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Computing Verisimilitude

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Abstract This paper continues the *power ordering approach* to verisimilitude. We define a parameterized verisimilar ordering of theories in the finite propositional case, both semantically and syntactically. The syntactic definition leads to an algorithm for computing verisimilitude. Since the power ordering approach to verisimilitude can be translated into a standard notion of belief revision, the algorithm thereby also allows the computation of membership of a belief-revised theory.

1 Introduction Verisimilitude (or truthlikeness) concerns the ordering of theories according to their closeness to the truth. In the context of this paper 'the truth' will be some preferred theory, and thus verisimilitude becomes a parameterized ordering. Originally the notion is due to Popper [11], for whom it was a necessary ingredient in his philosophy that science makes progress by discarding one theory in favor of another which is closer to the truth. On Popper's definition, one theory is closer to the truth than another if and only if it has more true consequences and fewer false consequences. However, Miller [10] and Tichý [16] later showed that Popper's ordering on theories was flawed in that all theories that have some false consequences are incomparable. A survey of the developments in verisimilitude since then can be found in Brink [3].

Somewhat tangential to the context of philosophy of science, Brink, Heidema, and Burger have introduced and studied the *power relations approach* to verisimilitude (see [5], [6], [8]). This approach seems quite general, being embedded in the study of power structures (as in Brink [4]), and having been linked up to domain theory (as in Brink, Vermeulen, and Pretorius [7]) and more recently by Ryan and Schobbens to belief revision (cf. [13]). Ryan and Schobbens show that given a suitable notion of verisimilitude we may define the standard notion of belief revision from it by saying that $\varphi \in T * \psi$ if and only if φ is in every theory containing ψ which is closest to T.

In this paper we extend the power relations approach to verisimilitude by exhibiting the parameterized version in both a semantic and a syntactic context and casting the syntactic version in the form of an algorithm for computing verisimilitude. This then also yields an algorithm for computing belief revision.

In Section 2section.2 we rephrase the relevant concepts from [5]. In Section 3section.3 we generalize the work in [5] by defining a semantic order on propositional sentences relative to an arbitrary propositional sentence. This may be seen as ordering sentences in a situation of incomplete information, or as ordering sentences relative to a given subjective truth. In Section 4section.4 we give the syntactic description of the semantic order described in Section 3section.3 and turn this into an algorithm. Section 5section.5 gives an application to belief revision. The work presented in Sections 2section.2 and 3section.3 has been extended to the case of infinitely many variables (see [6], [7]). Our discussion will however be restricted to the case of finitely many propositional variables, due to the inherently finite nature of the computational syntactic approach of Section 4section.4.

2 The power relations approach Let $\mathcal{L}$ denote the propositional language generated by finitely many propositional variables $p_1, p_2, \ldots, p_n$, and the usual connectives $\vee$, $\wedge$ and $\neg$. The tautology is written $\top$ and its negation $\bot$. Propositional sentences will be denoted by Greek symbols. Let δ be any conjunction of literals such that each variable appears exactly once in δ. That is, δ is a conjunction of propositional variables and negations of propositional variables of the form $\delta = [\neg]p_1 \wedge [\neg]p_2 \wedge \ldots \wedge [\neg]p_n$. δ makes a claim about the truth or falsity of each atomic fact in the language $\mathcal{L}$. Each propositional variable represents an atomic fact. If p_j appears in δ, then the truth of p_j is asserted, and if $\neg p_j$ appears in δ, then the falsity of p_j is asserted. We say that δ fully describes some *possible world*. We fix one such possible world, called t, as the *real world*. t corresponds to the Truth as known by an omniscient observer. t can also be written as a valuation $t : \mathcal{L} \to \mathbf{2}$, where $\mathbf{2}$ is the two-element Boolean algebra, with

$$t(p_i) = \begin{cases} 1 \text{ iff } p_i \text{ is true in the real world} \\ 0 \text{ iff } p_i \text{ is false in the real world.} \end{cases}$$

Each possible world can be identified in this way with some valuation $w : \mathcal{L} \to \mathbf{2}$. We will, for the remainder of this paper, regard possible worlds as valuations. The set of all valuations will be denoted by $\mathcal{W}$.

The set of valuations that satisfy a propositional sentence φ is written $M\varphi$. A valuation w is called a *model* of φ iff $w \in M\varphi$ iff $w(\varphi) = 1$. Two sentences φ and ψ are called *equivalent* iff they have the same models. Two valuations u and w are called *i-equivalent*, written $u \equiv_i w$, iff $u(p_j) = w(p_j)$ for all $j \neq i$.

Our aim in this section is to order sentences according to their closeness to the Truth, by ordering their sets of models. These sets of valuations will be ordered in terms of an order $\leq_t$ on elements of the sets according to their closeness to the real world t. Let u and w be valuations. $u \leq_t w$ reads "w is closer to t than u is," where "closer" is taken to include implicitly the possibility "or equal." It holds iff w agrees with t on at least all those propositional variables where u agrees with t, and is defined as follows:

Definition 2.1 Let t be the real world, and u and w be any valuations. The binary relation $\leq_t \subseteq \mathcal{W}^2$ is defined by:

$$u \leq_t w \text{ iff } (\forall p_i)(u(p_i) = t(p_i)) \Rightarrow (w(p_i) = t(p_i)).$$

This order on valuations can now be used to define a verisimilar pre-order on sets of models of sentences through the use of *power relations*. Brink [4] provides a detailed account on power relations in the more general context of power structures.

For any binary relation over a set A, one can define a power relation over the power set of A. We overload the symbol $\leq_t$ of Definition 2.1 to use it for the power relation it induces as well. (The elements related to each other disambiguate the two relations.) Let φ and ψ be sentences. $\varphi \leq_t \psi$ reads "ψ is closer to t than φ is."

