFU for our independence results. Given a model M' of $ZFU + AC$ which has A as its set of atoms, a permutation model M of ZFA is determined by a group G of permutations of A and a filter Γ of subgroups of G which satisfies

$$(\forall a \in A)(\exists H \in \Gamma)(\forall \psi \in H)(\psi(a) = a)$$

and

$$(\forall \psi \in G)(\forall H \in \Gamma)(\psi H \psi^{-1} \in \Gamma).$$

Each permutation of A extends uniquely to a permutation of M' by $\in$ induction and for any $\psi \in G$ we identify ψ with its extension. If H is a subgroup of G and $x \in M'$ and $(\forall \psi \in H)(\psi(x) = x)$ we say H *fixes* x. We will also use the following notation: If $E \subseteq A$ and H is a subgroup of G then $\text{fix}_H(E)$ will denote $\{\psi \in H \mid (\forall a \in E)(\psi(a) = a)\}$.

The permutation model M determined by M', G and Γ consists of all those $x \in M'$ such that for every y in the transitive closure of x, there is some $H \in \Gamma$ such that H fixes y. We refer the reader to page 46 of [4] for a proof that M is a model of ZFA.

Theorem 3.1 $ZFU \nvdash CF \longrightarrow DF{=}F$.

Proof: For this argument, we use the basic Fraenkel model described in [4]. We describe this model briefly: A is a countable set, G is the group of all permutations of A and

$$\Gamma = \{H \mid (\exists E \subseteq A)(E \text{ is finite and } \text{fix}_G(E) \subseteq H\}.$$

In this model M, A is an infinite, Dedekind finite set (see problem 4 on page 52 of [4]). Therefore $DF{=}F$ is false. Also, in M, every linearly ordered set is well-orderable [3]. It follows easily from this that every linearly ordered set has a cofinal sub-well-ordering in M.

Theorem 3.2 $ZFU \nvdash DF{=}F \longrightarrow DS$.

Proof: Let M' be a model of $ZFU + AC$ with a countable set A of atoms ordered by $\leq$ so that $(A, \leq)$ has the order type of the rationals. Let G be the group of all order automorphisms of A and let Γ be the filter of subgroups of G generated by the support groups $\mathrm{fix}_G(E)$ where E ranges over subsets of A that satisfy the following three conditions:

1. The set E is well ordered by $\leq$.
2. The set E is bounded in the ordering $\leq$ on A.
3. If $b : \alpha \to E$ is an order preserving bijection from an ordinal α onto E and $\lambda \leq \alpha$ is a limit ordinal then $\{b(\gamma) : \gamma < \lambda\}$ has no least upper bound in the ordering $(A, \leq)$. (That is, if we identify $(A, \leq)$ with the rational numbers then the least upper bound of $\{b(\gamma) : \gamma < \lambda\}$ is irrational.)

(We note that any order preserving permutation of a well ordered set E must fix E pointwise hence $\mathrm{fix}_G(E) = \{\varphi \in G \mid \varphi \text{ fixes } E \text{ pointwise}\}$.) Since the union of two subsets of A satisfying (1), (2) and (3) also satisfies (1), (2) and (3), every group in the filter contains a group of the form $\mathrm{fix}_G(E)$ where E satisfies (1), (2) and (3). Therefore if we let M be the model determined by the filter Γ, for every $x \in M$ there is a subset E of A satisfying (1), (2) and (3) such that

$$(\forall\varphi \in \mathrm{fix}_G(E))(\varphi(x) = x).$$

When this happens we say E is a support of x.

We show that DS is false in M by showing (by contradiction) that $(A, \leq)$ has no infinite descending sequences in M: Assume that E is a support of an infinite descending sequence $\{(n, a_n) \mid n \in \omega\}$ of elements of A. Then

(*) $(\forall\varphi \in \mathrm{fix}_G(E))(\forall n \in \omega)(\varphi(a_n) = a_n)$.

Since E is well-ordered by $\leq$ there is at least one $i \in \omega$ such that $a_i \notin E$. Condition (3) insures that there are two elements e_1 and e_2 of A such that a_i is in the open interval (e_1, e_2) and $(e_1, e_2) \cap E = \varnothing$. We can now obtain a one to one, order preserving function from (e_1, e_2) onto (e_1, e_2) which moves a_i (since (e_1, e_2) is order isomorphic to the rationals). This can be extended to a permutation φ of A which fixes $A - (e_1, e_2)$ pointwise. The permuation φ therefore moves a_i and fixes E, contradicting $(*)$.

DS is false in M since $(A, \leq)$ is an infinite linear order with no infinite descending sequences in M.

We will show $DF{=}F$ in M by showing that every infinite set in M has an infinite subset which is well-orderable in M. Assume X in an infinite set in M. Since X is in M, there is some subset E of A such that for every φ in $\mathrm{fix}_G(E)$, $\varphi(X) = X$ and such that E satisfies (1), (2) and (3). If $\mathrm{fix}_G(E)$ fixes every element of X, then X is well orderable in M and we are done. We may therefore assume that there is a $y \in X$ and a permutation $\varphi_0 \in \mathrm{fix}_G(E)$ such that $\varphi_0(y) \neq$ y. Suppose E' is a support of y such that $E \subseteq E'$ and let $F = E' - E$.

