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An Approach to Uncertainty via Sets of Truth Values

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Abstract An approach to the treatment of inference in the presence of uncertain truth values is described, based on representing uncertainties by sets of ordinary (certain) truth values. Both the algebraic and the logical aspects are studied for a variety of lattices used as truth value spaces in the domain of many-valued logic.

1 Introduction The paper is devoted to one facet of the problem of reasoning with imperfect information, namely when some of the available knowledge is *uncertain*. Uncertain truth values are modeled by sets of ordinary truth values. In order to explain the intuitions behind such an approach we need to recall some facts about classical logic.

1.1 The classical picture Let us begin with an outline of the classical logical doctrine concerning reasoning. According to it logic deals with *correct* reasoning, this notion being explicated as referring to transformations of statements which, if applied to true ones, lead to true statements, hence the importance of *truth values*. The basic thesis of classical formal logic concerning truth values seems to be that *in every epistemic situation a well-formed statement A is always assumed either true or false*, but not both, although sometimes the exact truth value is (temporarily!) unknown. Moreover, the truth value of a compound statement is recoverable from its syntactic structure and the truth values of the components (although this might lead to enquiries about other epistemic situations).

Thus,

1. the problem of how the truth values are obtained is radically separated from the ontological problem of their existence;
2. the definiteness (certainty) of truth values regardless of any difficulties in their actual establishing is assumed; and

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3. in a sense perfect information about every conceivable (even remote) situation is postulated, independent from the state of the *observer*.

For the formal implementation of the above doctrine one associates with an epistemic situation a *truth assignment* (a semantical evaluation function) ν which assigns each statement A a *definite* truth value from the set {**true**, **false**} ($\nu : Fml \to$ {**true**,**false**}). For typographical reasons we use below 1 instead of **true** and 0 instead of **false**. Assuming the usual interpretation of the classical connectives, i.e., assuming that all connectives are *truth-functional*, this set (the smallest possible logical matrix) is the Boolean algebra:

$$\mathbf{2} = \langle \{0, 1\}, \wedge, \vee, \neg, 0, 1\rangle.$$

In this way *classical semantics* is represented by some set H of homomorphisms into **2**. The definition of semantic consequence relation: $\Gamma \models A$ (where Γ is a set of statements and A is a statement), if $\forall \nu \in H(\forall B \in \Gamma(\nu(B) = 1) \to \nu(A) = 1)$, captures the basic intuition about sound inference: that it should transmit the truth forward, i.e., if all hypotheses of an inference are true (in a situation) then the conclusion should also be true (in the same situation). However, for reasoning involving intensional connectives (not truth-functional in **2**) like modalities $\Box_i$, tense operators, etc., a more sophisticated version is needed.

Example 1.1 Here we allow many epistemic situations, or *possible worlds*, with several accessibility relations between them (but keeping them all binary); thus we can accommodate most of the unary intensional connectives (modal, temporal, deontic, etc.) and some of the binary ones such as conditionals, data connectives, etc. In this approach a *frame* $\mathcal{F}$ is a tuple $\langle W, \{R_i\}_{i\in I}\rangle$, of which,

1. W is a nonempty set of *possible worlds*;
2. R_i are binary relations in W, i.e., $R_i \subseteq W \times W$.

A *model* $\mathcal{M}$ (on a frame $\mathcal{F}$) is a pair $\langle \mathcal{F}, \varphi\rangle$ where φ is a truth assignment (*valuation function*), i.e.,

$$\varphi : W \times Var(\mathcal{L}) \to \mathbf{2}.$$

In a model $\mathcal{M}$ the function φ can be extended to a mapping $\varphi_M : W \times \mathcal{L} \to \mathbf{2}$ by the well-known truth conditions for different connectives, for example (writing $\mathcal{M}, w \models A$ instead of $\varphi_M(w, A) = 1$):

$$\begin{aligned} \mathcal{M}, w \models A \wedge B \quad &\text{iff} \quad \mathcal{M}, w \models A \text{ and } \mathcal{M}, w \models B, \text{ or} \\ \mathcal{M}, w \models \Box_i A \quad &\text{iff} \quad \forall w'(wR_iw' \Rightarrow \mathcal{M}, w' \models A), \text{ etc.} \end{aligned}$$

