

Decidability and Completeness for Open Formulas of Membership Theories

DORELLA BELLÈ and FRANCO PARLAMENTO

Abstract We establish the decidability, with respect to open formulas in the first order language with equality $=$, the membership relation \in , the constant \emptyset for the empty set, and a binary operation w which, applied to any two sets x and y , yields the results of adding y as an element to x , of the theory NW having the obvious axioms for \emptyset and w . Furthermore we establish the completeness with respect to purely universal sentences of the theory $NW + E + R$, obtained from NW by adding the Extensionality Axiom E and the Regularity Axiom R , and of the theory $NW + AFA'$ obtained by adding to NW (a slight variant of) the Antifoundation Axiom AFA .

1 Introduction Investigations into the decision problem for “small axiomatic fragments of set theory,” to use Tarski’s wording in [15], date back, at least, to Tarski and Szmielew [14] (see also the ensuing Collins and Halpern [4]), which stated the interpretability of Robinson’s Arithmetic \mathbf{Q} into the theory having the axioms $(N) \forall x(x \notin \emptyset)$ and $(W) \forall x \forall y \forall z(x \in w(y, z) \leftrightarrow x \in y \vee x = z)$ as well as the Extensionality Axiom $(E) \forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$ to be henceforth denoted with NWE . Quite recently that interpretability result has been extended also to the theory NW having (N) and (W) as axioms (see Montagna and Mancini [7]). Although such results imply the essential undecidability of the theories NWE and NW , they are of little help to assess precisely to which class of sentences, classified according to the quantificational prefix, the undecidability of NW and NWE applies to. The obvious reduction, through the deduction theorem, to the decision problem for pure logic alone, extensively investigated in Dreben and Goldfarb [5] and Lewis [6], also does not yield any useful information in that respect.

Significant undecidability results for the above problem can be drawn from Parlamento and Policriti [12], which establishes the undecidability of the satisfiability in ω -models of ZF of formulas involving only restricted universal quantifiers in the lan-

Received February 1, 1994; revised July 6, 1995

guage $=, \in$ with the addition of the unary predicate “ x is a pair of distinct sets,” since such a predicate is easily definable by using \mathbf{w} and restricted universal quantifiers.

The present work provides a further contribution of a positive kind on the same problem, as it shows that, the derivability of universal closures of open formulas in $=, \in, \emptyset, \mathbf{w}$ with respect to NW is decidable. Actually such a decidability result holds also for many extensions of NW. In fact it holds for the result $\text{NW} + \text{E}_1 + \text{E}_2$ of adding to NW the following two equational consequences E_1 and E_2 of the Extensionality Axiom: $(\text{E}_1) x \mathbf{w} y \mathbf{w} z = x \mathbf{w} z \mathbf{w} y$ and $(\text{E}_2) x \mathbf{w} y \mathbf{w} y = x \mathbf{w} y$. Furthermore it holds for all the extensions of $\text{NWE} + \text{R}$, namely the theory obtained from NWE by adding the Regularity Axiom $(\text{R}) \forall x(x \neq \emptyset \rightarrow \exists y \in x \forall z \in y(z \notin x))$, and for all the extensions of the theory $\text{NW} + \text{AFA}'$ obtained by adding to NW a slight variant of the Antifoundation Axiom AFA discussed in Aczel [1], since we will show that both $\text{NWE} + \text{R}$ and $\text{NW} + \text{AFA}'$ are complete with respect to universal closures of open formulas in the language $=, \in, \emptyset, \mathbf{w}$.

We will prove the above results by exploiting the eliminability of the operator \emptyset and \mathbf{w} through the use of restricted universal quantifications in the language $=, \in$, which effectively correlates to every open formula an equisatisfiable $\forall_{0,0}$ -formula, namely a prenex formula in the language $=, \in$ involving only restricted universal quantification without occurrences of nesting of quantified variables, like the one which occurs in $\forall x \in y \forall z \in x$. In fact we show that the satisfiability of $\forall_{0,0}$ -formulas with respect to NW as well as the extensions of NW mentioned above is decidable, and furthermore that both NWER and $\text{NW} + \text{AFA}'$ are complete with respect to existential closures of $\forall_{0,0}$ -formulas. We stress that the decision methods developed in this work are extremely inefficient, as they rely on a blind exhaustive search for a finite structure satisfying suitable conditions, as well as on a further reduction to the case in which different variables are required to be interpreted with distinct “nonempty sets.” Although we are not interested in this paper with efficiency issues, we should mention that more efficient decision procedures, which deal directly with the language $=, \in, \emptyset, \mathbf{w}$, can be devised (cf. Bellé and Parlamento [2]).

2 Reduction to $\forall_{0,0}$ -formulas Following Parlamento and Policriti [9],[11] we give the following definition.

Definition 2.1 φ is a $(\forall)_{0,0}$ -formula iff it is equivalent to a conjunction $\varphi_1 \wedge \dots \wedge \varphi_m$ such that, $\forall i, 1 \leq i \leq m$,

$$\varphi_i = (\forall x_1^i \in y_1^i) \dots (\forall x_{j_i}^i \in y_{j_i}^i) (\ell_1^i \vee \dots \vee \ell_{h_i}^i),$$

where $y_k^i \neq x_k^i \forall k, 1 \leq k \leq j_i$ and ℓ_k^i , for $1 \leq k \leq h_i$, is a literal of the form $a \in b, a \notin b, a = b$ or $a \neq b$ with a and b variables. If no y_j^i is also a x_k^i we say that φ is a $(\forall)_{0,0}$ -formula.

The requirement $y_k^i \neq x_k^i \forall k, 1 \leq k \leq j_i$ rules out quantifications of the form $\forall x \in x$, which cannot be regarded as restricted quantifications; it also implies that a $\forall_{0,0}$ -formula has at least one free variable. Open formulas in the language with the functional symbols \emptyset and \mathbf{w} can be translated into equisatisfiable, with respect to extensions of NW, $\forall_{0,0}$ -formulas.

Proposition 2.2 *Every open formula ψ in $=, \in, \emptyset, \mathbf{w}$ can be effectively transformed into a $\forall_{0,0}$ -formula ψ_0 such that, for every extension T of NW, ψ is satisfiable with respect to T iff ψ_0 is satisfiable with respect to T .*

Proof: Given an open formula ψ , let \mathcal{T}_ψ be the closure under subterms of the set of terms that occur in ψ . Let $V_\psi = \{v_0, v_1, \dots, v_q\}$, where $q = |\mathcal{T}_\psi|$ if $\emptyset \notin \mathcal{T}_\psi$ and $q = |\mathcal{T}_\psi| - 1$ if $\emptyset \in \mathcal{T}_\psi$, a set of variables which do not occur in ψ ; y a variable which does not occur in ψ or in V_ψ , and $T_\psi = \{(t_1, t_2) : t_1 \mathbf{w} t_2 \in \mathcal{T}_\psi\}$. Let $\underline{\cdot} : \mathcal{T}_\psi \rightarrow V_\psi$ be a bijection such that $\underline{\emptyset} = v_0$ if $\emptyset \in \mathcal{T}_\psi$.