Definition 2.2 For any sentences φ and ψ, the pre-order $\leq_t$ is defined by:

$$\varphi \leq_t \psi \text{ iff } (\forall u \in M\varphi)(\exists v \in M\psi)[u \leq_t v] \text{ and } (\forall v \in M\psi)(\exists u \in M\varphi)[u \leq_t v].$$

Brink and Heidema [5] motivate this definition by showing that this order on sentences exhibits many of the desirable properties of a verisimilar order on theories as formulated by Popper and others.

3 Semantic generalization We regard a theory as an assertion phrased in the language of propositional logic over finitely many variables. The beliefs of the previous section were ordered relative to the belief of an omniscient observer whose beliefs describe a single possible world t. The verisimilar order then indicates which of any two beliefs are in closer agreement with that of the omniscient observer who knows the truth values of all atomic facts. In this section, we will order sentences according to how closely they agree with an arbitrary third sentence. We call the sentence relative to which we define the order τ, the models of τ being any set of valuations, instead of the singleton set $\{t\}$. If we wish to remain in a context where we still believe in the existence of an objectively observed Truth, τ may be seen as describing that truth partially. Any one of the possible worlds described by τ may be the real world. τ therefore provides incomplete information about the Truth, and other sentences are ordered relative to this incomplete description of the Truth. Alternatively, if we are not concerned with an objectively observed Truth, we can view the order as being a subjective ordering of beliefs relative to the beliefs stated by τ. Agents having different beliefs can use this order to compare the beliefs of other agents to their own.

An important concept in the ordering of theories is that of relevance. Schurz and Weingartner [14] noted that the case against Popper's original proposed order depends upon introducing certain irrelevancies as consequences in the construction of its argument. By placing certain relevance criteria on the classical deductive consequence relation, these irrelevant consequents can be disallowed. In [14] the relevance criteria are purely syntactical. For example, irrelevant disjunctive weakening is not allowed: The inference from $\alpha \vdash \beta$ to $\alpha \vdash \beta \vee \gamma$ is not valid, since γ may be irrelevant to both α and β. A verisimilar order on theories is then defined in terms of this restricted relevant deductive consequence relation, while remaining within the framework of classical logic.

In Section 2section.2 sentences were ordered relative to a single valuation t. If that is the case, the sentence with model t pronounces upon the truth of all atomic facts, and hence all atomic facts must be relevant when determining the verisimilar order. Further, since the truth values of all nonatomic facts are functionally dependent upon the truth of atomic facts, we need not be concerned about the truth values of any nonatomic consequences of τ. The order on valuations (and hence also the order on sentences) is therefore defined in terms of the truth values of all the atomic variables. If, however, τ is an arbitrary sentence, not all atomic variables are relevant.

Our notion of relevance is that of Ryan [12], where it is called *natural consequence*. It is based on the concept of *monotonicity*. We will use the concept of monotonicity to define two orders on valuations, namely $\preceq_\tau$ (in 3.4) and then $\leq_\tau$, which is based on $\preceq_\tau$ (in 3.11). The power order of $\leq_\tau$ then gives a parameterized verisimilar order on sentences (in Definition 3.12), similar to the power order defined in Definition 2.2. We will also relate the order on valuations defined in Definition 3.11 to an order on valuations defined in [12] and formulated in terms of the natural consequence relation. This is done in Theorem 3.19 and the three lemmas preceeding it.

Definition 3.1 ([12])

1. A sentence φ is *monotone* in an atomic proposition p_i iff $\forall u \in M\varphi$ and $\forall w \in \mathcal{W}$ such that $u \equiv_i w$, if $u(p_i) \leq w(p_i)$ then $w \in M\varphi$.
2. A sentence φ is *antitone* in an atomic proposition p_i iff $\forall u \in M\varphi$ and $\forall w \in \mathcal{W}$ such that $u \equiv_i w$, if $u(p_i) \geq w(p_i)$ then $w \in M\varphi$.

In other words, φ is monotone in a propositional variable p_i iff increasing the truth value of p_i in **2** preserves satisfaction of φ, and it is antitone in p_i iff decreasing the truth value of p_i in **2** preserves satisfaction of φ. (Note: [12] uses the term 'antimonotonic' where we've used 'antitone.') The variables in which a sentence is monotone (antitone) can also be characterized syntactically (see Barwise [2]).

Theorem 3.2 *A sentence φ is monotone (antitone) in a variable p_i iff there exists some sentence ψ, written in disjunctive normal form and logically equivalent to φ, such that p_i does not occur negatively (positively) in ψ.*

Proof: This is a consequence of Lyndon's Homomorphism theorem (see [2]), which states that a theory is logically equivalent to a positive set of sentences if and only if it is preserved under homomorphic images.

For example, $p \wedge q$ is monotone in both p and q, but antitone in neither; $p \vee \neg q$ is monotone in p and antitone in q; p is monotone in itself, and both monotone and antitone in q, and $p \vee \neg p$ is both monotone and antitone in both p and q.

Two sets of propositional variables that we will frequently refer to, are the following.