As in (3) assume b is an order preserving bijection from some ordinal α onto E. We will use the ordinals $\leq \alpha$ to index the intervals in $(A, \leq)$ determined by the set E as follows: For $0 < \eta < \alpha$,

$$I_\eta = \{a \in A : (\forall\beta < \eta)(b(\beta) < a \wedge a < b(\eta))\}.$$

And

$$I_\alpha = \{a \in A : (\forall \beta < \alpha)(b(\beta) < a)\}.$$

Each I_η is an interval (in the sense that if $a_1 < a_2 < a_3$ and a_1 and a_3 are in I_η then a_2 is in I_η) and the set of intervals $\{I_\eta : 0 \leq \eta \leq \alpha\}$ is a partition of A–E. Further by properties (1), (2) and (3), each of the sets I_η is non-empty.

Temporarily fix η, $0 \leq \eta \leq \alpha$, and let F_η be the intersection of the support of y with I_η. That is, $F_\eta = I_\eta \cap$ F. Since φ_0 is in $\operatorname{fix}_G(E)$, $\varphi_0(F_\eta) \subseteq I_\eta$. We claim that there are two elements s_η and t_η of I_η such that $F_\eta \cup \varphi_0(F_\eta) \subseteq (s_\eta, t_\eta)$. (Here (s_η, t_η) denotes the open interval in the ordering $(A, \leq)$.) The argument, which is omitted, uses the fact that E and E' satisfy property (3), the fact that the order $(A, \leq)$ is dense and (if $\eta = \alpha$) property (2) of E'. Let (a_0^η, c_0^η), (a_1^η, c_1^η), $\ldots$ be a sequence of open intervals in the ordering $(A, \leq)$, each contained in I_η and chosen so that

$$t_\eta < a_0^\eta < c_0^\eta < a_1^\eta < c_1^\eta < \ldots < b(\eta)$$

and so that the set $\{a_0^\eta, c_0^\eta, a_1^\eta, c_1^\eta, \ldots\}$ has no least upper bound in $(A, \leq)$ (and in addition if $\eta = \alpha$ we require that $\{a_0^\eta, c_0^\eta, a_1^\eta, c_1^\eta, \ldots\}$ be bounded). Finally, for each $i \in \omega$, let ψ_i^η be an element of $\operatorname{fix}_G(E)$ such that ψ_i^η fixes $A - I_\eta$ pointwise, and $\psi_i^\eta((s_\eta, t_\eta)) = (a_i^\eta, c_i^\eta)$.

Now we combine the permutations ψ_i^η, $0 \leq \eta \leq \alpha$, for each $i \in \omega$: For each $i \in \omega$, let ψ_i be the element of $\operatorname{fix}_G(E)$ that agrees with ψ_i^η on I_η for all η, $0 \leq \eta \leq \alpha$. The permutation ψ_i thus defined is in $\operatorname{fix}_G(E)$ and therefore fixes X. Hence $\psi_i(y) \in X$ for each $i \in \omega$. Further $\psi_i(F) \cup E$ is a support of $\psi_i(y)$ and

$$\psi_i(F) \subseteq \bigcup_{\eta \in \alpha+1} (a_i^\eta, c_i^\eta).$$

We also claim:

- A: $\bigcup_{i \in \omega} \psi_i(F)$ satisfies (1), (2) and (3). (From which it follows that the set $\{\psi_i(y) : i \in \omega\}$ is well orderable in M.)
- B: For all $i, j \in \omega, i \neq j$ implies $\psi_i(y) \neq \psi_j(y)$.(From which it follows that $\{\psi_i(y) : i \in \omega\}$ is infinite.)

We outline the proof of B: There is an element φ^* of $\operatorname{fix}_G(E)$ such that φ^* agrees with φ_0 on F and such that φ^* is the identity outside of $\bigcup_{\eta \in \alpha+1} (s_\eta, t_\eta)$. This uses the denseness of the ordering on A and the fact that F and $\varphi_0(F)$ are both subsets of $\bigcup_{\eta \in \alpha+1} (s_\eta, t_\eta)$.

Therefore $\varphi^*(y) = \varphi_0(y) \neq y$. For each $i \in \omega$, let

$$\varphi_i^* = \psi_i \circ \varphi^* \circ \psi_i^{-1}$$

then φ_i^* is the identity outside of $\bigcup_{\eta \in \alpha+1} (a_i^\eta, c_i^\eta)$. Therefore for $j \neq i$, φ_i^* restricted to $\bigcup_{\eta \in \alpha+1} (a_j^\eta, c_j^\eta)$ is the identity. So for $j \neq i$, φ_i^* fixes the support $\psi_j(F) \cup E$ of $\psi_j(y)$ pointwise and hence fixes $\psi_j(y)$. On the other hand

$$\varphi_i^*(\psi_i(y)) = \psi_i \varphi^* \psi_i^{-1} \psi_i(y) = \psi_i \varphi^*(y) \neq \psi_i(y)$$

since $\varphi^*(y) \neq y$. Since φ_i^* moves $\psi_i(y)$ and fixes $\psi_j(y)$ we conclude that $\psi_i(y) \neq \psi_j(y)$.

Theorem 3.3 $ZFU \nvdash LFC \longrightarrow CF$.

Proof: Let M' be a model of $ZFU + AC$ with a countable set A of atoms. For the construction of the model M we will assume that $a : \omega \times \mathbf{Z} \longrightarrow A$ is one to one and onto so that

$$A = \{ a(i, j) \mid i \in \omega \wedge j \in \mathbf{Z} \}$$

where ω is the set of natural numbers $\{0, 1, 2, \ldots\}$ and $\mathbf{Z}$ is the set of integers.