Writing $||A||_{\mathcal{M}}$ for $\{w : \mathcal{M}, w \models A\}$ we get a mapping of $\mathcal{L}$ into the intensional algebra $\mathbf{A}(\mathcal{F})$ of the frame $\mathcal{F}$, i.e., the algebra of all subsets $\langle \wp(W), \cap, \cup, \ldots, \{\Box_i\}_{i\in I}\rangle$, where the intensional (e.g., modal, temporal, etc.) operations are defined as, e.g., $\Box_i Z = \{w : \forall w'(wR_iw' \Rightarrow w' \in Z)\}$. $||.||_{\mathcal{M}}$ is a homomorphism: $||A \wedge B|| = ||A|| \cap ||B||, \ldots, ||\Box_i A|| = \Box_i ||A||$. We denote $||A||_{\mathcal{M}} = W$ by $\mathcal{M} \models A$ and the fact that for all models $\mathcal{M}$ based on $\mathcal{F}$, $\mathcal{M} \models$ A, by $\mathcal{F} \models A$.

The important point for our exposition is that a possible worlds frame $\mathcal{F}$ is synonymous with an intensional (modal) algebra $\mathbf{A}(\mathcal{F})$, whereas a model $\mathcal{M}$ corresponds to a homomorphism of $\mathcal{L}$ into $\mathbf{A}(\mathcal{F})$, i.e., a member of $\mathrm{Hom}(\mathcal{L}, \mathbf{A}(\mathcal{F}))$. In this way all connectives become again truth-functional, though in respect to another (more complex) logical matrix, in which the truth values are sets of possible worlds traditionally called *propositions*.

Given a class of such models there are several possibilities for defining the notion of *semantic consequence*, e.g.,

1. $\Gamma \models_0 A$ iff $\forall \mathcal{M} \forall w \in W(\forall B \in \Gamma(w \models B) \Rightarrow w \models A)$;
2. $\Gamma \models_1 A$ iff $\forall \mathcal{M}(\forall B \in \Gamma(\mathcal{M} \models B) \Rightarrow \mathcal{M} \models A)$.

Expressed in algebraic terms these conditions become:

1′. $\Gamma \models_0 A$ iff $\forall \mathcal{F} \forall h \in H \subseteq \mathrm{Hom}(\mathcal{L}, \mathbf{A}(\mathcal{F}))(h(\wedge\{B : B \in \Gamma\}) \leq h(A))$

2′. $\Gamma \models_1 A$ iff $\forall \mathcal{F} \forall h \in H \subseteq \mathrm{Hom}(\mathcal{L}, \mathbf{A}(\mathcal{F}))(\forall B \in \Gamma(h(B) = 1) \Rightarrow h(A) = 1)$.

As is well known, the first of these consequence operations is suitable for reasoning in relational models. In the present paper we concentrate on the second possibility, which is familiar mainly from the so-called *matrix approach* in the study of many-valued logics (cf. Brown and Suszko [4], Wojcicki [29].

1.2 Criticism The above notions of truth and semantic consequence can be questioned on several points.

The first goes back in time to the intuitionistic criticism of the classical approach to mathematical truth. It questions the rationality of assuming that one can *always* assign a truth value to a particular statement (and hold this as a methodological principle when dealing with still unsettled mathematical problems). Such a criticism leads to admitting statements which are *undefined*. Analyzing the notion of algorithm, in particular the statements one can make concerning their behavior, Kleene came up with the “strong Kleene truth tables” (cf. [17]) that included *undefined* as a third possibility, but even earlier Łukasiewicz had introduced the third value when investigating the status of statements about contingent future events (there is an obvious connection between these two concerns). This opened the door to considering the truth values as partial objects and to applications of fix point techniques. For example in the theory of truth developed by Kripke [18] and others (e.g., Visser [28]), the fixed points of certain monotone operators on the family of all truth assignments were studied. The importance of the relation of “being more defined” and its connection with the “being more true” relation began gradually to emerge.