Definition 3.3

$$\varphi_+ = \{p_i \mid \varphi \text{ is not antitone in } p_i\}, \text{ and } \varphi_- = \{p_i \mid \varphi \text{ is not monotone in } p_i\}.$$

Syntactically, φ_+ is the set of all propositional variables that occur positively in every disjunctive normal form of φ. φ_- are those variables that occur negatively in every disjunctive normal form of φ. Note that [12] defines the related sets φ^+ and φ^- as the sets of variables in which φ is monotone and antitone respectively. We will however mostly use their set-theoretic complements and have therefore taken these as definition for φ_+ and φ_- respectively. We use these sets to define an order on valuations relative to an arbitrary sentence τ:

Definition 3.4 Let τ be any sentence and u and w be valuations. Then

$$u \preceq_\tau w \text{ iff } (\forall p_i \in \tau_+)[u(p_i) = 1 \Rightarrow w(p_i) = 1] \text{ and } (\forall p_i \in \tau_-)[u(p_i) = 0 \Rightarrow w(p_i) = 0].$$

Lemma 3.5 *If $M\tau = \{t\}$, then $u \preceq_\tau w$ iff $u \leq_t w$.*

Proof: Definition 2.1 may be rewritten as follows:

$$u \leq_t w \quad \text{iff} \quad (\forall p_i)[(t(p_i) = 1 \Rightarrow (u(p_i) = 1 \Rightarrow w(p_i) = 1)) \text{ and } (t(p_i) = 0 \Rightarrow (u(p_i) = 0 \Rightarrow w(p_i) = 0))].$$

Recall that each propositional variable occurs exactly once in the primitive conjunction τ, either negated or unnegated. If p_i appears unnegated in τ, then $t(p_i) = 1$, and if p_i appears negated in τ, then $t(p_i) = 0$. Using the notation of Definition 3.3, the definition can therefore also be written as follows:

$$u \leq_t w \text{ iff } (\forall p_i \in \tau_+)[u(p_i) = 1 \Rightarrow w(p_i) = 1] \text{ and } (\forall p_i \in \tau_-)[u(p_i) = 0 \Rightarrow w(p_i) = 0].$$

Lemma 3.6 *$\preceq_\tau$ is a pre-order.*

Proof: To show reflexivity, we check that for any valuation u, $u \preceq_\tau u$. This follows from Definition 3.4, since $(\forall p_i \in \tau_+)[u(p_i) = 1 \Rightarrow u(p_i) = 1]$ and $(\forall p_i \in \tau_-)[u(p_i) = 0 \Rightarrow u(p_i) = 0]$.

To prove transitivity, let $u \preceq_\tau v$ and $v \preceq_\tau w$. That is, $(\forall p_i \in \tau_+)[u(p_i) = 1 \Rightarrow v(p_i) = 1]$ and $(\forall p_i \in \tau_-)[u(p_i) = 0 \Rightarrow v(p_i) = 0]\}$, and $(\forall p_i \in \tau_+)[v(p_i) = 1 \Rightarrow w(p_i) = 1]$ and $(\forall p_i \in \tau_-)[v(p_i) = 0 \Rightarrow w(p_i) = 0]\}$. Hence $(\forall p_i \in \tau_+)[u(p_i) = 1 \Rightarrow w(p_i) = 1]$ and $(\forall p_i \in \tau_-)[u(p_i) = 0 \Rightarrow w(p_i) = 0]\}$. Therefore $u \preceq_\tau w$.

In Definition 3.7 below, we define a sentence which we call u_τ. In Lemma 3.8 we show that the models of u_τ are those valuations that are closer to τ than u is according to the order $\preceq_\tau$.

Definition 3.7 For any sentence τ and valuation u, u_τ is defined by:

$$u_\tau = \bigwedge\{p_i \mid p_i \in \tau_+ \text{ and } u(p_i) = 1\} \wedge \bigwedge\{\neg p_i \mid p_i \in \tau_- \text{ and } u(p_i) = 0\}.$$

Lemma 3.8 *Mu_τ is the $\preceq_\tau$-upclosure of u.*

Proof: We have to show the following:

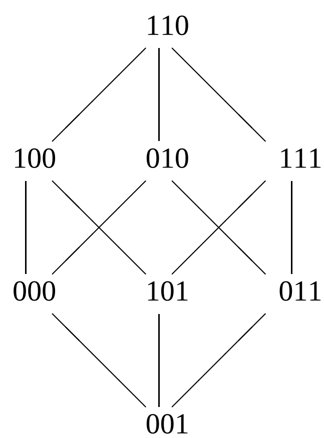
$$\begin{aligned} Mu_\tau &= \{v \mid u \preceq_\tau v\} \\ &= \{v \mid (\forall p_i \in \tau_+)[u(p_i) = 1 \Rightarrow v(p_i) = 1] \text{ and } \\ &\qquad (\forall p_i \in \tau_-)[u(p_i) = 0 \Rightarrow v(p_i) = 0]\}. \end{aligned}$$

u_τ is the conjunction of all literals that appear in every disjunctive normal form of τ and are satisfied by u. Therefore a valuation v satisfies u_τ iff for any literal l satisfied by u that appear in every disjunctive normal form of τ, $v(l) = 1$.

Lemma 3.9 *$M\tau$ is $\preceq_\tau$-upclosed.*

Proof: We have to show that, for any sentence τ and valuations u and w, if $u \preceq_\tau w$ and $u \in M\tau$, then $w \in M\tau$. Suppose $u \preceq_\tau w$ and $u \in M\tau$. By Definition 3.4, for all p_i such that τ is not antitone in p_i, if $u(p_i) = 1$ then $w(p_i) = 1$. And for all p_i such that τ is not monotone in p_i, if $u(p_i) = 0$ then $w(p_i) = 0$. Therefore, for any p_i such that $u(p_i) \neq w(p_i)$, one of two cases holds: either $u(p_i) = 1$ and τ is antitone in p_i, or $u(p_i) = 0$ and τ is monotone in p_i. Recall that u satisfies τ by assumption. In both cases satisfaction of τ is preserved when changing the value of $u(p_i)$ to that of $w(p_i)$. Since u and w can only differ in finitely many variables, $w \in M\tau$.

Example 3.10 Consider the sentence $\tau = (p \wedge \neg r) \vee (q \wedge \neg r)$, generated by propositional variables p, q and r. The models of τ is the set $M\tau = \{110, 111, 010, 011\}$. Further, $\tau_+ = \{p, q\}$ and $\tau_- = \{r\}$. The pre-order $\preceq_\tau$ on valuations relative to τ is obtained as follows: Let, for example, $u = 000$ and $v = 100$. Then, by Definition 3.4, we must check that if $u(p) = 1$ then $v(p) = 1$, if $u(q) = 1$ then $v(q) = 1$, and if $u(r) = 0$ then $v(r) = 0$. Any two valuations can be ordered in this fashion using Definition 3.4, giving rise to the pre-order below.