For each $i \in \omega$ let $\psi_i : A \longrightarrow A$ be the permutation defined by $\psi_i(a(i, j)) = a(i, j+1)$ and $\psi_i(a(k, j) = a(k, j)$ for $k \neq i$ and let G be the group of permutations generated by $\{ \psi_i \mid i \in \omega \}$. We note that each $\eta \in G$ is an order automorphism of $(A, \leq)$ where $\leq$ is the ordering on A induced by the lexicographic ordering on $\omega \times \mathbf{Z}$, that is $a(i, j) \leq a(m, n)$ if and only if $i < m$ or $(i = m$ and $j \leq n)$. For each finite subset $E \subseteq \omega$ we let $G_E = \{ \psi \in G \mid (\forall i \in E)(\forall k \in \mathbf{Z})(\psi(a(i, k)) = a(i, k)) \}$. Γ is the filter of subgroups of G generated by the groups G_E where E ranges over the finite subsets of ω. M is the permutation model determined by G and Γ.

The linear ordering $(A, \leq)$ defined above is in M since it is fixed by G. It is also the case that $(A, \leq)$ has no cofinal-sub-wellordering in M since no $H \in \Gamma$ fixes a cofinal subset of $(A, \leq)$ pointwise. Therefore CF is false in M.

We now argue that LFC is true in M. First note that the linear ordering $(A, \leq)$ does not have the fixed point property in M since the function f defined by $f(a(i, j)) = a(i, j+1)$ is order preserving, has no fixed points and is fixed by G and is therefore in M.

Now let $(C, \preceq)$ be any linear ordering in M. We will assume that $(C, \preceq)$ is not complete in M and construct a fixed point free order preserving function from $(C, \preceq)$ into $(C, \preceq)$ which is in M. Since $(C, \preceq)$ is not complete there is some subset $B \subseteq C$ with $B \in M$ and such that B has no least upper bound. We assume without loss of generality that B is closed downward (i.e., $(\forall c \in C)\,((\exists b \in B)(c \preceq b) \rightarrow c \in B)$.) It follows that if we let $D = C - B$, then $C = B \cup D$, $(\forall b \in B)(\forall d \in D)(b \preceq d)$, B has no least upper bound and D has no greatest lower bound. Let E be a finite subset of ω such that for all $\varphi \in G_E$, φ fixes $(C, \preceq)$, B and D.

Lemma 3.4 *M contains a fixed point free order preserving function on B and a fixed point free order preserving function on D.*

Proof: We will prove the lemma for B. The proof for D is similar.

We first partition B into two sets:

$$B_F = \{ b \in B \mid (\forall \varphi \in G_E)(\varphi(b) = b) \}$$

and

$$B_M = \{ b \in B \mid (\exists \varphi \in G_E)(\varphi(b) \neq b \}.$$

Case 1: B_F is cofinal in $(B, \preceq)$. In this case, since B_F is well-orderable in M (not necessarily by $\preceq$), $(B_F, \preceq)$ has a cofinal sub-well-ordering without greatest element which we call $(B'_F, \preceq)$. In this case the function $f : B \rightarrow B$ defined by

$$f(b) = \textit{the least element of } B'_F \textit{ which is} \succ b$$

is a fixed point free order preserving function on B.

Case 2: For some b_0 in B_M, $(\forall b \in B)(b_0 \preceq b \to b \in B_M)$. We first note that for each $b \in B_M$, only finitely many ψ_i, i in $\omega - E$, move b. (If E' is a support of b then no ψ_i for $i \notin E'$ moves b.) Choose, for each $b \in B_M$, a permutation η_b from the set

$$\{\, \psi_i \mid i \in \omega - E \,\} \cup \left\{\, \psi_i^{-1} \mid i \in \omega - E \right\}$$

so that $\eta_b(b)$ is as large as possible in the ordering $\preceq$ on B.

Now we claim that if $b \in B_M$ and $\varphi \in \mathrm{fix}_G(E)$, then $\eta_b(\varphi(b)) = \eta_{\varphi(b)}(\varphi(b))$. It is clear that $\eta_b(\varphi(b)) \preceq \eta_{\varphi(b)}(\varphi(b))$. Suppose that $\eta_b(\varphi(b)) \prec \eta_{\varphi(b)}(\varphi(b))$, then $\varphi(\eta_b(b)) \prec \varphi(\eta_{\varphi(b)}(b))$ since G is abelian. Therefore $\eta_b(b) \prec \eta_{\varphi(b)}(b)$ which contradicts our choice of η_b. This proves the claim.

Now we define a function $g : B \to B$ by

$$g(b) = \begin{cases} b & \text{if } b \in B_F \\ \eta_b(b) & \text{if } b \in B_M \end{cases}$$

Note that for $b \in B_M$, $b \prec g(b)$ since some ψ_i must move b and if $\psi_i(b) \prec b$ then $b \prec \psi_i^{-1}(b)$.

We show that $g \in M$ by showing that for all $\varphi \in \mathrm{fix}_G(E)$ and for all $b \in B$ that $\varphi(g(b)) = g(\varphi(b))$. This is clear if $b \in B_F$ since for such b, $b = \varphi(b) = g(b)$. If $b \in B_M$ then:

$$\varphi(g(b)) = \varphi(\eta_b(b)) = \eta_b(\varphi(b)) = \eta_{\varphi(b)}(\varphi(b)) = g(\varphi(b))$$

where the second to last equality uses the claim proved above.