Another point on which the classical view has been questioned is the contention, having its origin even before Aristotle, that no statement is both true and false (in one and the same epistemic situation). Arguments put forward by the like of Hegel, Wittgenstein, etc., seem to show that this is open to a discussion. Some recent publications give expositions of what can be done abandoning the view that “everything is consistent” and have spoken of the “consistency of the world” problem, cf., e.g., Priest [21],[22], Rescher and Brandon [24]. Nevertheless the assumption of such a consistency, equivalent to the well-known *law of noncontradiction*, is considered by the majority of logicians as the final and indisputable principle of logic beyond which there is absolutely no ground for a rational epistemic activity, cf. Lewis [19].

Philosophically speaking the consistency and completeness of knowledge are determined by its “correspondence” to the “outside world.” Thus contradictions may be the result of:

- defects in the correspondence,
- defects in the knowledge,
- defects in the world.

Concentrating on the defects in knowledge, it is an interesting problem what reasoning procedures can be developed in order to accommodate the possibility of contradictory statements. The simplest option is to permit statements to be both true and false and keep this as the *only* possibility beyond the classical assumptions. This leads to a picture where for a statement A and an epistemic situation we have just three ways with the truth value: *A is only true*; *A is only false*; *A is both true and false*. Formally this approach can be described by truth assignments into the set $\{\{0\}, \{1\}, \{0, 1\}\}$, as done e.g., by Priest in [21], [22]. The consequence relation generated by the algebra **3** above tolerates inconsistencies in the sense that there is no general inference of an arbitrary statement from a contradiction.

A further step is to combine the assumptions of partiality and contradictoriness. By this step we arrive at a class of assignments that have values in the set $\{\varnothing, \{1\}, \{0\}, \{1, 0\}\}$. In this case the corresponding consequence relation is also contradiction tolerant. The arising logic with two *designated* truth values, $\{0, 1\} = \mathbf{Both}$ and $\{1\} = \mathbf{True}$, is also well known and has been extensively studied, e.g., by Belnap [2],[3], etc. Recently this logic has found applications in computer science as a suitable basis for studying the semantics of the programming languages.

An obvious way to make Example 1.1 more "realistic" in the above respect is to admit either partial or contradictory models, or both. This has been done by many, in particular by Ginsberg [16], who recently promoted a notion incorporating most of the ideas discussed above. His *bilattices* (algebras with two complete lattice orders) were intended to combine model-theoretic and computational advantages in treating reasoning with imperfect information: they could be used either as conventional logical matrices, or as in denotational semantics as a background for fixed point calculations (in the latter case truth value assignments do not presuppose the truth functionality of any logical connective—an important point for nonmonotonic inference). Belnap's algebra **4** is the simplest bilattice. For its four elements Belnap indicated the "sets of ordinary truth values" interpretation: *if $a \in x$ for an $x \in \mathbf{4}$, then a was a possible ordinary truth value for the statement having value x*. The rules Belnap gave for manipulating logical connectives, though, are different from the rules generated by our interpretation below, mainly in the treatment of the value $\mathbf{None} = \varnothing$.

1.3 Truth-value spaces Along the path indicated by the above criticism of the classical semantical schema we arrive at the notion of *truth value space*. The classical spaces (spaces for classical logic) were in general Boolean algebras with additional operators representing intensional connectives occurring in the language. Early examples of nonclassical spaces were the pseudo-Boolean algebras, Post algebras, the unit interval $[0, 1]$ in fuzzy logic, etc.