One can see in this example that the set of models of τ form an upclosed set in the order, as proved in Lemma 3.9. In order to contract these models of τ so that they are equivalent, we refine the relation $\preceq_\tau$ to form the relation $\leq_\tau$ defined in Definition 3.11 below. Our justification for using the same notation in Definition 3.11 as in Definition 2.1 follows just below the definition.

Definition 3.11 Let τ be a sentence and u and w be valuations. Then $u \leq_\tau w$ iff $w \in M\tau \cup Mu_\tau$.

$M\tau \cup Mu_\tau$ is the set of models of the sentence $\tau \vee u_\tau$. If τ has a single model t, each propositional variable appears exactly once in τ, either negated or unnegated. In this case the sets τ_+ and τ_- therefore form a partition of the literals in τ. Namely, they are the subsets of positive and negative literals in τ. u_τ is the conjunction of all the literals in τ satisfied by u. Hence w is closer to t than u iff w agrees with t on at least those propositional variables where u agrees with t. This yields the same order on valuations as defined in Definition 2.1. Definition 2.1 is therefore a special case of Definition 3.11. This justifies our choice of $\leq_\tau$ as an order on valuations relative to an arbitrary sentence τ.

Now that we have an order on valuations, it is easy to obtain an order on sentences, as was done in Section 2section.2. The same power construction used in Definition 2.2 to order sets of valuations can be used.

Definition 3.12 Let τ, φ and ψ be sentences. The pre-order $\leq_\tau$ is defined by:

$$\varphi \leq_\tau \psi \text{ iff } (\forall u \in M\varphi)(\exists v \in M\psi)[u \leq_\tau v] \text{ and } (\forall v \in M\psi)(\exists u \in M\varphi)[u \leq_\tau v].$$

In the remainder of this section, we relate the order on valuations we defined in Definition 3.11 to the order defined in [12] and given in Definition 3.14 below.

Definition 3.13 ([12]) A sentence ψ is a *natural consequence* of a sentence φ iff $M\varphi \subseteq M\psi$, $\psi_- \subseteq \varphi_-$ and $\psi_+ \subseteq \varphi_+$.

Note, for example, that if α is a natural consequence of φ and p_i does not occur positively (negatively) in φ, then it may also not occur positively (negatively) in α. This prohibits the introduction of irrelevant disjuncts in consequences.

Definition 3.14 ([12]) Let τ be any sentence and u and w be valuations. Then w is *closer to* τ than u iff any natural consequence of τ satisfied by u is also satisfied by w.

As in Definition 3.12, this order on valuations can be lifted to a power order on sets of valuations, yielding a verisimilar order on propositional sentences.

Definition 3.15 A sentence ψ is called *logically stronger* than a sentence φ iff every valuation that satisfies ψ also satisfies φ, and *logically weaker* than φ iff every valuation that satisfies φ, also satisfies ψ.

In Lemma 3.18 below we show that for any sentence τ and valuation u, $\tau \vee u_\tau$ is the logically strongest natural consequence of τ satisfied by u. This implies, as we will prove in Theorem 3.19, that the orders defined in Definition 3.11 and Definition 3.14 coincide. Consequently, instead of considering all the natural consequences of a sentence as required in Definition 3.14, we need only consider the logically strongest natural consequence of the theory when determining which of two valuations are in closer agreement with the sentence.

Lemma 3.16 *The natural consequences of τ are precisely the $\preceq_\tau$-upclosed supersets of $M\tau$.*

Proof: Let φ be a natural consequence of τ. Then $M\varphi$ is a superset of $M\tau$. Let $u \in M\varphi$ and $u \preceq_\tau w$. We have to show that $w \in M\varphi$. By Definition 3.4, $(\forall p_i \in \tau_+)[u(p_i) = 1 \Rightarrow w(p_i) = 1]$ and $(\forall p_i \in \tau_-)[u(p_i) = 0 \Rightarrow w(p_i) = 0]\}$, and by Definition 3.13, $\varphi_+ \subseteq \tau_+$ and $\varphi_- \subseteq \tau_-$. Therefore $(\forall p_i \in \varphi_+)[u(p_i) = 1 \Rightarrow w(p_i) = 1]$ and $(\forall p_i \in \varphi_-)[u(p_i) = 0 \Rightarrow w(p_i) = 0]\}$. Therefore $u \preceq_\varphi w$. It follows from Lemma 3.9 that $w \in M\varphi$.

Conversely, let φ be any consequence of τ such that $M\varphi$ is a $\preceq_\tau$-upclosed set. Suppose φ is not a natural consequence of τ. Then either $\varphi_+ \not\subseteq \tau_+$ or $\varphi_- \not\subseteq \tau_-$. Let's assume $\varphi_+ \not\subseteq \tau_+$. That is, there exists some p_i such that τ is monotone in p_i but φ is not. Since φ is not monotone in p_i, there exist some valuations u and w such that $u \equiv_i w$ and $u(p_i) < w(p_i)$ and $u \in M\varphi$ and $w \notin M\varphi$. Since $M\varphi$ is upwardly closed in the order $\preceq_\tau$ and $u \in M\varphi$ and $w \notin M\varphi$, $u \not\preceq_\tau w$. It follows from the definition of $\preceq_\tau$

that $(\exists p_j \in \tau_+)[u(p_j) = 1 \wedge w(p_j) = 0]$ or $(\exists p_j \in \tau_-)[u(p_j) = 0 \wedge w(p_j) = 1]\}$. Since $u \equiv_i w$ and $u(p_i) < w(p_i)$, $p_i \in \tau_-$, contradicting the fact that τ is monotone in p_i. Therefore φ must be a natural consequence of τ.