It is straightforward to check that for every extension T of NW, ψ is satisfiable with respect to T iff the following formula is satisfiable with respect to T :

$$\begin{aligned} \psi_0 = \forall y [& (y \in v_0 \rightarrow y \neq y) \wedge \bigwedge_{(t_1, t_2) \in T_\psi} (y \in \underline{w(t_1, t_2)} \leftrightarrow (y \in \underline{t_1} \vee y = \underline{t_2})) \wedge \\ & \bigwedge_{(t_1, t_2), (k_1, k_2) \in T_\psi} ((\underline{t_1} = \underline{k_1} \wedge \underline{t_2} = \underline{k_2}) \rightarrow \underline{w(t_1, t_2)} = \underline{w(k_1, k_2)}) \wedge \underline{\psi}, \end{aligned}$$

where $\underline{\psi}$ is obtained from ψ by replacing every term t occurring in ψ with the variable \underline{t} . (Note that conjuncts of the third kind in ψ_0 are needed; for example, if they were omitted, the unsatisfiable formula $x = x_1 \wedge z = x_1 \mathbf{w} x \neq x_1 \mathbf{w} z$ would be translated into a satisfiable formula).

Furthermore, since every formula $\psi(x_1, \dots, x_n)$ with n free variables is logically equivalent to a disjunction of formulas of the form

$$y_1 \neq y_2 \wedge \dots \wedge y_{k-1} \neq y_k \wedge \varphi,$$

where y_1, \dots, y_k are taken from the free variables in ψ and φ is obtained from ψ by identifying some of them; we may restrict our attention to the satisfiability problem which requires that a formula be satisfied with distinct elements interpreting distinct variables, a problem which we will call 1-1-satisfiability problem.

Proposition 2.3 *For extensions of the theory NW, the satisfiability problem for open formulas in $=, \in, \emptyset, \mathbf{w}$ is reducible to the 1-1 satisfiability problem for $\forall_{0,0}$ -formulas.*

In view of the previous proposition, the rest of the paper will deal almost entirely with the decision problem for 1-1-satisfiability of $\forall_{0,0}$ -formulas. Actually our first positive result applies, thanks to the lack of the Extensionality Axiom in NW, to the full class \forall_0 .

3 NW

Proposition 3.1 *If $\psi(x_1, \dots, x_k)$ is a $(\forall)_0$ -formula, then $\psi(x_1, \dots, x_k)$ is 1-1-satisfiable with respect to NW iff there exists a finite structure $\mathcal{G} = (\{g_1, \dots, g_k\}, R_0)$ such that g_1, \dots, g_k satisfy $\psi(x_1, \dots, x_k)$ in \mathcal{G} .*

Proof: (\Leftarrow) Given \mathcal{G} fulfilling the stated conditions, we extend R_0 to a binary relation R on the Herbrand Universe \mathcal{H} over $g_1, \dots, g_k, \underline{0}$, \mathbf{w} where $\underline{0}$ is a new constant.

The elements of \mathcal{H} have the form $g_h w t_1 \dots w t_l$, $0 \leq h \leq k$, $l \in \omega$, where $g_0 \equiv \underline{0}$. For r and $g_h w t_1 \dots w t_l$ in \mathcal{H} we let

$$R(r, g_h w t_1 \dots w t_l) \text{ iff } R_0(r, g_h) \text{ or } r \equiv t_i \text{ for some } 1 \leq i \leq l.$$

It is straightforward to check that \mathcal{H} provides a (normal) model of NW with \emptyset interpreted as $\underline{0}$, \in interpreted as R , and \mathbf{w} having the canonical interpretation \mathbf{w}^H .

Furthermore $R(r, g_h)$ entails $R_0(r, g_h)$; in particular if $R(r, g_h)$ holds, then r must be among g_1, \dots, g_k . Since the R -predecessors of g_1, \dots, g_k are the same as the R_0 -predecessors of g_1, \dots, g_k , it is immediate that g_1, \dots, g_k satisfy ψ in \mathcal{H} .

(\Rightarrow) If $\mathcal{M} = (M, \in^M, \emptyset^M, \mathbf{w}^M)$ is a model of NW and $g_1, \dots, g_k \in M$ 1-1-satisfy $\psi(x_1, \dots, x_k)$, then letting $G = \{g_1, \dots, g_k\}$ and $R_0 = \in^M|_G$, it is obvious that ψ is satisfied by g_1, \dots, g_k in $\mathcal{G} = (G, R_0)$.

As an immediate consequence of Propositions 2.3 and 3.1 we have the following.

Proposition 3.2

1. The problem of establishing whether a \forall_0 -formula is satisfiable with respect to NW is decidable.
2. NW is decidable with respect to (the derivability of universal closures of) open formulas in $=, \in, \emptyset, \mathbf{w}$.

The decidability results given in the previous proposition apply also to the theory $\text{NW} + E_1 + E_2$, with the same decision test derived from Proposition 3.1 since we have the following.

Proposition 3.3 $\text{NW} + E_1 + E_2$ is a conservative extension of NW with respect to (the derivability of) universal closures of formulas in $=, \in$, involving only restricted existential quantifiers, as well as with respect to universal closures of open formulas in $=, \in, \emptyset, \mathbf{w}$.

Proof: It suffices to show that if a \forall_0 -formula ψ is satisfiable with respect to NW then it is satisfiable with respect to $\text{NW} + E_1 + E_2$ as well.

If a \forall_0 -formula $\psi(x_1, \dots, x_k)$ is satisfied in a model of NW then, because of Proposition 3.1, it is satisfied in the Herbrand model $\mathcal{H} = (H, \underline{0}, R, \mathbf{w}^H)$ defined in the \Leftarrow part of the proof of Proposition 3.1. Such a model can be refined by using as domain, instead of the full Herbrand Universe H , only the set of canonical terms, namely the values of the function *Canon* defined as follows, on the ground of a total order $<$ of the terms in H :

$$\begin{aligned} \text{Canon}(g_i) &= g_i \text{ for } 0 \leq i \leq k \\ \text{Canon}(g_h \mathbf{w} r_1 \mathbf{w} \dots \mathbf{w} r_l) &= g_h \mathbf{w} \text{Canon}(r_{i_1}) \mathbf{w} \dots \mathbf{w} \text{Canon}(r_{i_n}), \end{aligned}$$

where $\text{Canon}\{r_1, \dots, r_l\} = \text{Canon}\{r_{i_1}, \dots, r_{i_n}\}$ and, for $1 \leq j < k \leq n$, $\text{Canon}(r_{i_j}) < \text{Canon}(r_{i_k})$.