We show g is order preserving on $(B, \preceq)$. Assume that $b_1, b_2 \in B$ and that $b_1 \prec b_2$. If b_1 and b_2 are both in B_F, then $g(b_1) = b_1 \prec b_2 = g(b_2)$. If $b_1 \in B_F$ and $b_2 \in B_M$ then $g(b_1) = b_1 = \eta_{b_2}(b_1) \prec \eta_{b_2}(b_2) = g(b_2)$. Similarly if $b_1 \in B_M$ and $b_2 \in B_F$, $g(b_1) \prec g(b_2)$. If b_1 and b_2 are both in B_M then $g(b_1) = \eta_{b_1}(b_1) \prec \eta_{b_1}(b_2) \preceq \eta_{b_2}(b_2)$. The function g has fixed points if $B_F \neq \varnothing$. To get the fixed point free, order preserving function f on $(B, \preceq)$ we define f by

$$f(b) = \begin{cases} b_0 & \text{if } b \prec b_0 \\ g(b) & \text{if } b \succeq b_0 \end{cases}$$

It follows from our assumption $\{\, b \in B \mid b_0 \preceq b \,\} \subseteq B_M$ and the fact that for $b \in B_M$, $b \prec g(b)$ that f is fixed point free. It also follows, since g is order preserving, that f is order preserving. Finally, f is in M since it is definable from g, $(B, \preceq)$ and b_0 all of which are in M. This completes the proof of the lemma.

The proof of Theorem 3.3 is completed by combining the fixed point free order preserving functions on B and D to get a fixed point free order preserving function on C.

Theorem 3.5 $ZFU \nvdash DS \longrightarrow LFC.$

Proof: Let M' be a model of $ZFU + AC$ with a set of atoms A and an ordering $\leq$ on A such that $(A, \leq)$ is order isomorphic to the real numbers with their usual ordering. Let G be the group of all order automorphisms of $(A, \leq)$ and let

$$\Gamma = \{\, H \mid H \text{ is a subgroup of } G \wedge (\exists E \subseteq A)(E \text{ bounded } \wedge \mathrm{fix}_G(E) \subseteq H \,\}.$$

M is the model determined by M', G and Γ. If $z \in M$ then there is some bounded $E \subseteq A$ such that for all $\varphi \in \text{fix}_G(E)$, $\varphi(z) = z$ and as in the proof of Theorem 3.2 we will call such an E a *support* of z.

We first argue that in M, $(A, \leq)$ is a witness to the failure of LFC. The linear ordering $(A, \leq)$ is clearly not complete since A has no largest element. To show that $(A, \leq)$ has the fixed point property in M assume that $f : A \to A$ is an order preserving map on A which is in M. Suppose that f has support $E \subseteq A$. We may assume that $E = [a, b]$—some closed bounded interval in the ordering $(A, \leq)$. If f has a fixed point we are done. Otherwise for every $x \in A$, $f(x)$ must be in $[a, b]$. (If $f(x) \notin [a, b]$ then, since $x \neq f(x)$, there would be an element φ of $\text{fix}_G([a, b])$ such that $\varphi(x) = x$ but $\varphi(f(x)) \neq f(x)$. This would mean $\varphi(f) \neq f$ contradicting our choice of $[a, b]$ as a support of f.) This means that $f|[a, b] : [a, b] \to [a, b]$. Since $([a, b], \leq)$ is a complete linear ordering in M' where AC holds, $f|[a, b]$ has a fixed point which is also a fixed point of f in M.

To argue that DS is true in M, let $(X, \preceq)$ be a linear ordering in M which is not a well ordering. We will show that in M there is a sequence $\langle y_i \rangle_{i\in\omega}$ of elements of X such that $(\forall i \in \omega)(y_{i+1} \prec y_i)$. Let $E = [a, b]$ be a support of $(X, \preceq)$. Choose c and d in A so that $c < a < b < d$. Our plan is to find an infinite descending sequence $y_0 \succ y_1 \succ y_2 \cdots$ of elements of X such that each y_i has a support contained in $[c, d]$ (from which it will follow that the sequence $\langle y_i \rangle_{i\in\omega}$ is in the model M.) More specifically, let $\langle s_i \rangle_{i\in\omega}$ and $\langle t_i \rangle_{i\in\omega}$ be two sequences of elements of A satisfying

$$c < \cdots < s_2 < s_1 < s_0 < a < b < t_0 < t_1 < t_2 < \cdots < d.$$

We will construct y_i so that it has support $[s_i, t_i]$.

The construction is by induction on the subscript i. To construct y_0, choose any element $z_0 \in X$ and assume that $[s'_0, t'_0]$ is a support of z_0 which contains $[c, d]$. There is a φ_0 in $\text{fix}_G([a, b])$ such that $\varphi_0([s'_0, t'_0]) = [s_0, t_0]$ and we let $y_0 = \varphi_0(z_0)$. It follows that $[s_0, t_0]$ is a support of y_0.

Assume that y_i has been defined satisfying $y_i \in X$, y_i has support $[s_i, t_i]$ and $y_i \prec y_j$ for all $j \in \omega$, $j < i$. The element y_i is not least in X therefore we can choose an element $z_{i+1} \in X$ such that $z_{i+1} \prec y_i$. Assume that $[s'_{i+1}, t'_{i+1}]$ is a support of z_{i+1} containing [c,d]. There is a $\varphi_{i+1} \in \text{fix}_G([s_i, t_i])$ such that $\varphi_{i+1}([s'_{i+1}, t'_{i+1}]) = [s_{i+1}, t_{i+1}]$ and we let $y_{i+1} = \varphi_{i+1}(z_{i+1})$. Clearly $y_{i+1} \in X$ and has support $[s_{i+1}, t_{i+1}]$. Further

$$y_{i+1} = \varphi_{i+1}(z_{i+1}) \prec \varphi_{i+1}(y_i) = y_i$$

where the middle inequality holds because φ_{i+1} fixes $[a, b]$ pointwise and therefore fixes $\preceq$. This completes the proof of Theorem 3.5.

