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Levi Contractions and AGM Contractions: a Comparison

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Abstract A representation theorem is obtained for contraction operators that are based on Levi's recent proposal that selection functions should be applied to the set of saturatable contractions, rather than to maximal subsets as in the AGM framework. Furthermore, it is shown that Levi's proposal to base the selection on a weakly monotonic measure of informational value guarantees the satisfaction of both of Gärdenfors' supplementary postulates for contraction. These results indicate that Levi has succeeded in constructing a well-behaved operation of contraction that does not satisfy the postulate of recovery.

1 Introduction Much of the recent development in the study of belief change is based on ideas from Levi's early work, for example his [8]. In later years, the formal development has to a large part focused on a set of proposals from Alchourrón, Gärdenfors, and Makinson, (see their [1]), commonly referred to as the AGM model of belief change. In his recent book [9], Levi has proposed a way of performing belief contraction that differs in important respects from the AGM model. In this paper, we are going to present a formal development of Levi's ideas that allows for precise comparisons with the AGM model.

Both Levi and the AGM trio assume that belief states can be represented by a logically closed set of sentences, the "corpus" (Levi) or "belief set" (AGM). Operations of change, such as belief contraction, are applied to the corpus.

The basic mechanism for contraction in the AGM model is that of *partial meet contraction*. It is defined by the following identity:

$$K \div p = \cap\gamma(K\perp p).$$

$K\perp p$ is the set of all inclusion-maximal subsets of K that do not imply p. γ is a selection function, such that $\gamma(K\perp p)$ is a nonempty subset of $K\perp p$ unless the latter is empty, in which case $\gamma(K\perp p) = \{K\}$. Thus, in the principal case, the outcome of partial meet contraction is equal to the intersection of the maximally inclusive subsets of K that do not imply p.

A selection function, and the operator of partial meet contraction that it generates, are *relational* if and only if there is a relation $\leqq$ such that for all nonempty $K\perp p$:

$$\gamma(K\perp p) = \{K' \in K\perp p | K'' \underline{\ll} K' \text{ for all } K'' \in K\perp p\}.$$

It is *transitively relational* if and only if it is relational by a relation that is transitive.

AGM obtained a set of elegant representation theorems for partial meet contraction, referring to the following set of postulates, commonly called the Gärdenfors postulates:

(G–1)	$K \div p$ is logically closed if K is logically closed	(closure),
(G–2)	$K \div p \subseteq K$	(inclusion),
(G–3)	if $p \notin Cn(K)$ then $K \div p = K$	(vacuity),
(G–4)	if $p \notin Cn(\varnothing)$ then $p \notin K \div p$	(success),
(G–5)	if $p \leftrightarrow q \in Cn(\varnothing)$ then $K \div p = K \div q$	(extensionality),
(G–6)	$K \subseteq Cn((K \div p) \cup \{p\})$	(recovery),
(G–7)	$(K \div p) \cap (K \div q) \subseteq K \div (p\&q)$	(intersection),
(G–8)	if $p \notin K \div (p\&q)$ then $K \div (p\&q) \subseteq K \div p$	(conjunction).

An operator $\div$ on a belief set K is a partial meet contraction if and only if it satisfies the first six of these postulates, the "basic Gärdenfors postulates." It is a transitively relational partial meet contraction if and only if it also satisfies the remaining two postulates, the "supplementary" Gärdenfors postulates, see [1].

The most controversial among the basic Gärdenfors postulates is that of recovery. According to that postulate, if we contract p from K and then add p, nothing will be lost. In the presence of the other basic postulates recovery implies that we in this case indeed end up in K. Gärdenfors argues for the recovery postulate by appealing to informational economy: "information is in general not gratuitous, and unnecessary losses of information are therefore to be avoided," (see page 48 of his [4]).

In his book, Levi argues forcefully against the recovery postulate. His main point is that "measures of informational value ought to be carefully distinguished from measures of information" (see page 123 of [9]). Not all information is of value to the inquiring agent; hence, not every piece of information needs to be retained when moving from one belief state to another. However, the agent should retain as much as possible of the *valuable* information. Levi's recommendation is that we, instead of trying to minimize the loss of information, should try to minimize the loss of informational value. This may lead to violations of the postulate of recovery.

Several other authors have expressed doubts concerning the recovery postulate. For example, Makinson remarked that recovery is "the only one among the six [basic Gärdenfors postulates] that is open to query from the point of view of acceptability under its intended reading" (see page 385 of his [10]). For further criticism of the recovery postulate, see Hansson [5] and Niederée [11].

The intuitive doubtfulness of the recovery postulate provides a good reason to try to find alternative constructions satisfying the other basic Gärdenfors postulates but not recovery. In his book, Levi presented such an alternative construction, which is quite similar to partial meet contraction. It is based on a selection, not among the maximally inclusive subsets of K that fail to imply p, but among the "saturatable contractions," a larger set of subsets of K that fail to imply p.

In order to compare AGM's and Levi's contractions, they must be brought into the same formal apparatus. In this paper we are going to achieve this mainly by expressing what we believe to be Levi's basic ideas in the same formal framework that is used by AGM. This paper is a formal investigation of Levi's solution to the problem of *how* to contract. It should be emphasized that this is but one of many aspects of Levi's complete theory of contraction, which also deals at length with, for example, the problems of *whether* to contract in a given situation and, provided that contraction is admissable, *what* belief to remove.

2 Formal preliminaries In this section, some definitions and a postulate are given that will prove to be useful in the formal study of Levi-contractions.

Definition 2.1 Let $\mathcal{L}$ (the language) be a set of expressions that is closed under truth-functional operations. A consequence operation on $\mathcal{L}$ is a function Cn from $\mathcal{P}(\mathcal{L})$ to $\mathcal{P}(\mathcal{L})$ such that, for all subsets A and B of $\mathcal{L}$:

(i) $A \subseteq Cn(A)$ (inclusion),
(ii) if $A \subseteq B$, then $Cn(A) \subseteq Cn(B)$ (monotony),
(iii) $Cn(A) = Cn(Cn(A))$ (iteration).

A subset A of $\mathcal{L}$ is consistent if and only if there is no $x \in \mathcal{L}$ such that both $x \in Cn(A)$ and $\neg x \in Cn(A)$.

The elements of $\mathcal{L}$ will be denoted by lowercase letters. Subsets of $\mathcal{L}$ will be denoted by uppercase letters. The relational notation $A \vdash x$ will be used interchangeably with $x \in Cn(A)$. It will be assumed that the consequence operator includes classical truth-functional logic and satisfies the properties of deduction and compactness:

Proposition 2.2 *Cn satisfies the following three properties:*

(iv) if x can be derived from A by classical truth-functional logic, then $x \in Cn(A)$ (supraclassicality),
(v) $y \in Cn(A \cup \{x\})$ if and only if $(x \to y) \in Cn(A)$ (deduction),
(vi) if $x \in Cn(A)$, then $x \in Cn(A')$ for some finite subset $A' \subseteq A$ (compactness).

The following notation is adopted from the AGM literature (see Alchourrón and Makinson [2]):

Definition 2.3 Let $K \subseteq \mathcal{L}$ and $A \subseteq \mathcal{L}$. Then $X \in K \perp A$ if and only if:

(i) $X \subseteq K$,
(ii) $X \nvdash a$, for all $a \in A$,
(iii) if $X \subset Y \subseteq K$, then $Y \vdash a$ for some $a \in A$.