Lemma 3.17 *$\tau \vee u_\tau$ is a natural consequence of τ.*

Proof: The lemma follows directly from Lemmas 3.9, 3.8, and 3.16, as well as the fact that the union of two upclosed sets in a pre-order is upclosed.

Lemma 3.18 *$\tau \vee u_\tau$ is the logically strongest natural consequence of τ satisfied by u.*

Proof: The models of any natural consequence of τ form an upclosed set by Lemma 3.16. That set must contain $M\tau$ since it is a consequence of τ. Further, if it is satisfied by u, then it must contain Mu_τ, since Mu_τ is the upclosure of u. Any natural consequence of τ satisfied by u is therefore logically weaker than (or equally strong as) $\tau \vee u_\tau$. The result now follows from Lemma 3.17.

Theorem 3.19 *$u \leq_\tau w$ iff for every natural consequence ψ of τ, if u satisfies ψ, then so does w.*

Proof: Suppose $u \leq_\tau w$, that is, $w \in M\tau \cup Mu_\tau$. Let ψ be a natural consequence of τ satisfied by u. Then $(M\tau \cup Mu_\tau) \subseteq M\psi$ by Lemma 3.18. Hence $w \in M\psi$.

Conversely, suppose for every natural consequence ψ of τ, if u satisfies ψ, then so does w. In particular, if u satisfies $\tau \vee u_\tau$, then w satisfies $\tau \vee u_\tau$. Since $u \in Mu_\tau$, w satisfies $\tau \vee u_\tau$. That is, $u \leq_\tau w$.

4 Syntactic approach The verisimilar order on sentences defined in Definition 2.2 can also be defined in terms of two closure operations ∇_t (down-closure) and Δ_t (up-closure) on sentences (cf. [8]).

Definition 4.1 The down-closure $\nabla_t\varphi$ of a sentence φ with respect to a a valuation t is the sentence with models $\{u \mid (\exists v \in M\varphi)[u \leq_t v]\}$. The up-closure $\Delta_t\varphi$ is the sentence with models $\{v \mid (\exists u \in M\varphi)[u \leq_t v]\}$.

These operations can also be described in terms of the logical strength of φ. The up-closure of a sentence φ is the logically strongest sentence that is both logically weaker than φ and can be described using only positive literals. The down-closure of φ is the logically strongest sentence that is both logically weaker than φ and can be described using only negative literals. Lemma 4.2 describes how to obtain this description syntactically.

Lemma 4.2 ([8]) *If t is the valuation that assigns the value 1 to all positive literals, then the down-closure and up-closure of any sentence φ written in disjunctive normal form can be obtained as follows:*

1. If $\varphi \equiv \bot$, then $\nabla_t\varphi = \Delta_t\varphi = \bot$. Else:
2. Replace all positive literals in φ with $\top$ to obtain a sentence logically equivalent to $\nabla_t\varphi$.
3. Replace all negative literals in φ with $\top$ to obtain a sentence logically equivalent to $\Delta_t\varphi$.

Theorem 4.3 defines the verisimilar order of Definition 2.2 in terms of Definition 4.1.

Theorem 4.3 ([8]) *Let t be the real world, and let φ and ψ be any sentences.*

$$\begin{aligned}\varphi \leq_t \psi \quad & \text{iff } M\varphi \subseteq M\nabla_t\psi \text{ and } M\psi \subseteq M\Delta_t\varphi \\ & \text{iff } M\nabla_t\varphi \subseteq M\nabla_t\psi \text{ and } M\Delta_t\psi \subseteq M\Delta_t\varphi.\end{aligned}$$

The down-closure and up-closure of sentences can be described either semantically as in Definition 4.1 or syntactically as in Lemma 4.2. Theorem 4.3 therefore provides a description of the verisimilar order based on the syntactic form of sentences, as opposed to the definition in terms of valuations of Definition 2.2. In Algorithm 4.4 below we give this description. For any sentences φ and ψ, Algorithm 4.4 determines whether $\varphi \leq_t \psi$, where t is the valuation that satisfies the sentence $\tau = p_1 \wedge \ldots \wedge p_n$.

Algorithm 4.4 ([8])

1. Write φ and ψ in disjunctive normal form.
2. Derive the sentences $\nabla_t\varphi$, $\nabla_t\psi$, $\Delta_t\varphi$ and $\Delta_t\psi$, as described in Lemma 4.2.
3. Check if $\nabla_t\varphi \vdash \nabla_t\psi$ and $\Delta_t\psi \vdash \Delta_t\varphi$. If so, then $\varphi \leq_t \psi$ by Theorem 4.3.

Algorithm 4.4 works only if the order on valuations is relative to a single valuation satisfying all positive literals in the language. In this section we will give a similar algorithm whereby one can determine which of any two sentences are closer to an arbitrary third sentence. This will yield a description of the verisimilar order of Definition 3.12 based on the syntactic form of sentences. We first define the down-closure $\nabla_\tau\varphi$ and up-closure $\Delta_\tau\varphi$ of a sentence φ relative to an arbitrary third sentence τ, in terms of the pre-order $\preceq_\tau$ defined in Definition 3.4. The same notation may be used as in Definition 4.1 since we know from Lemma 3.5 that $\preceq_\tau$ and $\leq_t$ coincide when $M\tau$ is a singleton set. We then give an equivalent syntactic description of these closures in Lemma 4.6.

Definition 4.5 The down-closure $\nabla_\tau\varphi$ of a sentence φ with respect to a sentence τ is the sentence with models $\{u \mid (\exists v \in M\varphi)[u \preceq_\tau v]\}$, and its up-closure $\Delta_\tau\varphi$ is the sentence with models $\{v \mid (\exists u \in M\varphi)[u \preceq_\tau v]\}$.