Theorem 3.6 $ZFU \nvdash PDF \longrightarrow LDF{=}F$.

Proof: For the construction of the permutation model we begin with a model M' of $ZFU + AC$ with a countable set A of atoms and an ordering $\leq$ of A so that $(A, \leq)$ has the same order type as that of the rational numbers. We assume that A is the disjoint union $A = D_1 \cup D_2 \cup D_3$ of three dense subsets D_1, D_2 and D_3. We let G be the group of all order automorphisms φ of A such that $\varphi(D_i){=}D_i$, $i = 1, 2, 3$. The argument we give below will require the existence of several types of permutations in G. For example:

Lemma 3.7

A: *If $E_1, F_1 \subseteq D_1$, $E_2, F_2 \subseteq D_2$, $E_3, F_3 \subseteq D_3$, E_i and F_i are finite for $i = 1, 2$ and 3 and $\sigma : (E_1 \cup E_2 \cup E_3) \to (F_1 \cup F_2 \cup F_3)$ is one to one, onto, order preserving and satisfies $\sigma(E_1) = F_1$, $\sigma(E_2) = F_2$ and $\sigma(E_3) = F_3$ then there is a $\varphi \in G$ such that $\varphi|(E_1 \cup E_2 \cup E_3) = \sigma$.*

B: *If (a, b) is an interval in $(A, \leq)$ and ψ is a permutation in G which satisfies $(a, b) \cup \psi((a, b)) \subseteq (u, v)$ and if $s_1 < u < v < s_2$ then there is a $\psi' \in fix_G(A - (s_1, s_2))$ which agrees with ψ on (a, b).*

Lemmas of this type can be proved by the back and forth construction used to prove that any two countable dense linear orderings without first and last element are order isomorphic.

We will call a subset E of A a *support* if it satisfies the following conditions:

1. $E \cap D_1$ is finite.
2. $E \cap D_2$ is well ordered by $\leq$
3. If $b : \alpha \to E \cap D_2$ is an order preserving bijection from an ordinal α onto $E \cap D_2$ and $\lambda \leq \alpha$ is a limit ordinal then the least upper bound of $\{ b(\gamma) \mid \gamma < \lambda \}$ in $(A, \leq)$ exists and is in D_3.

Γ is the filter of subgroups H of G such that for some support E, $\text{fix}_G(E) \subseteq H$ and M is the permutation model determined by M', G and Γ.

If E is a support then for every $t \in D_1$, as long as t is not in the finite set $E \cap D_1$, there is a $\varphi \in \text{fix}_G(E)$ such that $\varphi(t) \neq t$. It follows that no well ordering of an infinite subset of D_1 is in M. Therefore $(D_1, \leq)$ is a linearly ordered, Dedekind finite, infinite set in M hence $LDF{=}F$ is false in M.

Now let X be any non-empty set in M. We will show that if $\mathcal{P}(X)$ is infinite in M then $\mathcal{P}(X)$ is Dedekind infinite in M from which it follows that PDF is true in M.

Assume that $\mathcal{P}(X)$ is infinite. It follows that X must be infinite. If X is well-orderable in M then $\mathcal{P}(X)$ is Dedekind infinite and we are done. We therefore assume that X is not well-orderable in M. Let E be a support of X.

Lemma 3.8 *There is a subset $Y \subseteq X$ such that*

1. $(\exists \psi \in fix_G(E))(\psi(Y) \neq Y)$
2. *Y has a support E' such that $E' - E \subseteq D_2$.*

Proof: Since X is not well-orderable in M there is an element $t \in X$ such that $\exists \varphi \in \text{fix}_G(E)$ with $\varphi(t) \neq t$. Assume that t has support $H' \supseteq E$ and let $H = H' - E$. Let $H \cap D_1 = \{d_1, d_2, \ldots, d_n\}$ and suppose H' is chosen so that $H \cap D_1$ has minimum cardinality.

If $H \cap D_1 = \varnothing$, then taking $Y = \{t\}$ and $E' = H'$ satisfies the lemma. If $H \cap D_1 \neq \varnothing$ then (by minimality) $\exists \varphi' \in \text{fix}_G(E)$ such that $\varphi' \in \text{fix}_G(H \cap D_2)$ and $\varphi'(t) \neq t$. Let b be a bijection for an ordinal α onto $(H \cap D_2) \cup E$ so that $(H \cap D_2) \cup E = \{ b(\gamma) \mid \gamma < \alpha \}$. This is possible by condition (2) in the definition of support. For each $\gamma < \alpha$ let

$$I_\gamma = \left\{ a \in A \mid (\forall \beta < \gamma)(c_\beta < a < c_\gamma \right\}$$

and let

$$I_\alpha = \left\{ a \in A \mid (\forall \gamma < \alpha)(c_\gamma < a) \right\}.$$

The I_γ for $\gamma \leq \alpha$ are the open intervals in $(A, \leq)$ determinted by the set $(H \cap D_2) \cup E$ and therefore

$$A - ((H \cap D_2) \cup E) = \bigcup_{\gamma \leq \alpha} I_\gamma.$$

It follows that $H \cap D_1 \subseteq \left(\bigcup_{\gamma \leq \alpha} I_\gamma\right)$. We also note that $\varphi'(I_\gamma) = I_\gamma$.