Thus, $K \perp A$ is the set of inclusion-maximal subsets of K that do not imply any of the sentences in A. We let $K \perp x$ abbreviate $K \perp \{x\}$. A simple consequence of Definition 2.3 is that if K is logically closed, and $X \in K \perp A$, then X is also logically closed. As a special case of this notational convention, $\mathcal{L} \perp^{\perp}$ is a convenient way to denote the set of maximally consistent subsets of $\mathcal{L}$.

Below, selection functions will be used as choice mechanisms, see Rott, [14]. The idea is that, given a set of possible states after contraction, the selection function should choose the optimal elements from that set. For example, if the input to the selection function is $K \perp x$, the inclusion-maximal subsets of K that do not imply x, the output should be a subset of $K \perp x$ containing only its optimal elements. We will return to the issue of standards of optimality in a subsequent section. The formal definition of a selection function amounts to the following:

Definition 2.4 Let K be a subset of $\mathcal{L}$. γ is a *selection function* on K iff γ is a function from $\mathcal{PP}(K)$ to $\mathcal{PP}(K)$, such that $\gamma(\Omega)$ is a nonempty subset of Ω, unless Ω is empty, in which case $\gamma(\Omega) = \{K\}$.

If Ω is empty, so that there is no possible contraction option, then the agent should remain in K. Technically, this is provided for by setting the output of the selection function equal to the singleton $\{K\}$ if Ω is empty.

Two limiting cases of selection functions are, first, those that (for all x) select a single optimal element of $K \perp x$ and, second, those that (for all x) pick out every element of $K \perp x$ as optimal. We can now introduce the three basic types of contraction that have been discussed by AGM: maxichoice, full meet and partial meet contraction:

Definition 2.5 Let γ be a selection function on K.

(i) γ is *opinionated* iff $\gamma(K \perp x)$ is a singleton for all x.
(ii) γ is *ignorant* iff $K \perp x \subseteq \gamma(K \perp x)$ for all x.

Furthermore:

(iii) $\div$ is a *maxichoice* AGM-*contraction operator* for K iff there exists an opinionated selection function γ on K such that for all $x \in \mathcal{L}$, $K \div x = \cap\gamma(K \perp x)$.
(iv) $\div$ is a *full meet* AGM-*contraction operator* for K iff there exists an ignorant selection function γ on K such that for all $x \in \mathcal{L}$, $K \div x = \cap\gamma(K \perp x)$.
(v) $\div$ is a *partial meet* AGM-*contraction operator* for K iff there exists a selection function γ on K such that, for all $x \in \mathcal{L}$, $K \div x = \cap\gamma(K \perp x)$.

If the selection function picks out more than one element of $K \perp x$ as optimal, then the agent is presented with a new decision problem: how should one choose between the different optimal elements? The solution which AGM offer to this problem is to take what is common to the all the optimal elements, i.e., the meet of these elements.

3 Levi-contractions It was noted already by Alchourrón and Makinson in [2] that if $x \in K$, then all elements X of $K \perp x$ have the property that $Cn(X \cup \{\neg x\}) \in \mathcal{L}\perp^{\perp}$. (For a proof, see Lemma 4.5 below of which this result is an immediate consequence.) In other words, if x is deleted using maxichoice contraction and $\neg x$ is then added, then the logical closure of the resulting set is a maximally consistent set. However, the elements of $K \perp x$ are not all the sets that have this property. Levi's basic proposal is that we, instead of restricting ourselves to $K \perp x$, should focus on a superset of $K \perp x$ consisting of all the logically closed subsets of K that have this property, i.e., all $X \subseteq K$ such that $X = Cn(X)$ and $Cn(X \cup \{\neg x\}) \in \mathcal{L}\perp^{\perp}$. These are, in Levi's terminology, the set of saturatable contractions of K by removing x. This set will, following Levi, be denoted by $S(K, x)$.

Definition 3.1 Let K be a logically closed subset of $\mathcal{L}$ and $x \in \mathcal{L}$. Then $X \in S(K, x)$ if and only if:

(i) $X = Cn(X)$,

(ii) $X \subseteq K$,

(iii) $Cn(X \cup \{\neg x\})$ is maximally consistent in $\mathcal{L}$.

The following lemma shows that $S(K, x)$ is indeed a superset of $K \perp x$.

Lemma 3.2 *Let K be a logically closed subset of $\mathcal{L}$, and let $x \in K$. Then $K \perp x \subseteq S(K, x)$.*

Proof: The proof is based on Observation 3.2 in [2]. If x is a tautology, then $K \perp x = \varnothing \subseteq S(K, x)$ and we are finished. Let $X \in K \perp x$ for a nontautological $x \in K$. We will show that $X \in S(K, x)$. It follows from the definition of $K \perp x$ that X is a logically closed subset of K so that (i) and (ii) are satisfied. For (iii), suppose to the contrary that there is some y such that $y \notin Cn(X \cup \{\neg x\})$ and $\neg y \notin Cn(X \cup \{\neg x\})$. Then $\neg x \rightarrow y \notin Cn(X)$ and $\neg x \rightarrow \neg y \notin Cn(X)$, and thus $\neg x \rightarrow y \notin X$ and $\neg x \rightarrow \neg y \notin X$. Since both $\neg x \rightarrow y$ and $\neg x \rightarrow \neg y$ follow logically from x, and $x \in K$, they are both elements of K. It follows from $\neg x \rightarrow y \in K \backslash X$ and $X \in K \perp x$ that $X \cup \{\neg x \rightarrow y\} \vdash x$, and similarly from $\neg x \rightarrow \neg y \in K \backslash X$ and $X \in K \perp x$ that $X \cup \{\neg x \rightarrow \neg y\} \vdash x$. It follows from $X \cup \{\neg x \rightarrow y\} \vdash x$ and $X \cup \{\neg x \rightarrow \neg y\} \vdash x$ that $X \vdash x$, contrary to $X \in K \perp x$.

Note that it follows from Definition 3.1 that if $X \in S(K, x)$, then $x \notin Cn(X)$. Note also that if x is a tautology, then there exists no saturatable contraction of K by removing x. Thus, $S(K, x)$ is empty if and only if $K \perp x$ is empty.

The following example, adopted from page 121 of Levi [9] should serve to give an intuitive idea of what saturatable contractions are. Let $\mathcal{L}$ be the language containing only the truth-functional compounds of p and q. Let $K = Cn(\{p, q\})$. The elements of $S(K, p)$ are the following:

(1) $Cn(\{p \leftrightarrow q\})$,

(2) $Cn(\{q\})$,

(3) $Cn(\{q \rightarrow p\})$, and

(4) $Cn(\{\neg q \rightarrow p\})$.

It is easily seen that, for instance, $Cn(\{q \rightarrow p\}) \in S(K, p)$. For it is the case that $Cn(Cn(\{q \rightarrow p\}) \cup \{\neg p\}) = Cn(\{\neg p, \neg q\})$, which is maximally consistent in $\mathcal{L}$. It can also be verified that $Cn(\{q \rightarrow p\})$ is not maxichoice, since $Cn(\{q \rightarrow p\}) \subset Cn(\{p \leftrightarrow q\}) \subseteq K$ and $Cn(\{p \leftrightarrow q\}) \nvdash p$. Thus, not all saturatable contractions are maxichoice contractions. The definition of saturatable contractions is not empty; in this example there are two nonsaturatable logically closed subsets of K that do not contain p : $Cn(\{p \rightarrow q\})$ and $Cn(\varnothing)$. However, the latter two sets are both meets of saturatable contractions. For $Cn(\{p \rightarrow q\})$ is the meet of options (1) and (2), and $Cn(\varnothing)$ is the meet of options (3) and (4).