Letting $H' = \{\text{Canon}(t) : t \in H\}$ and for $r, s \in H'$, $r \mathbf{w}^{H'} s = \text{Canon}(r \mathbf{w} s)$, since

$$\text{Canon}(\text{Canon}(t_1) \mathbf{w} \text{Canon}(t_2)) = \text{Canon}(t_1 \mathbf{w} t_2),$$

it is easy to verify that when \emptyset is interpreted as $\underline{0}$, \in as $R|_{H'}$, and \mathbf{w} as $\mathbf{w}^{H'}$, H' provides a model \mathcal{M}' of NW in which g_1, \dots, g_k satisfy the given \forall_0 -formula

$\psi(x_1, \dots, x_k)$. Furthermore it is obvious that \mathcal{M}' is also a model of the two equations E_1 and E_2 .

4 NW + Extensionality Axiom When the Extensionality Axiom E is added to NW the situation changes considerably. For one thing, we have that NWE is not a conservative extension of NW already with respect to universal closures of open formulas involving only $=$ and \mathbf{w} . For example the following formula,

$$(x \mathbf{w} d = y \mathbf{w} d \wedge y \mathbf{w} d = z \mathbf{w} d) \rightarrow x = y \vee x = z \vee y = z,$$

is derivable in NWE but its negation can be seen to be satisfiable with respect to NW.

Furthermore, when dealing with the satisfiability of \forall_0 -formulas with respect to NWE, there cannot be any decision test as the one provided by Proposition 3.1. In fact from results in Parlamento and Policriti [10] it follows that the following \forall_0 -formula $\varphi(x_1, x_2)$,

$$\begin{aligned} & x_1 \neq x_2 \wedge x_1 \notin x_2 \wedge x_2 \notin x_1 \wedge \\ & \forall x \in x_1 \forall y \in x(y \in x_2) \wedge \forall x \in x_2 \forall y \in x(y \in x_1) \wedge \\ & \forall x y \in x_1 \forall z w \in x_2 (x \in z \wedge z \in y \wedge y \in w \rightarrow x \in w) \wedge \\ & \forall x \in x_1 \forall y \in x_2 (x \in y \vee y \in x), \end{aligned}$$

is not satisfiable in a finite structure (G, R) by any pair of elements a, b of G , if R has to be extensional on every pair of elements of G containing either a or b ; in particular $\varphi(x_1, x_2)$ is not satisfiable in any finite structure that can be extended into a model of NWE without introducing new \in -predecessors to its members. As a matter of fact no decision method to check the satisfiability of \forall_0 -formulas with respect to NWE is presently known, and this decision problem might very well be unsolvable (see [11]). Definitely, for the time being, when dealing with satisfiability with respect to NWE we have to restrict our attention to $\forall_{0,0}$ -formulas, as we will do in the rest of this paper.

Proposition 4.1 *If $\psi(x_1, \dots, x_k)$ is a $(\forall)_{0,0}$ -formula then $\psi(x_1, \dots, x_k)$ is 1-1-satisfiable with respect to NW + E iff there exist a finite structure $\mathcal{G} = (G, R_0)$, with $|G| \leq 2k - 1$, and elements $g_1, \dots, g_k \in G$ such that:*

1. R_0 is extensional over g_1, \dots, g_k , i.e., for $i \neq j, g_i$ and g_j have different set of R_0 -predecessors in G ,
2. g_1, \dots, g_k satisfy ψ in \mathcal{G} .

Proof: (\Rightarrow) Let $\mathcal{M} = (M, \in^M, \emptyset^M, \mathbf{w}^M)$ be a model of NWE and suppose that $g_1, \dots, g_k \in M$ satisfy $\psi(x_1, \dots, x_k)$. Since $\mathcal{M} \models E$, \in^M is extensional on g_1, \dots, g_k , i.e., for $1 \leq i \neq j \leq k$ there is $d_i \in M$ such that

$$d_i \in^M e_i \text{ iff } d_i \notin^M e_j. \quad (1)$$

From the results in Parlamento, Policriti and Rao [13], it follows that there are $d_1, \dots, d_{k_d} \in M$, with $k_d < k$, such that $\{d_1, \dots, d_{k_d}\}$ acts as a differentiating set for $\{g_1, \dots, g_k\}$, namely, for every $1 \leq i \neq j \leq k$, a d_i satisfying Equation (1) can be found in $\{d_1, \dots, d_{k_d}\}$. Letting $G = \{g_1, \dots, g_k, d_1, \dots, d_{k_d}\}$ and $R_0 = \in^M \upharpoonright_G$ it is immediate that $\mathcal{G} = (G, R_0)$ fulfills Conditions (1) and (2).

(\Leftarrow) Assume we are given a finite structure $\mathcal{G} = (G, R_0)$ and g_1, \dots, g_k in G satisfying the Conditions (1) and (2) and let $G = \{g_1, \dots, g_k, d_1, \dots, d_{k_d}\}$ with $k_d < k$. Let \mathcal{H} be the Herbrand Universe over $g_1, \dots, g_k, d_1, \dots, d_{k_d}, \underline{0}, \mathbf{w}$, where $\underline{0}$ is a new constant. Among the elements of \mathcal{H} are the “numerals” \underline{n} , where $\underline{0}$ is the given new constant and $\underline{n+1}$ is inductively defined as $\underline{n} \mathbf{w} \underline{n}$. Let

$$\in' = R_0 \cup \{(j, d_i) : 1 \leq i \leq k_d, 2n+1+i \leq j \leq 3n+2+i\}$$

and \in^H be the closure over H of \in' with respect to the axiom W, namely let \in^H be the least binary relation over H such that:

- $\in' \subseteq \in^H$,
- $b \in^H a \mathbf{w} b$,
- if $c \in^H a$ then $c \in^H a \mathbf{w} b$.