Since $H \cap D_1$ is finite there are only finitely many $\gamma \leq \alpha$ for which $H \cap D_1 \cap I_\gamma \neq \varnothing$. For each such γ we carry out the following construction. Let $H \cap D_1 \cap I_\gamma = \left\{ d_1^\gamma, \ldots, d_{k_\gamma}^\gamma \right\}$ then for each i, $i = 1, 2, \ldots k_\gamma$, $\varphi'(d_i^\gamma) \in I_\gamma \cap D_1$. Choose $s_1^\gamma < s_2^\gamma$ in $I_\gamma \cap D_2$ so that for $i = 1, 2, \ldots, k_\gamma$, $s_1^\gamma < d_i^\gamma < s_2^\gamma$ and $s_1^\gamma < \varphi'(d_i^\gamma) < s_2^\gamma$. Using Lemma 3.7, we now choose a $\psi_\gamma \in \mathrm{fix}_G(A - I_\gamma)$ such that for $i = 1, 2, \ldots, k_\gamma$, $s_2^\gamma < \psi_\gamma(s_1^\gamma)$. This will insure that

$$\psi_\gamma^{-1}(s_1^\gamma) < \psi_\gamma^{-1}(s_2^\gamma) < s_1^\gamma < s_2^\gamma < \psi_\gamma(s_1^\gamma) < \psi_\gamma(s_2^\gamma)$$

and that for $i = 1, 2, \ldots, k_\gamma$,

$$\psi_\gamma^{-1}(s_1^\gamma) < \psi_\gamma^{-1}(d_i^\gamma) < \psi_\gamma^{-1}(s_2^\gamma), \;\; \psi_\gamma^{-1}(s_1^\gamma) < \psi_\gamma^{-1}(\varphi'(d_i^\gamma)) < \psi_\gamma^{-1}(s_2^\gamma)$$

$$\psi_\gamma(s_1^\gamma) < \psi_\gamma(d_i^\gamma) < \psi_\gamma(s_2^\gamma) \text{ and } \psi_\gamma(s_1^\gamma) < \varphi'(d_i^\gamma) < \psi_\gamma(s_2^\gamma).$$

By Lemma 3.7, there is a $\varphi''_\gamma \in \mathrm{fix}_G\left(A - \left(s_1^\gamma, s_2^\gamma\right)\right)$ such that for $i = 1, 2, \ldots, k_\gamma$, $\varphi''_\gamma(d_i^\gamma) = \varphi'(d_i^\gamma)$, where $\left(s_1^\gamma, s_2^\gamma\right)$ denotes the interval in $(A, \leq)$.

The above construction was carried out for γ for which $I_\gamma \cap H \cap D_1 \neq \varnothing$. Now we let

$$E' = (H \cap D_2) \cup E \cup \{ s_j^\gamma \mid I_\gamma \cap H \cap D_1 \neq \varnothing, \, j = 1 \text{ or } 2 \}$$

and let $Y = \left\{ \sigma(t) \mid \sigma \in \mathrm{fix}_G\left(E'\right) \right\}$. We claim that Y and E' satisfy the requirements of the lemma. Clearly $E' - E \subseteq D_2$. Also if $\eta \in \mathrm{fix}_G(E')$ then $\eta^{-1} \in \mathrm{fix}_G(E')$ so that both $\eta(Y) \subseteq Y$ and $\eta^{-1}(Y) \subseteq Y$. It follows from the second inclusion that $Y \subseteq \eta(Y)$ and we therefore can conclude that $Y = \eta(Y)$. This shows that E' is a support of Y and hence condition (2) of Lemma 3.8 is satisfied.

We must now show that there is a $\psi \in \mathrm{fix}_G(E)$ such that $\psi(Y) \neq Y$. Let ψ be the composition of the permutations ψ_γ such that $I_\gamma \cap H \cap D_1 \neq \varnothing$. (There are finitely many such ψ_γ and they move disjoint sets so the order in which they are composed does not matter.) Since each $\psi_\gamma \in \mathrm{fix}_G(E)$, $\psi \in \mathrm{fix}_G(E)$. Similarly, let φ'' be the composition of the permutations φ''_γ defined above for ordinals γ such that $I_\gamma \cap H \cap D_1 \neq \varnothing$. Since φ'' and φ' agree on H' (a support of t), $\varphi''(t) = \varphi'(t) \neq t$. We will show that $\psi(Y) \neq Y$ by showing that $\psi(t) \notin Y$.