Recall that a selection function is opinionated if it selects exactly one element from $K \perp x$ for all x, and ignorant if it selects all elements of $K \perp x$ for all x such that $K \perp x$ is nonempty. In a parallel fashion, we may call a selection function saturatably opionionated if it selects exactly one element from $S(K, x)$ for all x, and saturatably ignorant if it selects all elements of $S(K, x)$ for all x such that $S(K, x)$ is nonempty. Once these concepts have been introduced, we can define the counterparts in Levi's theory to maxichoice, full meet and partial meet contraction.

Definition 3.3 Let γ be a selection function on K.

(i) γ is *saturatably opinionated* iff $\gamma(S(K, x))$ is a singleton for all x.
(ii) γ is *saturatably ignorant* iff $S(K, x) \subseteq \gamma(S(K, x))$ for all x.

Furthermore:

(iii) $\div$ is a *maxichoice Levi-contraction operator* for K iff there exists a saturatably opinionated selection function γ on K such that for all $x \in \mathcal{L}$: either $x \in K$ and $K \div x = \cap\gamma(S(K, x))$ or $x \notin K$ and $K \div x = K$.
(iv) $\div$ is a *full meet Levi-contraction operator* for K iff there exists a saturatably ignorant selection function γ on K such that for all $x \in \mathcal{L}$: either $x \in K$ and $K \div x = \cap\gamma(S(K, x)$, or $x \notin K$ and $K \div x = K$.
(v) $\div$ is a *partial meet Levi-contraction operator* for K iff there exists a selection function γ on K such that, for all $x \in \mathcal{L}$: either $x \in K$ and $K \div x = \cap\gamma(S(K, x))$, or $x \notin K$ and $K \div x = K$.

Note that contrary to AGM contraction, Levi-contraction is not defined for belief bases, i.e., for sets of sentences that are not logically closed.

As Definition 3.3 reveals, defining the Levi-contraction operators is slightly more complicated than defining the corresponding AGM operators. At first, one might wish to define Levi-contraction so that $K \div x = \cap\gamma S(K, x)$ for all x, not just for $x \in K$. This is the way the corresponding definition for AGM-contraction is formulated, and Levi's text gives the impression that this is how he wants the definition. It turns out however that for $x \notin K$ this mode of defining saturatable contraction does not work. The reason for this is that for $x \notin K$ the postulate of vacuity (G–3) may not be satisfied, as can be seen from the following example. Let $\mathcal{L}$ be the language containing all truth-functional combinations of a and b. Let $K = Cn(\{a, b\})$ and let $K' = Cn(\{a\})$. Of course, $\neg b \notin K$. Since $\{K, K'\} \subseteq S(K, \neg b)$, it may well be the case that $\gamma S(K, \neg b) = \{K, K'\}$, and thus $\cap\gamma S(K, \neg b) = \cap\{K, K'\} = K'$.

Levi, as well as the originators of the AGM-theory, considers vacuity to be a desirable property of contraction, and he explicitly intends that $x \notin K$ should imply that $K \div x = K$. To quote from Levi: "... if we are instructed to give up $\underline{A}$ from $\underline{K}$ when $\underline{A}$ is not in $\underline{K}$, there is nothing we are instructed to give up ... So ... we should take the value of the contraction to be $\underline{K}$ itself," (See [9], p. 133; Levi uses underlined capitals to denote sentences as well as corpora). We concur with Levi that vacuity should be satisfied. In the definition of Levi-contractions, contrary to that of AGM-contractions, the case of vacuous contraction must be separately provided for.

Note that x being a tautology implies that $S(K, x)$ is empty, and hence that $\gamma(S(K, x)) = \{K\}$. In this case $K \div x = K$. This fully accords with Levi's intention: "In contraction, $\underline{A}$ is removed from $\underline{K}$. This can happen consistently with (K–1) if *and only if* $\underline{A}$ is not a logical truth," (page 133 of [9], our emphasis).

4 An axiomatic characterization Levi does not give an axiomatic characterization of his contractions. However, he notes that the postulates of closure, inclusion, vacuity, success and extensionality are satisfied by any partial meet Levi-contraction, (see page 134 of [9]). An operator that satisfies these five postulates is a *withdrawal* in the sense of Makinson, (page 388 of [10]). As we have seen, the withdrawal postulates together with the recovery postulate characterize partial meet AGM-contraction. As it turns out, just dropping the recovery postulate is not sufficient to characterize partial meet Levi-contraction. The following representation theorem is the main result of this section:

Theorem 4.1 *Let $K = Cn(K) \subseteq \mathcal{L}$. $\div$ is a partial meet Levi-contraction operator on K if and only if:*

(i)	$K \div p = Cn(K \div p)$	*(closure),*
(ii)	$K \div p \subseteq K$	*(inclusion),*
(iii)	*if* $\nvdash p$, *then* $K \div p \nvdash p$	*(success),*
(iv)	*if* $\vdash p \leftrightarrow q$, *then* $K \div p = K \div q$	*(extensionality),*
(v)	*if* $K \nvdash p$, *then* $K \div p = K$	*(vacuity),*
(vi)	*if* $\vdash p$, *then* $K \div p = K$	*(failure).*

In the proof, the following postulate will be referred to:

Proposition 4.2 (uniformity) *If* $\forall B[\, B \subseteq K \Rightarrow B \vdash p$ *iff* $B \vdash q\,]$, *then* $K \div p = K \div q$.

The following lemmas will be used in the proof:

Lemma 4.3 *Let K be a logically closed subset of $\mathcal{L}$. If $\div$ satisfies extensionality and vacuity, then $\div$ satisfies uniformity.*

Proof: Let K be a logically closed set and $\div$ an operator for K that satisfies extensionality and vacuity. In order to prove that uniformity is satisfied, let p and q be two sentences such that for all subsets B of K, $B \vdash p$ if and only if $B \vdash q$. We are going to show that $K \div p = K \div q$.

First, let us treat the case when $K \nvdash p$. It follows by vacuity that $K \div p = K$. Since K is a subset of itself, it follows from $K \nvdash p$ that $K \nvdash q$. By vacuity, $K \div q = K$, so that $K \div p = K \div q$.

Next, let us treat the principal case, in which $p \in K$. Since K is logically closed we have $Cn(\{p\}) \subseteq K$. Since $Cn(\{p\}) \subseteq K$ and $Cn(\{p\}) \vdash p$, $Cn(\{p\}) \vdash q$. Hence, $\vdash p \rightarrow q$. In a similar fashion we prove $Cn(q) \vdash p$ and consequently $\vdash q \rightarrow p$. It follows that $\vdash p \leftrightarrow q$. We may conclude by extensionality that $K \div p = K \div q$.