Let \sim be the minimal reflexive co-bisimulation over (H, \in^H) , namely the least binary reflexive relation on H such that,

$$\forall x \in^H a \exists y \in^H b (x \sim y) \wedge \forall y \in^H b \exists x \in^H a (x \sim y) \rightarrow a \sim b.$$

\sim is an equivalence relation (see [1]). For $a \in H$ we let $[a]$ be the \sim -equivalence class containing a . If we let

1. $H^\sim = H/\sim = \{[a] : a \in H\}$,
2. $\emptyset^\sim = [\underline{0}]$,
3. $[a] \in^\sim [b]$ iff there are $a', b' \in \mathcal{H}$ such that $a \sim a', b \sim b'$ and $a' \in^H b'$,
4. $[a] \mathbf{w}^\sim [b] = [a \mathbf{w} b]$,

it is easy to verify that $\mathcal{H}^\sim = (H^\sim, \emptyset^\sim, \in^\sim, \mathbf{w}^\sim)$ is a model of NWE. The canonical projection $\pi : H \rightarrow H^\sim$ is a system map from (H, \in^H) into (H^\sim, \in^\sim) , namely $\pi(a) = \{\pi(b) : b \in^H a\}^{H^\sim}$.

By induction it is easy to show that π is 1-1 on $N = \{\underline{0}, \underline{1}, \dots\}$, and furthermore that $\pi(i)$ has exactly i distinct elements. Since \in^H coincides with R_0 over $\{g_1, \dots, g_k, d_1, \dots, d_{k_d}\}$, g_1, \dots, g_k have at most n \in^H -predecessors. Therefore, since π is a system map $\pi(g_1), \dots, \pi(g_k)$ have at most n \in^\sim -predecessors. For $1 \leq i \leq k_d$, $\pi(d_i)$ has at least $n+1$ \in^\sim -predecessors, namely $\pi(\underline{2n+1+i}), \dots, \pi(\underline{3n+2+i})$. Therefore we have that

$$\pi(d_j) \neq \pi(g_i) \text{ for } 1 \leq j \leq k_d \text{ and } 1 \leq i \leq k. \quad (2)$$

Furthermore, since $\pi(\underline{3n+2+j})$ is an \in^\sim -predecessor of $\pi(d_j)$ but it is not an \in^\sim -predecessor of $\pi(d_i)$, we have that

$$\pi(d_i) \neq \pi(d_j) \text{ for } 1 \leq i \neq j \leq k_d. \quad (3)$$

Since \sim is the minimal reflexive co-bisimulation on (H, \in^H) and \in^H on $\{g_1, \dots, g_k, d_1, \dots, d_{k_d}\}$ is extensional over g_1, \dots, g_k , from Equations (2) and (3) it follows that π is 1-1 over all of $\{g_1, \dots, g_k, d_1, \dots, d_{k_d}\}$.

Since π is a system map, from the 1-1-ness of π , it follows that π is actually an isomorphism between $(\{g_1, \dots, g_k, d_1, \dots, d_{k_d}\}, R_0)$ and (H^\sim, \in^\sim) and furthermore that

$$\pi(g_i) = \{\pi(e) : e \in G \wedge R_0(e, g_i)\}^{H^\sim}. \quad (4)$$

It follows that $\pi(g_1), \dots, \pi(g_k)$ satisfy in \mathcal{H}^\sim the formula $\psi(x_1, \dots, x_k)$. For, suppose the following conjunct ψ_i of ψ ,

$$\forall x_1^i \in y_1^i \dots \forall x_j^i \in y_j^i (\ell_1^i \vee \dots \vee \ell_{h_i}^i),$$

is not satisfied by $\pi(g_1), \dots, \pi(g_k)$ in \mathcal{H}^\sim . Let also f_1, \dots, f_j be \in^\sim -elements in $\pi(g_1), \dots, \pi(g_k)$ for which $\neg \ell_1^i \wedge \dots \wedge \neg \ell_{h_i}^i$ holds. Because of Equation (4) there are elements e_1, \dots, e_j in G such that $f_1 = \pi(e_1), \dots, f_j = \pi(e_j)$. Since π is an isomorphism between \mathcal{G} and \mathcal{H}^\sim , e_1, \dots, e_j satisfy $\neg \ell_1^i \wedge \dots \wedge \neg \ell_{h_i}^i$ in \mathcal{G} ; therefore e_1, \dots, e_j witness the fact that g_1, \dots, g_k fail to satisfy ψ_i in \mathcal{G} , against the assumption that g_1, \dots, g_k satisfy ψ in \mathcal{G} .

As an immediate consequence of Propositions 2.3 and 4.1 we have the following.

Proposition 4.2

1. *The problem of establishing whether a $\forall_{0,0}$ -formula is satisfiable with respect to NWE is decidable.*
2. *NWE is decidable with respect to (the derivability of universal closures of) open formulas in $=, \in, \emptyset, \mathbf{w}$.*

5 NWE + Regularity Axiom When the Regularity Axiom is added to NWE, completeness in addition to decidability is achieved. In fact we have the following proposition.

Proposition 5.1

1. *A $(\forall)_{0,0}$ -formula $\psi(x_1, \dots, x_k)$ is 1-1-satisfiable with respect to $\text{NW} + \text{E} + \text{R}$ iff there exist a finite structure $\mathcal{G} = (G, R_0)$ with $|G| \leq 2k - 1$ and elements $g_1, \dots, g_k \in G$ such that:*
 - (a) *R_0 is extensional over g_1, \dots, g_k ,*
 - (b) *R_0 is well founded,*
 - (c) *g_1, \dots, g_k satisfy ψ in \mathcal{G} .*
2. *$\text{NW} + \text{E} + \text{R}$ is complete with respect to the existential closure of $(\forall)_{0,0}$ -formulas and of open formulas in $=, \in, \emptyset, \mathbf{w}$.*

Proof: The proof of the \Rightarrow part of (1) is essentially the same as for the \Rightarrow part of Proposition 4.1. The \Leftarrow part of (1) follows from the following fact.

If ψ is satisfied in $\mathcal{G} = (\{g_1, \dots, g_k, d_1, \dots, d_{k_d}\}, R_0)$ by g_1, \dots, g_k , and R_0 is well founded as well as extensional on g_1, \dots, g_k , then ψ is satisfied in (HF, \in) , HF being the collection of the ordinary hereditarily finite sets.