By our definition of Y this amounts to showing that for every $\sigma \in \mathrm{fix}_G(E')$, $\sigma(t) \neq \psi(t)$. Assume that $\sigma \in \mathrm{fix}_G(E')$. We will show that $\sigma(t) \neq \psi(t)$ by showing that the permutation $\psi\varphi''\psi^{-1}$ moves $\psi(t)$ but fixes $\sigma(t)$. The first part we prove by contradiction: Assume $\psi\varphi''\psi^{-1}(\psi(t)) = \psi(t)$, it follows that $\varphi''(t) = t$ which we have shown to be false. For the argument that $\psi\varphi''\psi^{-1}(\sigma(t)) = \sigma(t)$ we note that for each γ such that $I_\gamma \cap H \cap D_1 \neq \varnothing$, $\left\{ d_1^\gamma, \ldots, d_{k_\gamma}^\gamma \right\}$ is a subset of the interval $\left(s_1^\gamma, s_2^\gamma\right)$

and σ fixes both s_1^γ and s_2^γ. Therefore $\left\{\sigma(d_1^\gamma), \dots, \sigma(d_{k_\gamma}^\gamma)\right\} \subseteq \left(s_1^\gamma, s_2^\gamma\right)$. We conclude that $\sigma(H) \subseteq \bigcup \left\{ \left(s_1^\gamma, s_2^\gamma\right) \mid I_\gamma \cap H \cap D_1 \neq \varnothing \right\}$. Since $\sigma(H) \cup E$ is a support of $\sigma(t)$, any permutation in $\mathrm{fix}_G(E)$ that fixes $\bigcup \left\{ \left(s_1^\gamma, s_2^\gamma\right) \mid I_\gamma \cap H \cap D_1 \neq \varnothing \right\}$ pointwise, fixes $\sigma(t)$. But for each γ such that $I_\gamma \cap H \cap D_1 \neq \varnothing$, φ'' fixes $\left(\psi_\gamma^{-1}(s_1^\gamma), \psi_\gamma^{-1}(s_2^\gamma)\right)$ and therefore $\psi\varphi''\psi^{-1}$ fixes $\left(s_1^\gamma, s_2^\gamma\right)$. This completes the proof of Lemma 3.8.

Lemma 3.9 *If X has a subset satisfying the conditions of Lemma 3.8 then $\mathcal{P}(X)$ is Dedekind infinite.*

Proof: Assume $Y \subseteq X$ satisfies conditions (1) and (2) of Lemma 3.8 and let $F = E' - E \subseteq D_2$. As in the proof of Lemma 3.8 we assume that b is an order preserving bijection from an ordinal α onto E. Then $E = \{b(\gamma) \mid \gamma < \alpha\}$. We also define I_γ for $\gamma \leq \alpha$ as in the proof of Lemma 3.8. For each $\alpha < \gamma$, I_γ is an interval with right endpoint $b(\gamma) \in E$ and left endpoint in $E \cup D_3$. (We denote the left endpoint of I_γ by $b^-(\gamma)$.) It follows that $F \subseteq \bigcup_{\gamma \leq \alpha} I_\gamma$.

Fix $\gamma \leq \alpha$. By our assumption there are elements s_1^γ and s_2^γ of D_2 such that

$$b^-(\gamma) < s_1^\gamma < a < s_2^\gamma < b(\gamma)$$

for all $a \in F \cap I_\gamma$. (The set $F \cap I_\gamma$ has a least element by (2) in the definition of support and if $F \cap I_\gamma$ has no greatest element then by (3) in the definition of support, the least upper bound of $F \cap I_\gamma$ is in D_3 and is therefore $< b(\gamma)$.) In addition (and for similar reasons) we may assume that $s_1^\gamma < a < s_2^\gamma$ for all $a \in \psi(F) \cap I_\gamma$.

By Lemma 3.7 B there is a permutation $\psi' \in \mathrm{fix}_G(E)$ such that

$$\psi' \in \mathrm{fix}_G \left(A - \bigcup_{\gamma \leq \alpha} \left(s_1^\gamma, s_2^\gamma\right) \right)$$

and $\psi'(a) = \psi(a)$ for all $a \in F \cap \left(\bigcup_{\gamma \leq \alpha} I_\gamma\right)$. Since ψ and ψ' agree on a support of Y, we have $\psi'(Y) = \psi(Y) \neq Y$.

For each $\gamma \leq \alpha$ choose a sequence of intervals $\left\langle\left(r_i^\gamma, q_i^\gamma\right)\right\rangle_{i \in \omega}$ in the ordering $(A, \leq)$ and a point $t_\gamma \in A$ so that

$$s_2^\gamma < r_1^\gamma < q_1^\gamma < r_2^\gamma < q_2^\gamma < \cdots < t_\gamma < b(\gamma) \tag{1}$$

$$\sup\left\{r_i^\gamma \mid i \in \omega\right\} = sup\left\{q_i^\gamma \mid i \in \omega\right\} = t_\gamma \tag{2}$$

$$r_i^\gamma \text{ and } q_i^\gamma \in D_2 \text{ for } i \in \omega \tag{3}$$

$$t_\gamma \in D_3 \tag{4}$$

By Lemma 3.7 A, for each $i \in \omega$ there is a permutation $\eta_i^\gamma \in \mathrm{fix}_G\left(A - I_\gamma\right)$ such that $\eta_i^\gamma(s_1^\gamma) = r_i^\gamma$, $\eta_i^\gamma(s_2^\gamma) = q_i^\gamma$ and η_i^γ fixes the interval $\left[r_{i+1}, b(\gamma)\right)$ pointwise.

For each $i \in \omega$ let η_i be the composition of the permutations η_i^γ for $\gamma \leq \alpha$. Since for each $\gamma \leq \alpha$, η_i^γ is the identity outside of I_γ, we have $\eta_i(x) = \eta_i^\gamma(x)$ for all x in

I_γ. Let $Y_i = \eta_i(Y)$. Since η_i fixes X and $Y \in \mathcal{P}(X)$, we have $Y_i \in \mathcal{P}(X)$. Further, since Y has support $E \cup F$ and η_i fixes E pointwise, Y_i has support $E \cup \eta_i(F)$. We will complete the proof of Lemma 3.9 by proving the following two assertions:

$$(\forall i, j \in \omega)\left(i \neq j \;\;\rightarrow\;\; Y_i \neq Y_j\right) \tag{5}$$

$$\bigcup_{i\in\omega} (E \cup \eta_i(F)) \text{ is a support.} \tag{6}$$

From (5) it follows that $\{ Y_i \mid i \in \omega \}$ is infinite and by (6) it follows that $\{ Y_i \mid i \in \omega\}$ is in M and is well orderable in M.