Lemma 4.4 *Let K be a logically closed subset of $\mathcal{L}$ and x a sentence. If $K' \in K\perp x$, then $K' \in K\perp y$ for any $y \in K$ such that $y \notin K'$.*

Proof: Suppose that $K' \in K\perp x$ and $y \in K\backslash K'$. To show that $K' \in K\perp y$, it suffices to show that, whenever $K' \subset K'' \subseteq K$, $y \in Cn(K'')$. Let K'' be such that $K' \subset K'' \subseteq K$. Because $K' \in K\perp x$, we have $x \in K''$. Now suppose that $x \rightarrow y \notin K'$. It then follows from $K' \in K\perp x$ that $(x \rightarrow y) \rightarrow x \in K'$, and thus $x \in K'$, yielding a contradiction. We may conclude that $x \rightarrow y \in K'$. Since $x \in K''$ and $x \rightarrow y \in K'$, we can conclude from $K' \cup \{x\} \subseteq K''$ that $y \in Cn(K'')$.

Lemma 4.5 *Let $p \in K$ and $X \in K\perp p$. Then for all sentences r, either $r \in Cn(X \cup \{\neg p\})$ or $\neg r \in Cn(X \cup \{\neg p\})$.*

Proof: Let $p \in K$ and $X \in K\perp p$. It is clearly sufficient to show that for every sentence r, either $p \vee r \in X$ or $p \vee \neg r \in X$. It follows from $X \in K\perp p$ that $\nvdash p$. Suppose for contradiction that $p \vee r \notin X$ and $p \vee \neg r \notin X$. Because $X \in K \perp p$, it follows that $X \cup \{p \vee r\} \vdash p$ and likewise $X \cup \{p \vee \neg r\} \vdash p$. Hence $X \cup \{r\} \vdash p$ and $X \cup \{\neg r\} \vdash p$, and thus $X \vdash p$. But this contradicts our assumption that $X \in K \perp p$.

Proof: (Theorem 4.1) Let $\div$ be a partial meet Levi-contraction operator. We first show that $\div$ satisfies the properties given in the theorem.

- *Closure:* By definition $K \div p = \cap\gamma(S(K, p))$. Since, by definition, saturatable contractions are logically closed, so are intersections of saturatable contractions.
- *Inclusion:* Trivial.
- *Success:* It follows directly from the definition of $S(K, p)$ that if $\nvdash p$, then $p \notin X$ for all $X \in S(K, p)$.
- *Extensionality:* Assume that $\vdash p \leftrightarrow q$. It suffices to show that $\cap\gamma(S(K, p)) = \cap\gamma(S(K, q))$. We prove that $S(K, p) = S(K, q)$. Then the desired result follows, since γ is a function. To prove that $S(K, p) \subseteq S(K, q)$, assume that $X \in S(K, p)$. It follows from $\vdash p \leftrightarrow q$ that $Cn(X \cup \{\neg p\}) = Cn(X \cup \{\neg q\})$. Thus, $Cn(X \cup \{\neg q\})$ is maximally consistent in $\mathcal{L}$. By definition, X is a logically closed subset of K. Thus, $X \in S(K, q)$. Similarily, we prove that $S(K, q) \subseteq S(K, p)$.
- *Vacuity:* Directly from the definition.
- *Failure:* If $\vdash p$, then $S(K, p) = \varnothing$ and thus $\gamma S(K, p) = \{K\}$.

For the other direction of the theorem, let $\div$ be an operation that satisfies (i)–(vi). To show that $\div$ is a partial meet Levi-contraction operator we need to find a selection function γ on K such that $K \div p = \cap\gamma(S(K, p))$ if $p \in K$. (The case when $p \notin K$ follows trivially since vacuity holds.) Let γ be such that:

$$
\begin{array}{lll}
(1) & \gamma(S(K, p)) = \{K\}, & \text{if } S(K, p) = \varnothing, \\
(2) & \gamma(S(K, p)) = \{X \in S(K, p) | K \div p \subseteq X\}, & \text{otherwise.}
\end{array}
$$

We have to prove: (a) γ is a well-defined function; (b) γ is a selection function; and, (c) for all $p \in K : \cap\gamma(S(K, p)) = K \div p$.

(a) To prove that γ is well-defined, we have to show that if $S(K, p) = S(K, q)$, then $\gamma(S(K, p)) = \gamma(S(K, q))$. Let $S(K, p) = S(K, q)$. We are going to show that if $B \subseteq K$ then $B \vdash p$ iff $B \vdash q$. Suppose for contradiction that $B \vdash q$ but $B \nvdash p$. Then there exists a B' such that $B \subseteq B'$ and $B' \in K \perp p$. Thus, by Lemma 3.2, $B' \in S(K, p)$. But $B' \notin S(K, q)$ since $B' \vdash q$. We have a contradiction. Since $\div$ satisfies extensionality and vacuity, it follows from Lemma 4.3 that $K \div p = K \div q$. By the definition of γ, $\gamma(S(K, p)) = \gamma(S(K, q))$.

(b) Next we show that γ is a selection function. By definition, if $S(K, p) = \varnothing$, then $\gamma(S(K, p)) = \{K\}$. We have to show that if $S(K, p) \neq \varnothing$, then $\gamma(S(K, p)) \neq \varnothing$. Suppose $S(K, p) \neq \varnothing$. Then $\nvdash p$. Success implies that $K \div p \nvdash p$. By inclusion $K \div p \subseteq K$. Thus there exists an X such that $K \div p \subseteq X \in K \perp p$. Hence, by Lemma 3.2, $X \in S(K, p)$. It follows from the definition of γ that $X \in \gamma S(K, p)$, and thus $\gamma S(K, p) \neq \varnothing$.

(c) Finally, we must prove that, for all $p \in K$, $\cap\gamma S(K, p) = K \div p$.

Case 1: $S(K, p) = \emptyset$. It follows from $S(K, p) = \emptyset$ and Lemma 3.2 that $K \perp p = \emptyset$, hence $\vdash p$. It follows from failure that $K \div p = K$. Furthermore $\cap \gamma S(K, p) = K$ by clause (1) in the definition of γ. Thus the desired result holds in this case.

Case 2: $S(K, p) \neq \emptyset$. $K \div p \subseteq \cap \gamma S(K, p)$ holds since $K \div p \subseteq X$, for every $X \in \gamma(S(K, p))$. For the other direction we show that if $q \notin K \div p$, then $q \notin \cap \gamma S(K, p)$. This holds if $q \notin K$, since $\cap \gamma S(K, p) = K$.
Assume $q \in K \backslash (K \div p)$. It suffices to show that there exists an X such that:

(1) $q \notin X$,
(2) $K \div p \subseteq X \subseteq K$,
(3) $X = Cn(X)$,
(4) $Cn(X \cup \{\neg p\}) \in \mathcal{L} \perp p$.

Subcase 1: $K \div p \nvdash p \vee q$. Let X be any set such that $K \div p \subseteq X \in K \perp p \vee q$. It follows directly that (1), (2) and (3) are satisfied. It remains to show that (4) is satisfied. It follows from Lemma 4.4 that $X \in K \perp p$. By Lemma 4.5, $Cn(X \cup \{\neg p\})$ is a maximal consistent subset of the language, and since it does not contain p we have $Cn(X \cup \{\neg p\}) \in \mathcal{L} \perp p$.