Let $n = k + k_d$. Since R_0 is well founded on $G = \{g_1, \dots, g_k, d_1, \dots, d_{k_d}\}$, we can define by induction on R_0 a map $^* : G \rightarrow HF$ by letting

$$\begin{aligned} g_i^* &= \{e^* : R_0 e g_i\} \\ d_i^* &= \{e^* : R_0 e d_i\} \cup \{\{i, n - 1\}\}. \end{aligned}$$

Since $\text{rank}(\{i, n - 1\}) = n$, it is immediate that $\text{rank}(d_i^*) > n$ for $1 \leq i \leq k_d$. Furthermore, if in the R_0 -transitive closure of g_j there is no element in $\{d_1, \dots, d_{k_d}\}$, then

$rank(g_j^*) < k \leq n$; if, on the other hand, the R_0 -transitive closure of g_j contains some element in $\{d_1, \dots, d_{k_d}\}$, then $rank(g_j^*) > n$. Thus, for every $e \in G$, $rank(e^*) \neq n$, and so $e^* \neq \{i, n-1\}$ for all $1 \leq i \leq k_d$.

From that it follows very easily that $*$ is a 1-1 map and in turn that

$$\text{for all } e_1, e_2 \in G, e_1 R_0 e_2 \text{ iff } e_1^* \in e_2^*. \quad (5)$$

We claim that ψ is satisfied by g_1^*, \dots, g_k^* in HF . Assume by way of contradiction that ψ is not satisfied by g_1^*, \dots, g_k^* in HF , and let ψ_i be a conjunct,

$$\forall x_1^i \in y_1^i \dots \forall x_j^i \in y_j^i (\ell_1^i \vee \dots \vee \ell_{h_i}^i),$$

which is not satisfied by g_1^*, \dots, g_k^* . Let f_1, \dots, f_j be elements in g_1^*, \dots, g_k^* for which $\neg \ell_1^i \wedge \dots \wedge \neg \ell_{h_i}^i$ hold. Due to the definition of $*$ on $\{g_0, \dots, g_k\}$, f_1, \dots, f_j are actually values of $*$ itself, say $f_1 = e_1^*, \dots, f_j = e_j^*$. Because of Equation (5), e_1, \dots, e_j would provide a counterexample to the assumed satisfiability of ψ by g_0, \dots, g_k .

(2) If ψ is satisfied with respect to $NW + E + R$, combining the \Leftarrow and \Rightarrow part of (1) we have that ψ is satisfied in (HF, \in) . Since (HF, \in) is isomorphically embedded as an \in -initial part into every model of NW , it follows that ψ is satisfiable in every model of $NW + E + R$; therefore the existential closure of ψ is derivable from $NW + E + R$.

As an immediate consequence of Propositions 2.3 and 5.1 we have the following.

Proposition 5.2

1. *The problem of establishing whether a $\forall_{0,0}$ -formula is satisfiable with respect to $NWE + R$ is decidable; actually $NWE + R$ is complete with respect to $\forall_{0,0}$ -formulas.*
2. *$NWE + R$ is decidable with respect to (the derivability of universal closures of) open formulas in $=, \in, \emptyset, \mathbf{w}$; actually $NWE + R$ is complete with respect to universal closures (equivalently existential closures) of open formulas in $=, \in, \emptyset, \mathbf{w}$.*

Remark 5.3 It is clear from the proof that in Proposition 5.1 the Regularity Axiom R can be weakened to the schema:

$$R^s: \neg \exists x_1 \dots x_n (x_1 \in x_2 \wedge \dots \wedge x_{n-1} \in x_n \wedge x_n \in x_1),$$

which is derivable in NW from R . The example we have provided to show that NWE is not a conservative extension of $NW + E_1 + E_2$ actually establishes also that $NWE + R$ is not a conservative extension of $NW + E_1 + E_2 + R$ for universal closures of formulas in $=$ and \mathbf{w} alone. Thus the completeness result in Proposition 5.1.2 does not hold any more if $NWE + R$ or $NWE + R^s$ are weakened to $NW + E_1 + E_2 + R$ or $NW + E_1 + E_2 + R^s$.

For each n , in NW it is possible to introduce via an explicit definition the n -tuple operator $\{x_1, \dots, x_n\}_n = \emptyset \mathbf{w} x_1 \mathbf{w} \dots \mathbf{w} x_n$, and derive the following schema:

$$T: x \in \{x_1, \dots, x_n\}_n \leftrightarrow x = x_1 \vee \dots \vee x = x_n.$$

$NW + E_1 + E_2$ implies all the formulas in the following schema:

$$E^T: \{x_1, \dots, x_k\}_k = \{y_1, \dots, y_n\}_n,$$

where y_1, \dots, y_n lists without repetitions all the variables in the sequence x_1, \dots, x_k . Letting NT be the theory in the language containing \emptyset and for each n the n -ary functional symbol $\{\dots\}_n$ whose axioms are N and all the formulas in the schema T ; it can be shown that $\text{NT} + E^T + R^s$ is complete with respect to existential closure of open formulas in its own language.

$\text{NT} + E^T + R^s$ appears to be a minimal theory which, on one hand, ensures the existence of the hereditarily finite sets and, on the other hand, enjoys the completeness property established in Proposition 5.1.2 (cf. also Ville [16]).

6 NW + Antifoundation Axiom Let us recall from [1] that the anti-foundation axiom AFA states that every graph (G, R) has a unique decoration, namely a function f whose domain is G and such that $\forall a \in G, f(a) = \{f(b) : bRa\}$. The uniqueness of the decoration whose existence is stated in AFA entails a strong form of extensionality that can be analyzed using the notion of bisimulation, which in [1] is defined as follows. A binary relation R is a bisimulation if

$$aRb \Rightarrow \forall x \in a \exists y \in b (xRy) \wedge \forall y \in b \exists x \in a (xRy).$$

The uniqueness condition stated in AFA is equivalent (in ZF) to the nonexistence of bisimulations different from the identity (proper bisimulations).

The Axiom of Regularity together with the Axiom of Extensionality readily entails the nonexistence of bisimulations relating different sets. In particular no such bisimulations can exist on HF . On the other hand, in ZF minus the Axiom of Regularity, the axiom of Extensionality immediately follows from the nonexistence of proper bisimulations. However that it is not the case in NW as it is shown by the following model.

Let M be the closure under \mathbf{w} of $HF \cup \{v_1, v_2\}$, where v_1, v_2 are distinct objects not belonging to HF ; let \in' be the expansion of \in over M obtained by letting $e \in' v_1$ and $e \in' v_2$ for all $e \in HF \cup \{v_1, v_2\}$; let \in^M be the least expansion of \in' such that for all a, b, c in M , $b \in_M a \mathbf{w} b$; and if $c \in^M a$ then $c \in^M a \mathbf{w} b$.