From the very beginning deviations from the classical scheme were justified by appealing not only to the *uncertainty* of information (on the basis of which the decision to declare something true is taken) but to the *indefiniteness* of data, *vague-*

ness (fuzziness) of notions, i.e., all kinds of imperfections in the available knowledge or lack of suitable knowledge due to difficulties in understanding (subjective nonsignificance), and even objective nonsignificance (as for example did Bochvar, cf. Finn et al. [7], who studied propositions in the foundations of mathematics that destroyed any theory they appeared in). The truth-value spaces reflected in their internal structure different views and assumptions (philosophical, mathematical, logical, pragmatic, etc.) concerning truth and inference. But there seem to be some general features common to all known examples of truth spaces: they represent methods of evaluation of information, i.e., truth values of statements are determined on the basis of the available information. We can even in general identify them with the *available relevant information* (about the state of affairs described or referred to by the statement).

This information can be characterized in two ways:

truth degree: reflecting the truth content of a statement. No doubt here we need a theory of truth (e.g., correspondence theory, or any other coherent view on how information is to be considered true, on the necessity of an external world, etc.) but clearly truth degrees generate a partial order among truth values.

degree of knowledge: reflecting the definiteness of information or the completeness of the knowledge about the truth value (this could involve an estimation how reliable the information is, indications whether we find it plausible, etc.).

1.4 Sets of truth values as generalized uncertain truth values (background) In this framework a way to account for the *uncertainty* of knowledge is to consider *sets* of truth values, e.g., sets of propositions, as representatives of the "not perfectly known" truth-value of a statement. We find analogous ideas in fields like fuzzy set theory and logic (cf. Atanassov and Gargov [1]), probabilistic logic (cf. Calabrese [5], Dempster [6], Gärdenfors [10]), AI (cf. Sandewall [25]), many-valued logic (cf. Gargov [11],[12]), etc. Here we propose a codification of such uses in the notion of set expansion of a given truth value space. Set expansions consist of sets representing the *possible truth values a statement can have according to the available information*; consequently degree of knowledge ordering is inverse set inclusion, while truth degree ordering is somewhat more complicated and depends on the concrete problem.

Our idea stems from an early attempt of Vakarelov [26] who explored certain schema for obtaining relative semantics, later developed and applied to various nonclassical systems in Gargov [11],[12],[14], Gargov and Radev [15], and Vakarelov [27].

Put very briefly, the schema consisted in the following: take a propositional language $\mathcal{L}$ and let $\mathcal{L}_1$ be any other language with counterparts to all the connectives of $\mathcal{L}$ (and possibly some additional ones). Assume that **Sem** is a semantics for $\mathcal{L}_1$, i.e., that for $s \in$ **Sem** we may in principle decide whether a formula a is *true at s* (denoted by $s \models a$) or not, transfer **Sem** to formulas of $\mathcal{L}$ by means of (finite) sets of $\mathcal{L}_1$ formulas using *interpretation functions i* which assign to each $\mathcal{L}$-formula a set of $\mathcal{L}_1$-formulas, the following condition being satisfied:

for a connective $o(A_1, \ldots, A_n)$ of $\mathcal{L}$,

$$i(o(A_1, \ldots, A_n) = \{o(a_1, \ldots, a_n) : a_k \in i(A_k), k = 1, \ldots, n\}.$$

Let *Int* denote the set of all interpretation functions and Int_0 the set of those interpretation functions which do not contain the empty set in their ranges. Call a pair (s, i) an *interpretation index*. Formulas in $\mathcal{L}$ can be evaluated at an index according to one of the following rules (but there certainly are other possibilities for evaluation, some of which were considered in [14], [15], among them the *majority strategy* according to which A is accepted if the majority of members of $i(A)$ are true).

$$A \text{ is true}_1 \text{ at } (s, i) \quad \text{iff} \quad \forall \alpha \in i(A)\ s \models \alpha;$$
$$A \text{ is true}_0 \text{ at } (s, i) \quad \text{iff} \quad \exists \alpha \in i(A)\ s \models \alpha.$$