Subcase 2: $K \div p \vdash p \vee q$. Let X be any set such that $K \div p \subseteq X \in K \perp \{p, q\}$. It follows that (1), (2) and (3) are satisfied. It remains to be shown that (4) is satisfied. Let $r \notin Cn(X \cup \{\neg p\})$. We are going to show that $\neg r \in Cn(X \cup \{\neg p\})$. It follows from $r \notin Cn(X \cup \{\neg p\})$, by the deduction property of Cn, that $p \vee r \notin Cn(X)$ and consequently, since X is logically closed, $p \vee r \notin X$. Since $p \in K$, and K is logically closed, we have $p \vee r \in K$. It follows from $p \vee r \in K \backslash X$ and $X \in K \perp \{p, q\}$ that either $X \cup \{p \vee r\} \vdash p$ or $X \cup \{p \vee r\} \vdash q$.
Suppose that $X \cup \{p \vee r\} \vdash q$. It then follows that $X \vdash p \rightarrow q$. This, however, is incompatible with $X \vdash p \vee q$ and $X \nvdash q$, that follow immediately from our definition of X. We can conclude that $X \cup \{p \vee r\} \nvdash q$, and consequently $X \cup \{p \vee r\} \vdash p$. It follows from $X \cup \{p \vee r\} \vdash p$ that $X \vdash r \rightarrow p$, and thus $X \vdash \neg p \rightarrow \neg r$, from which we can conclude that $\neg r \in Cn(X \cup \{\neg p\})$, as desired. Just as in the first subcase it follows that $Cn(X \cup \{\neg p\})$ is a maximal consistent subset of the language, and since it does not contain p we have $Cn(X \cup \{\neg p\}) \in \mathcal{L} \perp p$.

It follows from this theorem that all partial meet AGM contractions are partial meet Levi-contractions, as Levi indeed indicates, see pages 125–126 in his [9]. The postulate of failure, that was introduced in Fuhrmann and Hansson [3], serves to ensure that contraction by a tautology leads to no change of the corpus (belief set). The motivation for this is that logical truths are elements of all corpora. The postulate of failure is a formal means of saying that when instructed to do the impossible, you do nothing.

Partial meet contraction satisfies failure. As was pointed out to us by an anonymous referee, to see that failure is not implied by the postulates (i)–(v) mentioned in the theorem, we can let $\div$ be such that $K \div p = K$ whenever $p \notin K$ and $K \div p =$

$Cn(\varnothing)$ whenever $p \in K$. (This example can also be used to show that the further addition of the postulates (G–7) and (G–8) to (i) – (v) does not guarantee the satisfaction of the failure postulate.)

For an example showing that saturatable contraction does not in general satisfy recovery, let $\mathcal{L}$ be the language containing only the truth-functional compounds of p and q, and let $K = Cn(\{p, q\})$. Then $Cn(\{q \rightarrow p\}) \in S(K, p)$, and we can let γ be such that $\gamma S(K, p) = \{Cn(\{q \rightarrow p\})\}$. We see that K is not a subset of $Cn(Cn(\{q \rightarrow p\}) \cup \{p\})) = Cn(\{p\})$, which means that recovery fails.

The following postulate of core-retainment was introduced in [5] as a weaker alternative to recovery. The intuition is that if q is excluded from K when p is removed, then q plays some role for the fact that K implies p.

Proposition 4.6 (Core-retainment) *If $q \in K$ and $q \notin K \div p$, then there is some subset A of K such that $A \cup \{q\} \vdash p$ and $A \nvdash p$.*

However, as was shown in [5], the seemingly much weaker notion of core-retainment implies recovery in the presence of the postulates of closure, inclusion, success and preservation. The following example illustrates that saturatable contraction does not satisfy core-retainment. Let $\mathcal{L}$ consist of p, q and their truth-functional combinations and (as was shown above to be possible) set $K \div p$ equal to $Cn(\{q \rightarrow p\})$. Then $p \rightarrow q \in K \backslash (K \div p)$. Assume for contradiction that core-retainment is satisfied. Then there exists an $A \subseteq K$ such that $A \cup \{p \rightarrow q\} \vdash p$ and $A \nvdash p$. But, by the deduction property, $A \cup \{p \rightarrow q\} \vdash p$ implies $A \vdash (p \rightarrow q) \rightarrow p$, which entails $A \vdash p$. And thus we have a contradiction.

5 The limiting cases All maxichoice AGM-contractions are maxichoice Levi contractions:

Theorem 5.1 *If $\div$ is a maxichoice AGM-contraction operator for K, then $\div$ is a maxichoice Levi-contraction operator for K.*

Lemma 5.2 *Let $p \in K$ and $q \in K$. If $S(K, p) = S(K, q)$, then $K \bot p = K \bot q$.*

Proof: (Lemma 5.2) Suppose to the contrary that $S(K, p) = S(K, q)$ and $K \bot p \neq K \bot q$. Without loss of generality, we may assume that there is some X such that $X \in K \bot p$ and $X \notin K \bot q$.

It follows from $X \in K \bot p$ by Lemma 3.2 that $X \in S(K, p)$ and thus $X \in S(K, q)$, so that $q \notin X$. From this and $X \in K \bot p$ it follows by Lemma 4.4 that $X \in K \bot q$. This contradiction concludes the proof.

Proof: (Theorem 5.1) Let $\div$ be a maxichoice AGM-contraction that is based on the selection function γ. Let γ' be the selection function such that for all p, $\gamma' S(K, p) = \gamma(K \bot p)$. It follows from Lemmas 3.2 and 5.2 that γ' is a well-defined selection function. Let $\div'$ be the Levi-contraction that is based on γ'. Then clearly $K \div p = K \div' p$ for all p.

However, Theorem 5.1 cannot be strengthened to say that if γ generates a maxichoice AGM-contraction operator for K, then it also generates a maxichoice Levi-contraction operator for K. To see this, let γ generate a maxichoice AGM-contraction operator for K. This means that γ maps $K \bot x$ on a single element of $K \bot x$. It may nevertheless be the case that γ maps the wider set $S(K, x)$ on a subset of $S(K, x)$ with more than one member. If this is the case, then the Levi-contraction operator generated by γ is not maxichoice, although γ generates a maxichoice AGM-operator. For an example, let the language consist of p, q and their truth-functional combinations. Let $K = Cn(\{p\&q\})$, let $\gamma(K\bot p) = \{Cn(\{q\})\}$, and let $\gamma(S(K, p)) = \{Cn(\{p \leftrightarrow q\}), Cn(\{p \vee q\})\}$. In this example $\gamma(K\bot p)$ is a singleton but $\gamma(S(K, p))$ is not a singleton.

In their [2], Alchourrón and Makinson proved the following result for full meet AGM contraction:

$$\cap(K\bot p) = K \cap Cn(\{\neg p\}).$$

Hence, full meet AGM contraction involves a radical deformation of the belief set. As Gärdenfors observes, "full meet [AGM-] contraction in general results in contracted belief sets that are far too small," (see page 79 of [4]). It is therefore not regarded to be a realistic operation of change. As the following theorem shows, full meet Levi contraction fares still worse in this respect. (It should be emphasized that Levi does not himself propose the application of ignorant selection functions to saturatable contractions.)

Theorem 5.3 *If $\div$ is a full meet Levi-contraction operator for K, then $K \div p = Cn(\varnothing)$ for all nontautological $p \in K$.*

Proof: Let $\div$ be a full meet Levi-contraction operator for K. Let $p \in K \backslash Cn(\varnothing)$. Then $K \div p = \cap\gamma S(K, p) = \cap S(K, p)$. We will prove that $\cap S(K, p) = Cn(\varnothing)$. Since all $X \in S(K, p)$ are logically closed, it follows that $Cn(\varnothing) \subseteq \cap S(K, p)$. It remains to be shown that $\cap S(K, p) \subseteq Cn(\varnothing)$, i.e., that if $x \notin Cn(\varnothing)$ then $x \notin \cap S(K, p)$. This is trivial unless $x \in K$. Let $x \in K \backslash Cn(\varnothing)$.