Lemma 4.6 *The down-closure and up-closure with respect to a sentence τ of any sentence φ written in disjunctive normal form can be obtained as follows:*

1. If $\varphi \equiv \bot$, then $\nabla_\tau\varphi = \Delta_\tau\varphi = \bot$. Else:
2. Replace with $\top$ every positive literal in φ in which τ is antitone but in which φ is not antitone. Replace further with $\top$ every negative literal in which τ is monotone but in which φ is not monotone. The resulting sentence is logically equivalent to $\Delta_\tau\varphi$.
3. Replace with $\top$ all positive literals in φ in which τ is monotone but in which φ is not monotone. Replace further with $\top$ all negative literals in which τ is antitone but in which φ is not antitone. The resulting sentence is logically equivalent to $\nabla_\tau\varphi$.

Proof: (1) If φ is the contradiction, the result is immediate by definition. For the remainder of the proof we assume that φ is consistent.

(2) Call the resulting sentence ξ. We will prove that $M\Delta_\tau\varphi \subseteq M\xi$ and then that $M\xi \subseteq M\Delta_\tau\varphi$. Let $w \in M\Delta_\tau\varphi$. Then $\exists u \in M\varphi$ with $u \preceq_\tau w$, i.e., $w \in Mu_\tau$. By Lemma 3.8,

$$(\forall p_i \in \tau_+)[u(p_i) = 1 \Rightarrow w(p_i) = 1] \text{ and } (\forall p_j \in \tau_-)[u(p_j) = 0 \Rightarrow w(p_j) = 0].$$

In the formation of ξ, all the positive occurrences of variables in which φ is not antitone and τ is antitone were removed from φ. ξ is in disjunctive normal form because φ is, and these literals do not appear in ξ, so in all those positive literals in which ξ is not antitone, τ is not antitone in either. Therefore $\xi_+ \subseteq \tau_+$. Similarly, $\xi_- \subseteq \tau_-$. Therefore,

$$(\forall p_i \in \xi_+)[u(p_i) = 1 \Rightarrow w(p_i) = 1] \text{ and } (\forall p_j \in \xi_-)[u(p_j) = 0 \Rightarrow w(p_j) = 0].$$

That is, $w \in Mu_\xi$. Each replacement during the formation of ξ is a replacement of a literal with $\top$ in a sentence written in disjunctive normal form and therefore weakens the sentence logically. Therefore $M\varphi \subseteq M\xi$. Since $u \in M\varphi$, $u \in M\xi$. Any sentence is a natural consequence of itself. Since $u \in M\xi$, ξ is a natural consequence of itself satisfied by u. It follows from Lemma 3.18 that $Mu_\xi \subseteq M\xi$. Therefore $w \in M\xi$. So $M\Delta_\tau\varphi \subseteq M\xi$.

Conversely, suppose $\varphi = \delta_0 \vee \delta_1 \vee \ldots \vee \delta_{n-1}$, where each disjunction δ_i is some primitive conjunction, say $\delta_i = p_0 \wedge p_1 \wedge \ldots \wedge p_{m-1}$. Consider any such conjunction δ_i. Replace any positive literal $+p_j$ in δ_i in which τ is antitone and in which φ is not antitone with $\top$ to form φ'. Let $w \in M\varphi'$. If $w \in M\varphi$ then $w \in M\Delta_\tau\varphi$. Otherwise, if $w \notin M\varphi$, then w is j-equivalent to some $u \in M\delta_i \subseteq M\varphi$. Since $p_j \notin \tau_+$ and $u(p_j) = 1$, by Definition 3.4 $u \preceq_\tau w$. Therefore $w \in M\Delta_\tau\varphi$. The argument for the deletion of negative literals is the same.

(3) Again, call the resulting sentence ξ. Let $u \in M\nabla_\tau\varphi$. Then $\exists w \in M\varphi$ with $u \preceq_\tau w$, i.e., $w \in Mu_\tau$. By Lemma 3.8,

$$(\forall p_i \in \tau_+)[u(p_i) = 1 \Rightarrow w(p_i) = 1] \text{ and } (\forall p_j \in \tau_-)[u(p_j) = 0 \Rightarrow w(p_j) = 0],$$

i.e.,

$$(\forall p_j \in \tau_-)[w(p_j) = 1 \Rightarrow u(p_j) = 1] \text{ and } (\forall p_i \in \tau_+)[w(p_i) = 0 \Rightarrow u(p_i) = 0].$$

Since $\xi_+ \subseteq \tau_-$ and $\xi_- \subseteq \tau_+$,

$$(\forall p_j \in \xi_+)[w(p_j) = 1 \Rightarrow u(p_j) = 1] \text{ and } (\forall p_i \in \xi_-)[w(p_i) = 0 \Rightarrow u(p_i) = 0].$$

That is, $u \in Mw_\xi$. Since $w \in M\varphi$ and $M\varphi \subseteq M\xi$, $w \in M\xi$. It follows from this and Lemma 3.18 that $Mw_\xi \subseteq M\xi$. Therefore $u \in M\xi$, and thus $M\nabla_\tau\varphi \subseteq M\xi$.

Conversely, let $\varphi = \delta_0 \vee \delta_1 \vee \ldots \vee \delta_{n-1}$, where each disjuct δ_i is some primitive conjunct $\delta_i = p_0 \wedge p_1 \wedge \ldots \wedge p_{m-1}$. Consider any such conjunct δ_i. To form δ_i', replace with $\top$ any positive literal $+p_j$ in δ_i in which τ is monotone and in which φ is not antitone. Let $u \in M\delta_i'$. Then u is j-equivalent to some $w \in M\delta_i \subseteq M\varphi$. Since $p_j \notin \tau_-$ and $w(p_j) = 1$, by Definition 3.11 $w \in Mu_\tau$, i.e., $u \preceq_\tau w$. Therefore $u \in M\nabla_\tau\varphi$. The argument for the deletion of negative literals is similar. So $M\xi \subseteq M\nabla_\tau\varphi$.