For Γ, a set of $\mathcal{L}$-formulas, and A, an $\mathcal{L}$-formula, say that A is a $(\mathbf{Sem}, Int)_n$ consequence of Γ $(n = 0, 1)$, if for all indices (s, i), if all $B \in \Gamma$ are true$_n$ at (s, i), then A is also true$_n$ at (s, i). There are several intuitions behind the schema. For example, truth$_1$ can be associated with the notion of *disambiguation* (treated by Lewis [19]): *a proposition is assumed true if all its possible disambiguations are true*. In general disambiguations are statements formulated in a language different from the original one, but on the other hand they follow closely the structure of the proposition disambiguated. For truth$_0$ one has the notion of *justification*: a statement may be considered true iff there is at least one true justification of this statement (cf. [14], [15]). The justifications of a statement can be formulated in a completely different language, but the conditions upon the interpretation functions presuppose a very strict correspondence between the propositional structure of a statement and the structure of its justifications.

When applied to the classical propositional language (equipped with the ordinary **2**-semantics) the schema gives consequence relations related to some three- and four-valued logics (cf. Gargov [13]). For instance $(\mathbf{2}, Int_0)_1$ is Kleene's original three-valued consequence relation (coinciding with Łukasiewicz' for the basic language, cf. [26]), in [11] it was proved that $(\mathbf{2}, Int_0)_0$ is the consequence operation of the three-valued logic of Priest [21]. $(\mathbf{2}, Int)_1$ and $(\mathbf{2}, Int)_0$ were studied in [13] where the corresponding logics were formalized in a natural deduction style.

1.5 What is to follow In the present paper we treat in the spirit of Rasiowa and Sikorski [23] the mathematics (part 1) and logic (part 2, which for simplicity of presentation we restrict to propositional languages) of the set expansions. The many-valued logics determined by different subclasses of these depend on a number of parameters. One of the goals of the paper is to present a classification of the corresponding logics.

2 Set expansions of lattices of truth values: algebraic aspects Let $\mathbf{L}$ be a bounded lattice $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$. Elements of $\mathbf{L}$ represent truth values, e.g., they can be elements of an abstract logical matrix, or sets of possible worlds, etc. We assume also any number of additional *finitary* operations $o(x_1, \ldots, x_n)$. For example, we could have unary operations like negation (in the case of de Morgan lattices and Boolean algebras), modalities, or a binary operation of implication when we consider Heyting algebras, etc.

The basic idea explored in this paper is to view subsets $X, Y, Z, \ldots$ of L as new, "expanded" truth values. A set X could be said to embody the knowledge an observer has about the "real" truth value of a statement A, so $||A|| \in X$, where $||A||$ is the "real" truth value. Consequently, it can be viewed as an infinitary disjunction $\bigvee\{||A|| = x : x \in X\}$. In this section we develop the algebraic aspects of such an approach.

2.1 Internal operations Set expansions of lattices will be introduced step by step. As a first step we define a class of operations which are expanded counterparts of the operations in the basic algebra.

Definition 2.1 The set expansion $\mathbf{L}^{\mathbf{set}}$ of $\mathbf{L} = \langle L, \wedge, \vee, 0, 1, \{o_i\}_{i \in I}\rangle$ is an algebra based on the set of all subsets of $L - \varrho(L)$ and having the following *internal* operations:

$$o(X_1, \dots, X_n) = \{o(x_1, \dots, x_n) : x_k \in X_k, k = 1, \dots, n\},$$

where $o(x_1, \dots, x_n)$ is an operation of $\mathbf{L}$, e.g.,

$$\begin{aligned} X \wedge Y &= \{x \wedge y : x \in X, y \in Y\} \\ X \vee Y &= \{x \vee y : x \in X, y \in Y\} \\ 1 &= \{1\} \\ 0 &= \{0\}. \end{aligned}$$

And if, say, the lattice $\mathbf{L}$ has a negation $\neg$ or a modality operator $\Box$, then

$$\neg X = \{\neg x : x \in X\}$$
$$\Box X = \{qx : x \in X\}.$$

The set expansion contains another pair of remarkable elements:

$$\top = \varnothing, \quad \bot = L.$$

These are called *external* constants of $\mathbf{L}^{\mathbf{set}}$ in contrast to the internal constants $0, 1$.