Case 1: $x \vee p \notin Cn(\varnothing)$. Then $K\bot(x \vee p)$ is nonempty. Let $Y \in K\bot(x \vee p)$. Clearly, $x \notin Y$. It follows by Lemma 4.4 that $Y \in K\bot x$. It follows by Lemma 3.2 that $x \notin Y \in S(K, p)$.

Case 2: $x \vee p \in Cn(\varnothing)$. Let $Z \in K\bot p$ and $Y \in Z\bot x$. Since $\vdash x \vee p$, we have $Y \cup \{\neg p\} \vdash x$, and thus $Cn(Y \cup \{x\}) \subseteq Cn(Y \cup \{\neg p\})$. By the recovery property, $Cn(Y \cup \{x\}) = Z$, and consequently $Z \subseteq Cn(Y \cup \{\neg p\})$. Then $Cn(Z \cup \{\neg p\}) \subseteq Cn(Y \cup \{\neg p\})$. Since $Cn(Y \cup \{\neg p\})$ is consistent ($p \notin Cn(Y)$), and $Cn(Z \cup \{\neg p\}) \in \mathcal{L}\bot^{\bot}$ (by Lemma 4.5), we have $Cn(Y \cup \{\neg p\}) = Cn(Z \cup \{\neg p\})$, so that $Cn(Y \cup \{\neg p\}) \in \mathcal{L}\bot^{\bot}$ and thus $x \notin Y \in S(K, p)$.

We may conclude that there is no nontautological sentence in $\cap S(K, p)$, and, consequently, that $\cap S(K, p) \subseteq Cn(\varnothing)$. This fact and the previous observation that $Cn(\varnothing) \subseteq \cap S(K, p)$ yields the desired conclusion.

6 Informational value Both Levi's account and that of the AGM trio are based on the assumption that the "best" or most valuable contraction should be chosen. Let $\mathcal{V}$ be a measure on the set of logically closed subsets of K. Following Levi, we will consider $\mathcal{V}$ to represent the informational value of various belief sets smaller than K. Levi distinguishes between two monotonicity requirements on $\mathcal{V}$, (see page 82 of his [9]):

if $A \subset B$, then $\mathcal{V}(A) < \mathcal{V}(B)$ (strong monotonicity),
if $A \subset B$, then $\mathcal{V}(A) \leq \mathcal{V}(B)$ (weak monotonicity),

Levi argues that contraction should be guided by some measure $\mathcal{V}$ of informational value that satisfies weak monotonicity. He provides two equivalent formulations of how contraction can be based on $\mathcal{V}$. One of these formulations is that $K \div p$ is the meet of all saturatable contractions of K removing p that minimize the loss of informational value, (see page 130 of [9]). In other words, $K \div p = \cap\gamma(S(K, p))$ where the selection function γ is defined so that (when $S(K, p) \neq \varnothing$):

(1) $\gamma(S(K, p)) = \{X \in S(K, p) \mid \mathcal{V}(Y) \leq \mathcal{V}(X) \text{ for all } Y \in S(K, p)\}$.

As Levi notes, the problem with contraction based on informational value is that the recommended contraction strategy need not be optimal. To see this, recall from the example in Section 3 that the meet of two saturatable contractions might well be nonsaturatable and, hence, suboptimal. Since Levi adheres to the decision-theoretically motivated idea that the recommended contraction alternative should be optimal, he offers an alternative way to interpret contraction. The alternative formulation is based on a modification of the measure $\mathcal{V}$. The modified measure $\mathcal{V}_D$ (damped informational value) assigns to each logically closed subset X of K the minimum (greatest lower bound) of $\mathcal{V}(Y)$ for any saturatable contraction Y such that $X \subseteq Y$. In this formulation, $K \div p$ is the meet of all logically closed subsets of K removing p that minimize the loss of *damped* informational value, see page 128 of [9]. Levi concludes that although the definition of $\div$ using informational value and the one that appeals to damped informational value are formally interchangeable, the notion based on damped informational value is better motivated from a decision-theoretical point of view, (see page 129 of [9]).

Since the two formulations yield the same result, we are going to use the formulation in terms of $\mathcal{V}$, that better brings out the similarities between Levi's approach and that of AGM.

Definition 6.1 $\div$ is a value-based Levi-contraction iff:

(i) if $p \in K$, then $K \div p = \cap\gamma(S(K, p))$ where $\gamma(S(K, p)) = \{X \in S(K, p) \mid \mathcal{V}(Y) \leq \mathcal{V}(X)$ for all $Y \in S(K, p)\}$ and $\mathcal{V}$ is a real-valued measure on the logically closed subsets of K satisfying weak monotonicity;

(ii) if $p \notin K$, then $K \div p = K$.

Definition 6.1 does not, however, have the full structure of Levi's proposal for belief contraction. The measure $\mathcal{V}$ is intended to be probability-based, an aspect that will not be covered in this paper.

The transitively relational contractions of AGM are based on a transitive relation $\underline{\ll}$ defined over the set $\mathcal{U}_K = \{X \mid X \in K \perp p \text{ for some } p \in K\}$. Such a relation gives rise to a selection function according to the following relationship:

(2) $\gamma(K \perp p) = \{X \in K \perp p \mid Y \underline{\ll} X \text{ for all } Y \in K \perp p\}$.

It was shown by AGM that a selection function is based in the manner of (2) on some transitive relation $\underline{\ll}$ if and only if it is based in that way on some transitive and connected relation $\underline{\ll}'$, see [1]. Since all transitive and connected relations can be represented by real-valued measures, see pages 110–111 of Roberts [12], there is a measure $\mathcal{V}$ such that:

(3) $\gamma(K\perp p) = \{X \in K\perp p \mid \mathcal{V}(Y) \leq \mathcal{V}(X) \text{ for all } Y \in K\perp p\}$.

Since it holds for all distinct $X, Y \in \mathcal{U}_K$ that X is not a subset of Y (see [1]), $\mathcal{V}$ (vacuously) satisfies strong monotonicity.