Furthermore, let \emptyset^M be $\emptyset \in HF$, and

$$\mathbf{w}^M(a, b) = \begin{cases} a \cup \{b\} & \text{if } a, b \in HF \\ a \mathbf{w} b & \text{otherwise.} \end{cases}$$

$\mathcal{M} = (M, \in^M, \emptyset^M, \mathbf{w}^M)$ is a model of NW.

v_1 and v_2 as well as all those elements of M which are obtained by starting with v_1 or v_2 and applying \mathbf{w}^M have all the elements in HF among their \in^M -predecessors. Furthermore they are the only elements of M which have infinitely many \in^M -predecessors. Since v_1 and v_2 have the same \in^M -predecessors and they are distinct, the Axiom of Extensionality fails in \mathcal{M} . Nevertheless

$$\mathcal{M} \models \forall ab \neg \exists R (R \text{ is a bisimulation} \wedge (a, b) \in R \wedge a \neq b)$$

For, suppose $a, b, R \in M$ are such that $\mathcal{M} \models (R \text{ is a bisimulation} \wedge (a, b) \in R \wedge a \neq b)$. If $a, b \in HF$ then from R we could easily obtain a bisimulation on (HF, \in)

relating a and b ; but, as we noticed, no such bisimulation on HF can exist. On the other hand if, for example, $a \in M \setminus HF$ then either v_1 or v_2 , say v_1 , is in the \in^M -transitive closure of a . To every \in^M -predecessor of v_1 has to correspond an M -ordered pair $(x, y)^M$ such that $(x, y)^M \in^M R$, and to different \in^M -predecessors of v_1 correspond different M -ordered pairs \in^M -related to R . Since v_1 has infinitely many \in^M -predecessors, every element in HF is \in^M -related to R , in particular there are elements \in^M -related to R which are not M -ordered pairs, so the assumption that $\mathcal{M} \models R$ is a bisimulation is contradicted.

A simple change in the notion of bisimulation suffices to give to AFA a form which in one hand is equivalent to the previous one with respect to ZF minus the Regularity Axiom (ZF^-), and on the other hand is appropriate when working with NW. In fact it yields to NW enough strength to derive the Extensionality Axiom and to share with $ZF^- + AFA$ the completeness property to be proved in the following. We say that a binary relation R is a weak bisimulation if

$$aRb \Rightarrow \forall x \in a(x \in b \vee \exists y \in b(xRy)) \wedge \forall y \in b(y \in a \vee \exists x \in a(xRy)).$$

Furthermore R is proper if it contains at least one pair (a, b) with $a \neq b$. Note that in ZF, due to the existence of transitive closures, this new version of the definition of bisimulation is equivalent to the old one.

We formulate the strong extensionality axiom SE as follows

(SE) there are no proper weak bisimulations,

and AFA' as the conjunction of SE and

(AFA₁) every graph has at least one decoration.

Note that in NW, SE entails E; in fact if a and b are different and have the same predecessors then $\{(a, b)\}$, which exists in NW, is a proper weak bisimulation.

The role previously played by the structure HF of the hereditary finite sets is played in the present context by the structure V_f of the hereditary finite hyperset, which we will define following the construction of a model for $ZF^- + AFA$ described in [1].

Let us recall from [1] that an accessible pointed graph (apg) is a graph with a distinguished node, called its point, from which any (other) node can be reached through a finite path. Let V_{0f} be the class of all the finite apgs. For $a, b \in V_{0f}$ let $a \in_{0f} b$ hold iff a is a subgraph of b generated by one of the predecessor in b of the point of b . If for $a, b \in V_{0f}$ we let $a \sim_{V_{f0}} b$ mean that there is a bisimulation R on V_{0f} such that $(a, b) \in R$, then $\sim_{V_{f0}}$ is an equivalence relation.

We let V_f be the quotient of V_{0f} with respect to $\sim_{V_{f0}}$ and \in_f be the relation induced over V_f by \in_{0f} . (V_f, \in_f) is strongly extensional in the sense that no proper bisimulation with respect to \in_f exists on V_f , and it is called the strongly extensional quotient of V_f . (V_f, \in_f) is a model of NW; actually it is a model of ZF deprived of the Foundation and Infinity axioms.

For every finite graph $\mathcal{G} = (G, R_0)$ there is a unique system map from G into V_f , namely a function $\pi_{\mathcal{G}} : G \rightarrow V_f$ such that

$$\begin{aligned} aR_0b &\Rightarrow \pi_{\mathcal{G}}(a) \in_f \pi_{\mathcal{G}}(b) \\ c \in_f \pi_{\mathcal{G}}(b) &\Rightarrow \exists a \in G aR_0b \wedge c = \pi_{\mathcal{G}}(a). \end{aligned}$$

$\pi_{\mathcal{G}}$ is a strongly extensional quotient of \mathcal{G} in the sense that $\pi_{\mathcal{G}}$ induces on G an equivalence relation which is the maximum bisimulation on \mathcal{G} . (V_f, \in_f) is isomorphically embedded, as an \in -initial part, into every model of $\text{NW} + \text{AFA}'$.

If $\mathcal{M} = (M, \in^M, \emptyset^M, \mathbf{w}^M)$ is a model of NW , for $a \in M$ we let Ma denote the \in^M -transitive closure of a , i.e.,

$$Ma = \{b \in M : \text{there is a finite } \in^M \text{-chain } a_0 \in^M a_1 \in^M \dots \in^M a_n, \\ \text{such that } a_0 = b \text{ and } a_n = a\},$$

and let

$$hf(\mathcal{M}) = \{a \in M : Ma \text{ is finite } \}.$$

A simple adaptation of result in [1] leads to the following.

Proposition 6.1 *If $\mathcal{M} \models \text{NW} + \text{AFA}'$ then $(hf(\mathcal{M}), \in^M)$ is isomorphic to (V_f, \in_f) .*

Proof: If $a \in hf(\mathcal{M})$ then $Ma \in V_{0f}$ and the map that assigns Ma to $a \in M$ is clearly a system map which, composed with the strongly extensional quotient $\pi_f : V_{0f} \rightarrow V_f$ yields a system map $\pi : hf(\mathcal{M}) \rightarrow V_f$. $hf(\mathcal{M})$ is strongly extensional, hence π is injective, as it follows by Theorem 2.19 in [1].

If $a \in V_f$ then $V_f a$ is an apg and, since it is finite and $\mathcal{M} \models \text{NW}$, there exists in \mathcal{M} the corresponding graph g . Furthermore since $\mathcal{M} \models \text{AFA}'$ there exists in \mathcal{M} a decoration $d^{\mathcal{M}}$ of g , from which we obtain an \mathcal{M} -decoration d of $V_f a$. Then $\pi \circ d : V_f a \rightarrow V_f$ is a system map. As V_f is strongly extensional $\pi \circ d$ must be the identity map on $V_f a$, since the identity is a system map on $V_f a$ and there is only one system map on $V_f a$ ([1] Thm. 2.19). In particular $a = \pi(da)$. Thus π is surjective as well as injective, hence it is an isomorphism.