For the proof of (5), assume $i, j \in \omega$ and that $i < j$. The permutation $\eta_i \psi' \eta_i^{-1}$ fixes $Y_j = \eta_j(Y)$ since the support $E \cup \eta_i(F)$ of Y_j is contained in $E \cup \left(\bigcup_{\gamma\leq\alpha} \left[r^\gamma_{i+1}, b(\gamma)\right)\right)$ which ψ' and η_i both fix pointwise. On the other hand, the equation $\eta_i\psi'\eta_i^{-1}(Y_i) = Y_i$ is equivalent to

$$\eta_i\psi'\eta_i^{-1}(\eta_i(Y)) = \eta_i(Y)$$

which in turn implies the contradiction $\psi'(Y) = Y$. Since $\eta_i\psi'\eta_i^{-1}$ fixes Y_j and moves Y_i we conclude that $Y_j \neq Y_i$.

For the proof of (6), let $S = \bigcup_{i\in\omega} (E \cup \eta_i(F)) = E \cup \left(\bigcup_{i\in\omega} \eta_i(F)\right)$. We argue that S satisfies the three conditions in the definition of support which follows Lemma 3.7. First note that $F \subseteq D_2$, hence for $i \in \omega$, $\eta_i(F) \subseteq D_2$. Therefore $S \cap D_1 = E \cap D_1$ which is finite since E is a support.

For the argument that S is well ordered let S' be a non-empty subset of S. If the least element of $S' \cap E$ is least in S' then we are done. Otherwise let γ be the least ordinal such that $I_\gamma \cap S \neq \varnothing$. Then $\varnothing \neq S' \cap I_\gamma = S' \cap \left(\bigcup_{i\in\omega}\left(r^\gamma_i, q^\gamma_i\right)\right)$. Let i be the least natural number such that $S' \cap \left(r^\gamma_i, q^\gamma_i\right) \neq \varnothing$. Then

$$\varnothing \neq S' \cap \left(r^\gamma_i, q^\gamma_i\right) = \eta_i(F) \cap I_\gamma.$$

Since F is well ordered by $\leq$, $\eta_i(F)$ is also well ordered by $\leq$. If we let c be the least element of $\eta_i(F) \cap I_\gamma$ then c is the least element of S'.

It only remains to show that if $w : \lambda \to S$ where w is one to one and order preserving and λ is an ordinal, then the least upper bound of $\{ w(\beta) \mid \beta < \lambda \}$ is in D_3. We prove this by looking at several cases. If $\{ w(\beta) \mid \beta < \lambda \}$ has a cofinal subsequence in E then, since E is a support, the least upper bound of $\{ w(\beta) \mid \beta < \lambda \} \in D_3$. If $\{ w(\beta) \mid \beta < \lambda \}$ has no cofinal subsequence in E then we may assume that $\{ w(\beta) \mid \beta < \lambda \} \subseteq \bigcup_{\gamma\leq\alpha} I_\gamma$. If there is a limit ordinal λ' such that

$$(\forall\gamma < \lambda')\left(\{ w(\beta) \mid \beta < \lambda \} \cap I_\gamma \neq \varnothing\right)$$

and $\{ w(\beta) \mid \beta < \lambda \} \cap I_{\lambda'} = \varnothing$, then the least upper bound of $\{ w(\beta) \mid \beta < \lambda \}$ will be the same as the least upper bound of $\{ b(\gamma) \mid \gamma < \lambda' \}$ which is in D_3.

The only remaining possiblity is that there is a largest $\gamma \leq \alpha$ which is such that $\{ w(\beta) \mid \beta < \lambda \} \cap I_\gamma \neq \varnothing$. Since $\{ w(\beta) \mid \beta < \lambda \} \subseteq S$, $\{ w(\beta) \mid \beta < \lambda \} \cap I_\gamma \subseteq \bigcup_{i\in\omega}\left(r^\gamma_i, q^\gamma_i\right)$. If the set $\left\{ j \mid \left(r^\gamma_j, q^\gamma_j\right) \cap \{ w(\beta) \mid \beta < \lambda \} \neq \varnothing \right\}$ is infinite then

$$lub\,\{ w(\beta) \mid \beta < \lambda \} = lub\left\{r^\gamma_i \mid i \in \omega\right\} = t_\gamma \in D_3$$

by (2) and (4). Finally, if there is a largest i such that $(r_i^\gamma, q_i^\gamma) \cap \{ w(\beta) \mid \beta < \lambda \} \neq \varnothing$ then $\eta_i^{-1}\left((r_i^\gamma, q_i^\gamma) \cap \{ w(\beta) \mid \beta < \lambda \}\right) \subseteq F$ and since F is a support the least upper bound of $\eta_i^{-1}\left((r_i^\gamma, q_i^\gamma) \cap \{ w(\beta) \mid \beta < \lambda \}\right)$ is in D_3. Since $\eta_i \in G$ we conclude that the least upper bound of $(r_i^\gamma, q_i^\gamma) \cap \{ w(\beta) \mid \beta < \lambda \}$ is in D_3.

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