It is essential for value-based Levi-contraction that the weakly monotonic measure $\mathcal{V}$ is applied only to the elements of $S(K, p)$. (If $S(K, p)$ is replaced by $C(K, p)$, i.e., the set of logically closed subsets of K not implying p, in (1), then $\mathcal{V}$ will have to be replaced by the more complex measure $\mathcal{V}_D$ to yield the same result.) On the other hand, it is not difficult to show that it makes no difference if a strongly monotonic measure chooses between the elements of $K\perp p$ (as in the AGM theory) or between the elements of $S(K, p)$ or those of $C(K, p)$:

Theorem 6.2 *Let $\mathcal{V}$ be a measure on the logically closed subsets of K satisfying strong monotonicity. Then:*

$$\begin{aligned} & \cap\{X \in K\perp p \mid \mathcal{V}(Y) \leq \mathcal{V}(X) \textit{ for all } Y \in K\perp p\} \\ = \; & \cap\{X \in S(K, p) \mid \mathcal{V}(Y) \leq \mathcal{V}(X) \textit{ for all } Y \in S(K, p)\} \\ = \; & \cap\{X \in C(K, p) \mid \mathcal{V}(Y) \leq \mathcal{V}(X) \textit{ for all } Y \in C(K, p)\}. \end{aligned}$$

Proof: Let $\mathcal{V}$ be a measure satisfying strong monotonicity. It suffices to show that $\{X \in C(K, p) | \mathcal{V}(Y) \leq \mathcal{V}(X) \text{ for all } Y \in C(K, p)\} \subseteq K\perp p$. If p is a tautology, then this follows from $C(K, p) = \varnothing$. Assume that p is nontautological and let $X \in \{X \in C(K, p) | \mathcal{V}(Y) \leq \mathcal{V}(X) \text{ for all } Y \in C(K, p)\}$. It follows immediately that $X \subseteq K$ and not $X \vdash p$. It remains to show that if $X \subset Z \subseteq K$, then $Z \vdash p$. Assume $X \subset Z \subseteq K$. Assume for contradiction that $Z \in C(K, p)$. By the definition of X, $\mathcal{V}(Z) \leq \mathcal{V}(X)$. However, strong monotonicity and $X \subset Z$ entails that $\mathcal{V}(X) < \mathcal{V}(Z)$. Thus we have a contradiction. We may conclude that $Z \notin C(K, p)$. We may conclude that $Z \vdash p$. Hence, $X \in K\perp p$ as desired.

The main results of this section are the following two theorems, that show that partial meet Levi-contraction satisfies Gärdenfors' supplementary postulates.

Theorem 6.3 *Value-based Levi-contraction satisfies:*

$$(K \div a) \cap (K \div b) \subseteq K \div (a\&b).$$

The following lemmas will be used in the proof:

Lemma 6.4 $S(K, a\&b) \subseteq S(K, a) \cup S(K, b)$.

Proof: (Lemma 6.4) Let $X \in S(K, a\&b)$. Then $Cn(X \cup \{\neg a \vee \neg b\})$ is maximally consistent. Since it is maximally consistent and contains $\neg a \vee \neg b$ it contains either $\neg a$ or $\neg b$.

Case 1: $\neg a \in Cn(X \cup \{\neg a \vee \neg b\})$. Then $Cn(X \cup \{\neg a\}) = Cn(X \cup \{\neg a \vee \neg b\})$, so that $Cn(X \cup \{\neg a\})$ is maximally consistent. Since we already know that $X = Cn(X) \subseteq K$, we can conclude that $X \in S(K, a)$.

Case 2: $\neg b \in Cn(X \cup \{\neg a \vee \neg b\})$. It follows in the same way that $X \in S(K, b)$.

Lemma 6.5 *Let γ be a selection function that is based on a weakly monotonic measure. Then if $Z \in \gamma S(K, p)$ and $Z \subseteq Z' \in K \perp p$ then $Z' \in \gamma S(K, p)$*

Proof: (Lemma 6.5) Assume that $Z \in \gamma S(K, p)$ and that $Z \subseteq Z' \in K \perp p$. It follows from the weak monotonicity of $\mathcal{V}$ that $\mathcal{V}(Z) \leq \mathcal{V}(Z')$. It follows from $Z \in \gamma S(K, p)$ and $Z' \in S(K, p)$ that $\mathcal{V}(Z') \leq \mathcal{V}(Z)$. Thus, $\mathcal{V}(Z') = \mathcal{V}(Z)$. We can conclude that $Z' \in \gamma S(K, p)$.

Proof: (Theorem 6.3) Let γ be the selection function on which $\div$ is based.

Case 1: $a \in Cn(\varnothing)$. We start with the left side of the equation. $K \div a = \cap\gamma(\varnothing) = \cap\{K\} = K$ so that $K \div a \cap K \div b = K \div b$. For the right side of the equation, note that if $a \in Cn(\varnothing)$ then b and $a\&b$ are logically equivalent. Since extensionality is satisfied (Theorem 4.1), $K \div (a\&b) = K \div b$.

Case 2: $b \in Cn(\varnothing)$. This case is symmetrical to Case 1.

Case 3: $a \notin K$. We start with the left side. If $a \notin K$, then it follows from the definition of $\div$ that $K \div a = K$. Consequently, the left side is equal to $K \div b$. Concerning the right side, we note that $a \notin K$ entails $a\&b \notin K$. Hence, from the definition of $\div$, $K \div (a\&b) = K$. Since inclusion is satisfied, $K \div b \subseteq K$.

Case 4: $b \notin K$. This case is symmetrical to Case 3.

Case 5: $a, b \in K \backslash Cn(\varnothing)$. Let $e \in (K \div a) \cap (K \div b)$. We have to prove that $e \in K \div (a\&b)$. It follows from $e \in (K \div a) \cap (K \div b)$ that if $X \in \gamma S(K, a)$ or $X \in \gamma S(K, b)$, then $e \in X$. Now let $Y \in \gamma S(K, a\&b)$. It follows from Lemma 6.4 that either $Y \in S(K, a)$ or $Y \in S(K, b)$. Without loss of generality, we may assume that $Y \in S(K, a)$. There is then some Y' such that $Y \subseteq Y' \in K \perp a$. By Lemma 4.4, $Y' \in K \perp (a\&b)$. By Lemma 6.5, $Y' \in \gamma S(K, a\&b)$. We are going to prove that $Y \in \gamma S(K, a)$. To do this, it is sufficient to prove that if $Z \in S(K, a)$, then $\mathcal{V}(Z) \leq \mathcal{V}(Y)$. Suppose to the contrary that $Z \in S(K, a)$ and $\mathcal{V}(Y) < \mathcal{V}(Z)$. It follows from $Z \in S(K, a)$ that there is some Z' such that $Z \subseteq Z' \in K \perp a$. By Lemma 4.4, $Z' \in K \perp (a\&b)$. By weak monotonicity, $\mathcal{V}(Z) \leq \mathcal{V}(Z')$. We therefore have $\mathcal{V}(Y') = \mathcal{V}(Y) < \mathcal{V}(Z) \leq \mathcal{V}(Z')$, i.e., $\mathcal{V}(Y') < \mathcal{V}(Z')$. This, however, cannot hold since $Y' \in \gamma S(K, a\&b)$ and $Z' \in S(K, a\&b)$. We can conclude from this contradiction that if $Z \in S(K, a)$, then $\mathcal{V}(Z) \leq \mathcal{V}(Y)$, and consequently that $Y \in \gamma S(K, a)$. It follows from $Y \in \gamma S(K, a)$ that $e \in Y$.

Theorem 6.6 *Let $\div$ be a value-based Levi-contraction on K. Then if $a \notin K \div (a\&b)$, then $K \div (a\&b) \subseteq K \div a$.*

The following lemma will be used in the proof:

Lemma 6.7 *Let a, b and d be elements of K. If $d \notin Y \in S(K, a)$, then there is some Z such that $Y \subseteq Z \in S(K, a\&b)$ and $d \notin Z$.*

Proof: (Lemma 6.7) Let $d \notin Y \in S(K, a)$. There are three cases:

Case 1: $a \vee d \notin Y$. Let $Z = Cn(Y \cup \{a \rightarrow b\})$. It follows that $Cn(Z \cup \{\neg a \vee \neg b\}) = Cn(Y \cup \{\neg a\})$. Since $Y \in S(K, a)$, we have $Cn(Y \cup \{\neg a\}) \in \mathcal{L} \perp^{\perp}$, and thus $Cn(Z \cup \{\neg a \vee \neg b\}) \in \mathcal{L} \perp^{\perp}$. Suppose that $d \in Z$. Then, since $Z \subseteq Cn(Y \cup \{\neg a\})$, we have $Y \cup \{\neg a\} \vdash d$, and thus $Y \vdash a \vee d$, contrary to the condition for this case. Thus $d \notin Z$.