The key notion to obtain our further decidability as well as completeness results is expressed by the following definition, in which the notion of R being a proper bisimulation refers to the binary relation R_0 rather than \in .

Definition 6.2 A graph (G, R_0) is said to be strongly extensional on $\{g_1, \dots, g_k\} \subseteq G$ if there is no proper weak bisimulation R on $\{g_1, \dots, g_k\}$ such that $g_i R g_j$ and g_i and g_j have the same R_0 -predecessors in $G \setminus \{g_1, \dots, g_k\}$ for some $1 \leq i \neq j \leq k$.

Proposition 6.3 *A $(\forall)_{0,0}$ -formula $\psi(x_1, \dots, x_k)$ is 1-1-satisfiable with respect to $\text{NW} + \text{SE}$ iff there exist a finite structure $\mathcal{G} = (G, R_0)$, with $|G| \leq 2k - 1$ and elements $g_1, \dots, g_k \in G$ such that:*

1. R_0 is strongly extensional over g_1, \dots, g_k ,
2. g_1, \dots, g_k satisfy ψ in \mathcal{G} .

Proof: (\Rightarrow) Suppose \mathcal{M} is a model of $\text{NW} + \text{SE}$ and g_1, \dots, g_k are elements in \mathcal{M} satisfying $\psi(x_1, \dots, x_k)$. For $0 \leq i, j \leq k$, we let

$$g_i \sim g_j \quad \text{if } \text{Pred}_{\in^M}(g_i) \setminus \{g_1, \dots, g_k\} = \text{Pred}_{\in^M}(g_j) \setminus \{g_1, \dots, g_k\},$$

where $\text{Pred}_{\in^M}(a) = \{b \in M : b \in^M a\}$.

Since $D = \bigcup \{\text{Pred}_{\in^M}(g_i) \setminus \{g_1, \dots, g_k\} : 0 \leq i \leq k\}$ is a differentiating set for \sim , i.e., if $g_i \not\sim g_j$ then there is $d \in D$ such that $d \in^M g_i \leftrightarrow d \notin^M g_j$, by [13]

there are elements d_1, \dots, d_{k_d} in D such that $k_d < k$ and $\{d_1, \dots, d_{k_d}\}$ is a differentiating set for \sim . Letting $R_0 = \in^M \upharpoonright_{\{g_1, \dots, g_k, d_1, \dots, d_{k_d}\}}$ and $\mathcal{G} = (G, R_0)$ where $G = \{g_1, \dots, g_k, d_1, \dots, d_{k_d}\}$; it is obvious that g_1, \dots, g_k satisfy ψ in \mathcal{G} .

Furthermore we claim that \mathcal{G} is strongly extensional on $\{g_1, \dots, g_k\}$. Assume that R is a weak bisimulation on $\{g_1, \dots, g_k\}$ such that if $g_i R g_j$, then g_i and g_j have the same R_0 -predecessors on $G \setminus \{g_1, \dots, g_k\}$. Since \mathcal{G} is finite and $\mathcal{M} \models \text{NW}$, there is an element R^M in \mathcal{M} such that $(a, b)^M \in^M R^M$ iff $(a, b) \in R$, where $(a, b)^M$ is the natural interpretation in \mathcal{M} of the ordered pair operation, i.e., $(a, b)^M$ is $\emptyset^M \mathbf{w}^M a \mathbf{w}^M b$ ($\emptyset^M \mathbf{w}^M a \mathbf{w}^M b$). Due to the choice of d_1, \dots, d_{k_d} it is immediate that R^M is also a weak bisimulation from the point of view of \mathcal{M} . Since $\mathcal{M} \models \text{SE}$, R^M must be an identity relation; i.e., if $(a, b)^M \in^M R^M$ then $a = b$. From that it follows that R is the identity relation on $\{g_1, \dots, g_k\}$ and our claim is proved.

(\Leftarrow) Let $G = \{g_1, \dots, g_k, d_1, \dots, d_k\}$ with $k_d < k$. Let H be the Herbrand universe over $g_1, \dots, g_k, d_1, \dots, d_k, \underline{0}, \mathbf{w}$ where $\underline{0}$ is a new constant and \in^H is defined on H as in the proof of Proposition 4.1. Let π be the strongly extensional quotient of (H, \in^H) in (V_f, \in_f) , namely the unique system map from (H, \in^H) into (V_f, \in_f) . Since π is a system map, for every $a \in H$ $|\pi(a)| \leq |\{b \in H : b \in^H a\}|$. Furthermore, since there is no bisimulation relating two distinct hereditarily finite sets, for $i, j \in \omega$, if $i \neq j$ then $\pi(i) \neq \pi(j)$ and $|\pi(i)| = i$. From that as in the proof of Proposition 4.1, it follows that

- (*) π is 1-1 on $\{d_1, \dots, d_k\}$ and
- (**) $\pi(d_j) \neq \pi(g_h)$ for $1 \leq j \leq k_d$ and $1 \leq h \leq k$.

Assume that $\pi(g_h) \neq \pi(g_\ell)$, $1 \leq h, \ell \leq k$. Since π is a system map, from $\pi(g_h) \neq \pi(g_\ell)$ it follows that there exists a weak bisimulation R on H such that $g_h R g_\ell$. The restriction of R to $\{g_1, \dots, g_k\}$ is a weak bisimulation. That easily follows from the following straightforward consequences of (*) and (**):

1. if $e_0 \in \{g_1, \dots, g_k\}$, $e_1 \in G$ and $e_0 R e_1$ then $e_1 \in \{g_1, \dots, g_k\}$;
2. if $g_i R g_j$, $1 \leq i, j \leq k$, then g_i and g_j have the same predecessors in $\{d_1, \dots, d_k\}$.

Since $\in^H \upharpoonright_G = R_0$, $\in^H \upharpoonright_G$ is strongly extensional over $\{g_1, \dots, g_k\}$ in \mathcal{G} ; from (2) it follows that R must be the identity over $\{g_1, \dots, g_k\}$, hence $g_h = g_\ell$. Hence π is 1-1 over all of $\{g_1, \dots, g_k, d_1, \dots, d_k\}$. The same argument used in the proof of Proposition 4.1 and Proposition 5.1 then shows that $\pi(g_1), \dots, \pi(g_k)$ satisfy ψ in (V_f, \in_f) .

As an immediate consequence of Propositions 2.3 and 6.3 we have the following.