Theorem 4.7 shows that the verisimilar order on sentences defined in Definition 3.12, can be described in terms of the up-closure and down-closure of theories defined in Definition 4.5. The theorem does not always hold when φ is the contradiction; it is therefore formulated and proved only for consistent sentences. (If φ is the contradiction, the left hand side holds only when ψ is also the contradiction, whereas the right hand side holds whenever $\Delta_\tau\psi \vdash \tau$.)

Theorem 4.7 *For any consistent sentences φ, ψ and τ,*

$$\begin{array}{lll} \varphi \leq_\tau \psi & \text{iff} & (M\nabla_\tau\psi \cap M\tau = \{\} \text{ implies } M\nabla_\tau\varphi \subseteq M\nabla_\tau\psi) \\ & & \text{and } (M\Delta_\tau\psi \subseteq M\Delta_\tau\varphi \cup M\tau) \\ & \text{iff} & ((\nabla_\tau\psi) \wedge \tau \vdash \bot \text{ implies } \nabla_\tau\varphi \vdash \nabla_\tau\psi) \text{ and } \Delta_\tau\psi \vdash (\Delta_\tau\varphi) \vee \tau. \end{array}$$

Proof: Suppose $\varphi \leq_\tau \psi$ and $M\nabla_\tau\psi \cap M\tau = \{\}$. Let $u \in M\nabla_\tau\varphi$. Then $\exists v \in M\varphi$ with $u \preceq_\tau v$. Since $\varphi \leq_\tau \psi$ by assumption, $\exists w \in M\psi$ with $v \leq_\tau w$. That is, $w \in Mv_\tau \cup M\tau$. Since $M\nabla_\tau\psi \cap M\tau = \{\}$ and $M\psi \subseteq M\nabla_\tau\psi$, $M\psi \cap M\tau = \{\}$. Therefore $w \notin M\tau$. Hence $w \in Mv_\tau$, that is, $v \preceq_\tau w$. Therefore $u \preceq_\tau w$ by Lemma 3.6. Hence $u \in M\nabla_\tau\psi$.

Second, suppose $\varphi \leq_\tau \psi$. Let $w \in M\Delta_\tau\psi$. Then $\exists v \in M\psi$ with $v \preceq_\tau w$. So $\exists u \in M\varphi$ with $u \leq_\tau v$, i.e., $v \in Mu_\tau \cup M\tau$, i.e., $u \preceq_\tau v$ or $v \in M\tau$. If $u \preceq_\tau v$ then $u \preceq_\tau w$ by Lemma 3.6, and therefore $w \in M\Delta_\tau\varphi$. Else $v \in M\tau$, and hence so is w by Lemma 3.9. Therefore $w \in M\Delta_\tau\varphi \cup M\tau$.

Conversely, suppose that $(M\nabla_\tau\psi \cap M\tau = \{\}$ implies $M\nabla_\tau\varphi \subseteq M\nabla_\tau\psi)$ and that $(M\Delta_\tau\psi \subseteq M\Delta_\tau\varphi \cup M\tau)$. Let $u \in M\varphi$. If $M\nabla_\tau\psi \cap M\tau \neq \{\}$ then $\exists v \in M\nabla\psi \cap M\tau$. Since $v \in M\tau$, $u \leq_\tau v$. Else $M\nabla_\tau\psi \cap M\tau = \{\}$. Therefore $u \in M\varphi \subseteq M\nabla_\tau\varphi \subseteq M\nabla_\tau\psi$. So $\exists v \in M\psi$ with $u \preceq_\tau v$. Hence $u \leq_\tau v$.

Second, let $w \in M\psi$. Since $M\Delta_\tau\psi \subseteq M\Delta_\tau\varphi \cup M\tau$ by assumption, $w \in M\Delta_\tau\varphi$ or $w \in M\tau$. If $w \in M\Delta_\tau\varphi$, then $\exists u \in M\varphi$ with $u \preceq_\tau w$, hence $u \leq_\tau w$. Else $w \in M\tau$, and $u \leq_\tau w$ for any $u \in M\varphi$.

Theorem 4.7 may be regarded as a generalization of Theorem 4.3. For suppose the special case holds where τ is the conjunction of all positive literals, and $M\tau = \{t\}$. We have to show that $[((\nabla_\tau\psi) \wedge \tau \vdash \bot$ implies $\nabla_\tau\varphi \vdash \nabla_\tau\psi)$ and $\Delta_\tau\psi \vdash (\Delta_\tau\varphi) \vee \tau]$ iff $[\nabla_\tau\varphi \vdash \nabla_\tau\psi$ and $\Delta_\tau\psi \vdash (\Delta_\tau\varphi)]$, using either Definition 4.1 or 4.5 for the up-closure and down-closure operations, since we have already shown that they are the same. Since $t \in \Delta_\tau\varphi$, $(\Delta_\tau\varphi) \vee \tau = \Delta_t\varphi$. So $\Delta_\tau\psi \vdash (\Delta_\tau\varphi) \vee \tau$ iff $\Delta_\tau\psi \vdash \Delta_\tau\varphi$. Further, if $t \in M\psi$, then $\nabla_\tau\psi = \top$ and hence both $((\nabla_\tau\psi) \wedge \tau \vdash \bot$ implies $\nabla_\tau\varphi \vdash \nabla_\tau\psi)$ and $(\nabla_\tau\varphi \vdash \nabla_\tau\psi)$ are true. Else, if $t \notin M\psi$, then $(\nabla_\tau\psi) \wedge \tau \vdash \bot$ and hence $((\nabla_\tau\psi) \wedge \tau \vdash \bot$ implies $\nabla_\tau\varphi \vdash \nabla_\tau\psi)$ iff $(\nabla_\tau\varphi \vdash \nabla_\tau\psi)$. Theorem 4.7 is therefore a generalization of Theorem 4.3.