Case 2: $b \rightarrow d \notin Y$. Let $Z = Cn(Y \cup \{a \rightarrow b\})$. It follows just as in Case 1 that $Cn(Z \cup \{\neg a \vee \neg b\}) \in \mathcal{L}\perp^{\perp}$. Suppose that $d \in Z$. Then $Y \vdash (a \rightarrow b) \rightarrow d$, from which follows $Y \vdash b \rightarrow d$, contrary to the condition. Thus $d \notin Z$.

Case 3: $\{a \vee d, b \rightarrow d\} \subseteq Y$. Let $Y' = Cn(Y \cup \{a, d \rightarrow b\})$. We are going to show that $Y' \nvdash a\&b$. Suppose to the contrary that $Y' \vdash a\&b$. Then $Y' \cup \{\neg a \vee \neg b\}$ is inconsistent, i.e., $Y \cup \{a, d \rightarrow b, \neg a \vee \neg b\} \vdash \perp$. However, since $\{a \vee d, b \rightarrow d\} \subseteq Y$, we have $Cn(Y \cup \{a, d \rightarrow b, \neg a \vee \neg b\}) \subseteq Cn(Y \cup \{\neg d\})$, so that $Cn(Y \cup \{\neg d\}) \vdash \perp$, contrary to $d \notin Y$. We can conclude from this contradiction that $Y' \nvdash a\&b$.

Since $Y' \subseteq K$, there is some Z such that $Y' \subseteq Z \in K\perp(a\&b)$. It follows from Lemma 3.2 that $Z \in S(K, a\&b)$. It remains to be shown that $d \notin Z$. Suppose to the contrary that $d \in Z$. Since $\{a, d \rightarrow b\} \subseteq Z$ we then have $a\&b \in Z$, contrary to $Z \in S(K, a\&b)$. We can conclude from this contradiction that $d \notin Z$.

Proof: (Theorem 6.6) Let $a \notin K \div (a\&b)$. There are five cases:

Case 1: $a \notin K$. Then, by the definition of $\div$, $K \div a = K$. Also $a\&b \notin K$, and consequently $K \div (a\&b) = K$. Hence $K \div (a\&b) \subseteq K \div a$ as desired.

Case 2: $b \notin K$. Thus, $a\&b \notin K$, so that $K \div (a\&b) = K$. Since by hypothesis $a \notin K \div (a\&b)$, it follows that $a \notin K$. By the same reasoning as in Case 1, we conclude that $K \div (a\&b) \subseteq K \div a$.

Case 3: $a \in Cn(\varnothing)$. Then $a \in K \div (a\&b)$ so that the theorem is vacuously true.

Case 4: $b \in Cn(\varnothing)$. Then $a\&b$ is equivalent to a, and since extensionality is satisfied (Theorem 4.1), $K \div (a\&b) = K \div a$.

Case 5: $a, b \in K \backslash Cn(\varnothing)$. Suppose that $a \notin K \div (a\&b)$. Then there is some Z such that $a \notin Z \in \gamma S(K, a\&b)$. We have $Cn(Z \cup \{\neg a \vee \neg b\}) \in \mathcal{L}\perp^{\perp}$, and thus either a or $\neg a$ is an element of $Cn(Z \cup \{\neg a \vee \neg b\})$. Suppose that $a \in Cn(Z \cup \{\neg a \vee \neg b\})$. Then $Z \vdash \neg a \vee \neg b \rightarrow a$, or equivalently $Z \vdash a$, contrary to the conditions. It follows that $\neg a \in Cn(Z \cup \{\neg a \vee \neg b\})$ or equivalently $a \rightarrow b \in Z$. Since $a \rightarrow b \in Z$ we have $Cn(Z \cup \{\neg a \vee \neg b\}) = Cn(Z \cup \{\neg a\})$, and thus $Z \in S(K, a)$. Now let X be any element of $S(K, a)$. Then there is some X' such that $X \subseteq X' \in K\perp a$. Since, by Lemma 4.4, $K\perp a \subseteq K\perp a\&b$, we also have $X' \in K\perp a\&b$, and thus, by Lemma 3.2, $X' \in S(K, a\&b)$. It follows from $X \subseteq X'$ that $\mathcal{V}(X) \leq \mathcal{V}(X')$, and from $X' \in S(K, a\&b)$ and $Z \in \gamma S(K, a\&b)$ that $\mathcal{V}(X') \leq \mathcal{V}(Z)$. Thus $\mathcal{V}(X) \leq \mathcal{V}(Z)$. Since this holds for all $X \in S(K, a)$, we can conclude that $Z \in \gamma S(K, a)$. We are now ready to show that $K \div (a\&b) \subseteq K \div a$, i.e., that $\cap\gamma S(K, a\&b) \subseteq \cap\gamma S(K, a)$. Let $d \notin \cap\gamma S(K, a)$. Then there is some $Y \in \gamma S(K, a)$ such that $d \notin Y$. It follows from Lemma 6.7 that there is some W such that $Y \subseteq W \in S(K, a\&b)$ and $d \notin W$. It follows from $Z, Y \in \gamma S(K, a)$ that $\mathcal{V}(Z) = \mathcal{V}(Y)$ and from $Y \subseteq W$, by weak monotonicity, that $\mathcal{V}(Y) \leq \mathcal{V}(W)$. Thus, $\mathcal{V}(Z) \leq \mathcal{V}(W)$. It follows from $\mathcal{V}(Z) \leq \mathcal{V}(W)$, $Z \in \gamma S(K, a\&b)$ and $W \in S(K, a\&b)$ that $W \in \gamma S(K, a\&b)$. Since $d \notin W$, we can conclude that $d \notin \cap\gamma S(K, a\&b)$, as desired.

7 Conclusions This paper contains two major formal results: (1) an axiomatic characterization of partial meet Levi contractions in terms of the postulates of closure, inclusion, vacuity, success, extensionality, and failure; and, (2) a demonstration that value-based Levi-contractions satisfy the two supplementary Gärdenfors postulates.

Levi has not to our knowledge commented on the intuitive reasonableness of the supplementary postulates. However, it seems to us that these postulates are fairly reasonable demands on belief contraction, that are well in tune with his basic ideas on the relation between belief contraction and informational value.

In summary, value-based Levi-contractions satisfy seven out of Gärdenfors' eight postulates for contraction, with recovery as the sole exception. The same applies to at least two other constructions that have been proposed in the belief revision literature, namely: (1) Rott's entrenchment-based contraction (see his [13]); and, (2) the operators of contraction on a belief set that are generated by transitively, maximizingly relational partial meet contraction on a finite and disjunctively closed base for that belief set, see Hansson [7]. The further interrelations between these three classes of operations remain to be investigated.

The formal results of this paper confirm that Levi has succeeded in constructing a well-behaved operation of contraction that does not satisfy recovery.

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