Proposition 6.4

1. The problem of establishing whether a $\forall_{0,0}$ -formula is satisfiable with respect to $\text{NW} + \text{SE}$ is decidable.
2. $\text{NW} + \text{SE}$ is decidable with respect to (the derivability of universal closures of) open formulas in $=, \in, \emptyset, \mathbf{w}$.

Since (V_f, \in_f) is a model of $\text{NW} + \text{AFA}'$, the proof of Proposition 6.3 has as byproduct the following conservativity result.

Proposition 6.5 $\text{NW} + \text{AFA}'$ is a conservative extension of $\text{NW} + \text{SE}$ with respect to (the derivability of) universal closures of formulas in $=, \in$, involving only restricted

existential quantifiers, as well as with respect to universal closures of open formulas in $=, \in, \emptyset, \mathbf{w}$.

Therefore the decidability results established in Proposition 6.4 apply to $\text{NW} + \text{AFA}'$ as well, with the same decision tests.

Finally, combining the proof of Proposition 6.3 with Proposition 6.5 we have as an immediate consequence the following completeness result.

Proposition 6.6 *NW + AFA' is complete with respect to existential closures of $(\forall)_{0,0}$ -formulas and with respect to universal closures (equivalently existential closures) of open formulas in $=, \in, \emptyset, \mathbf{w}$.*

Remark 6.7 The above completeness result does not hold for the theory $\text{NW} + \text{E} + \text{AFA}$.

Let N be the closure under \mathbf{w} of $V_f \cup \{v_1, v_2\}$, with $v_1 \neq v_2$, $v_1, v_2 \notin V_f$. Let \in' be the expansion of \in_f over N obtained by letting $a \in' v_1$ and $a \in' v_2$ for every $a \in V_f$ and $v_1 \in' v_2$ as well as $v_2 \in' v_1$ and let \in^N the least expansion of \in' such that for all a, b, c in N , $b \in^N a \mathbf{w} b$ and if $c \in^N a$ then $c \in^N a \mathbf{w} b$.

As in the proof of Proposition 4.1 the quotient N^\sim of N with respect to the minimal reflexive co-bisimulation on (N, \in^N) , yields a model \mathcal{N} of NWE. Since \in_f is extensional on V_f , $a \in V_f$ can be identified with its equivalence class $[a]$. Therefore (V_f, \in_f) can be seen as a substructure of \mathcal{N} .

The same argument previously given for the model \mathcal{M} shows that for $G \in N^\sim$, if \mathcal{N} is a model of “ G is a graph,” then G has only finitely many \in^{N^\sim} -predecessors; to such a $G \in N^\sim$ corresponds a finite graph which has a decoration d in V_f . Since d is finite and $N^\sim \models \text{NW}$, in N^\sim there is an element d^{N^\sim} which satisfies, in N^\sim , the property of being a decoration of G . Furthermore there is no other element \hat{d}^{N^\sim} in N^\sim which is also a decoration of G , since otherwise d^{N^\sim} and \hat{d}^{N^\sim} would give rise to an element R of N^\sim satisfying in N^\sim the property of being a bisimulation relation among two different elements of N^\sim (see [1]); the existence of such an R can be ruled out using the same argument given for the model \mathcal{M} .

The elements v_1 and v_2 satisfy in N^\sim the following $\forall_{0,0}$ -formula ψ :

$$\forall x \in u_1 (x \neq u_2 \rightarrow x \in u_2) \wedge \forall x \in u_2 (x \neq u_1 \rightarrow x \in u_1) \wedge \\ u_1 \in u_2 \wedge u_2 \in u_1 \wedge u_1 \neq u_2.$$

However ψ is not satisfiable in V_f . Therefore the existential closure of ψ is neither provable nor refutable in $\text{NW} + \text{E} + \text{AFA}$. Note that in N^\sim , $\{(v_1, v_2)^{N^\sim}\}^{N^\sim}$ satisfies the condition of being a proper weak bisimulation, so that $\mathcal{N} \not\models \text{AFA}'$.

Finally let us remark that, although every model of $\text{NW} + \text{SE}$ is extensional, the strong extensionality of a model of NW cannot be ensured by the validity of any set of first order sentences. In fact, if T is a theory consistent with NW then, by compactness, the theory

$$\text{NW} + T \cup \{c_i = c_{i+1} \mathbf{w} c_{i+1} : i \in \omega\} \cup \{c_i \neq c_j\},$$

where $\{c_i : i \in \omega\}$ is a set of new constants, has a model \mathcal{M} . Since the relation $\{(c_{i+1}^{\mathcal{M}}, c_i^{\mathcal{M}}) : i \in \omega\}$ is clearly a proper bisimulation on \mathcal{M} , \mathcal{M} is not strongly extensional.

- [10] Parlamento, F., and A. Policriti, “Expressing infinity without foundation,” *The Journal of Symbolic Logic*, vol. 56 (1991), pp. 1230–1235. [Zbl 0744.03051](#) [MR 93a:03052](#) 4
- [11] Parlamento, F., and A. Policriti, “The decision problem for restricted universal quantification in set theory and the axiom of foundation,” *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 38 (1992), pp. 143–156. [Zbl 0794.03016](#) [MR 95c:03029](#) 2, 4
- [12] Parlamento, F., and A. Policriti, “Undecidability results for restricted universally quantified formulae of set theory,” *Communication on Pure and Applied Mathematics*, vol. 46 (1993), pp. 57–73. [Zbl 0797.03005](#) [MR 94d:03088](#) 1
- [13] Parlamento, F., A. Policriti and K. P. S. B. Rao, “Witnessing differences without redundancies,” Technical Report 23/93, Università di Udine, 1993. [Zbl 0857.03027](#) [MR 97d:04003](#) 4, 6
- [14] Tarski, A., and W. Szmielew, “Mutual interpretability of some essentially undecidable theories,” p. 734 in *Proceedings of the International Congress of Mathematicians*, vol. 1, Cambridge University Press, Cambridge, 1950. 1
- [15] Tarski, A., A. Mostowsky and R. M. Robinson, *Undecidable Theories*, North-Holland, Amsterdam, 1953. [Zbl 0053.00401](#) [MR 15,384h](#) 1
- [16] Ville, F., “Décidabilité des formules existentielles en théorie des ensembles,” pp. 513–516 in *C. R. Acad. Sc. Paris*, 1971. [Zbl 0218.02044](#) [MR 43:4659](#) 5

Dipartimento di Matematica e Informatica

Università di Udine

Via delle Scienze 206

33100 Udine

Italy

email: belle@dimi.uniud.it

email: parlamen@dimi.uniud.it