To conclude, here is the algorithm to determine which of two consistent sentences φ and ψ are closer to an arbitrary sentence τ. Contradictions may be dealt with separately as a special case.

Algorithm 4.8

1. Write φ and ψ in disjunctive normal form.
2. Calculate τ_+, τ_-, φ_+, φ_-, ψ_+ and ψ_-.
3. Derive the sentences $\nabla_\tau\varphi$, $\nabla_\tau\psi$, $\Delta_\tau\varphi$ and $\Delta_\tau\psi$ as described in Lemma 4.6.

4. Check whether $((\nabla_\tau\psi) \wedge \tau \vdash \bot$ implies $\nabla_\tau\varphi \vdash \nabla_\tau\psi)$ and $\Delta_\tau\psi \vdash (\Delta_\tau\varphi) \vee \tau$. If so, then $\varphi \leq_\tau \psi$ by Theorem 4.7.

There are two potential problems with this algorithm. First, for the calculation in (2) it is assumed that literals do not appear redundantly in any sentence. That is, we depend upon the syntactical form of a sentence when determining its monotonicities. Second, if (4) is to be computed using a resolution-based theorem prover, the sentences should have been in conjunctive normal form. In the case of single propositional sentences this presents no real problem apart from efficiency, but it does not bode well for generalization to theories as sets of sentences, or to the predicate case. A theorem prover based on negation normal form (as in Andrews [1]) could eliminate this problem.

5 Application to belief revision In Definition 3.11, we defined an order on valuations relative to an arbitrary propositional sentence. In the context of default reasoning, such an order is known as a *preference relation*. $u \leq_\tau w$ means that w is preferred to u with respect to the satisfaction of τ. A preference relation $\leq_\tau$ can be used to define a nonmonotonic (not to be confused with our definition of monotone!) inference relation $\mid\!\sim_\tau$ (cf. Shoham [15]). As in the previous sections, we restrict ourselves to propositional sentences, although the definition is usually stated more generally.

Definition 5.1 Let τ, φ and ψ be sentences. Then $\varphi \mid\!\sim_\tau \psi$ iff every $\leq_\tau$-maximal model of φ is a model of ψ.

Note that $\leq_\tau$ is defined in the opposite direction from that usually used in the literature on preference relations; all minimal elements in such papers therefore become our maximal elements.

In Makison and Gärdenfors [9], the link between belief revision and default reasoning is argued to be the following: if $\tau * \varphi$ is the sentence obtained by revising τ with φ, then $\psi \in M(\tau * \varphi)$ iff $\varphi \mid\!\sim_\tau \psi$. Using this as definition of belief revision, one can define a belief revision operator in terms of a preference relation $\leq_\tau$ (cf. [13]).

Definition 5.2 Let τ, φ and ψ be sentences. $\psi \in M(\tau * \varphi)$ iff every $\leq_\tau$-maximal model of φ is a model of ψ.

In the notation and context of this paper, the belief revision operator induced by the preference relation $\leq_\tau$ can also be written in terms of the verisimilar order on sentences, provided $\leq_\tau$ is antisymmetric. In order to achieve this, we have to change the pre-orders defined in Definitions 3.4 and 3.11 into partial orders in the usual way: define an equivalence relation by $u \equiv v$ iff $u \preceq_\tau v$ and $v \preceq_\tau u$ (and similarly for $\leq_\tau$), and order valuations according to the equivalence classes to which they belong. In the next theorem, we assume that this has been done.

Theorem 5.3 *Let τ, φ and ψ be sentences. $\psi \in M(\tau * \varphi)$ iff $\varphi \leq_\tau \varphi \wedge \psi$.*

Proof: ψ is a consequence of $\tau * \varphi$

iff every $\leq_\tau$-maximal model of φ is a model of ψ
iff $(\forall u$ maximal in $M\varphi)(\exists w \in M\psi)[u = w]$
iff $(\forall u$ maximal in $M\varphi)(\exists w \in M(\varphi \wedge \psi))[u = w]$
iff $(\forall u \in M\varphi)(\exists w \in M(\varphi \wedge \psi))[u \leq_\tau w]$.

Comparing this reformulation to the verisimilar order of Definition 3.12, one can see that the former is the first half of the definition of the order $\varphi \leq_\tau (\varphi \wedge \psi)$. Since $M(\varphi \wedge \psi) \subseteq M\varphi$, the latter half of the definition follows trivially.

In the terminology of Definition 4.5, and as a consequence of Theorem 4.7, $\tau * \varphi \vdash \psi$ iff the following condition is met: $\varphi \wedge \psi$ must be consistent, and whenever $\varphi \wedge \psi \wedge \tau$ is a contradiction, then $\nabla_\tau \varphi \vdash \nabla_\tau(\varphi \wedge \psi)$. Algorithm 4.8 may in this way be applied to check whether a sentence ψ is a consequence of $\tau * \varphi$, where the belief revision operator $*$ is induced by the preference relation $\leq_\tau$.

In the previous sections we used the preference relation $\leq_\tau$ to define a parameterized verisimilar order, also called $\leq_\tau$, as a generalization of [5]. In this section we rewrote the belief revision membership relation of Definition 5.2 and noted that it closely resembles this verisimilar order. This similarity enabled us to apply the algorithm obtained for computing verisimilitude to also compute membership of a belief-revised theory. Verisimilitude seems the more general notion, since every belief-revision calculation of the form $\psi \in \tau * \varphi$ corresponds to a verisimilar ordering calculation of the form $\varphi \leq_\tau \xi$ where ξ is logically stronger than φ, but not all verisimilar orderings correspond directly to a belief revision calculation.

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