Remark 2.2 The internal expansions of the lattice operations reflect a certain view on the interaction of information about the truth values of components of compound sentences, namely on how the structure of the compound formula guides us as to the set of possibilities for its "real" truth value. Put briefly: *individual possible truth values interact completely independently from each other*.

Below we list several elementary properties of the introduced operations.

Proposition 2.3

1. $X \wedge Y = Y \wedge X,\ X \vee Y = Y \vee X$;
2. $X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z$,
   $X \vee (Y \vee Z) = (X \vee Y) \vee Z$;
3. $X \wedge 1 = X,\ X \vee 0 = X$.
   In the case when **L** *is a de Morgan lattice we would have also:*
4. $\neg\neg X = X$,
   $\neg\,(X \wedge Y) = \neg X \vee \neg Y$,
   $\neg\,(X \vee Y) = \neg X \wedge \neg Y$,
   etc.

Proof: The proof is easy: investigate the form of a typical element of the left-hand side and show that it can be transformed into an element of the right-hand side and vice versa.

Unfortunately not all identities concerning internal operations and valid in **L** are preserved in $\mathbf{L}^{\mathbf{set}}$, most notably the lattice laws of idempotence, absorption, and (if **L** happens to be distributive) distributivity. Let us look into this problem more closely. We start with a partial list of identities that are not in general preserved in the set expansion of a lattice **L**:

$$X \wedge 0 = 0, \;\; X \vee 1 = 1$$

do not hold in the set expansion of **2**, whereas identities like

$$\begin{aligned} X \wedge X &= X, \;\; X \vee X = X, \\ X \wedge (X \vee Y) &= X, \;\; X \vee (X \wedge Y) = X, \\ X \wedge (Y \vee Z) &= (X \wedge Y) \vee (X \wedge Z), \\ X \vee (Y \wedge Z) &= (X \vee Y) \wedge (X \vee Z), \;\text{etc.}, \end{aligned}$$

can all be refuted in the set expansion of the four element Boolean algebra. Such observations lead to a natural question: *which identities in the above operations are preserved under set expansion*?

2.2 Preservation of identities In order to formulate a partial answer we need some definitions and basic facts. Consider *internal* terms $s, t, \ldots$, i.e., terms built from variables $v_1, v_2, \ldots$, and symbols of internal operations and constants. Such terms can be evaluated both in the lattice **L** and the expansion $\mathbf{L}^{\mathbf{set}}$. If an evaluation function ν assigns to v_i an element X_i of $\varrho(L)$ we write the value of a term $s(v_1, v_2, \ldots, v_n)$ under ν as $s(X_1, X_2, \ldots, X_n)$. An expression like $s(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n \in L$, should be understood in the same way. Note that for the value of an internal term $s(X_1, \ldots, X_n)$, if $X_i = \top$, then $s = \top$, and vice versa; if all $X_i \neq \top$, then $s \neq \top$. The variables occurring in a term t form a set $\mathit{Var}(t)$. Call a term *linear*, if all its variables have single occurrences.

Let now **K** be a class of lattices. An equation $s = t$ is a **K**-identity if it is valid in all members of **K**. A characterization of all $\mathbf{K}^{\mathbf{set}}$-identities in terms of **K**-identities can be obtained for certain classes of lattices. Clearly a $\mathbf{K}^{\mathbf{set}}$-identity is also a **K**-identity since any valuation ν for a lattice **L** can be transformed into a valuation ν^* for $\mathbf{L}^{\mathbf{set}}$ by putting $\nu^*(X) = \{\nu(x)\}$, i.e., taking the corresponding singletons. This construction has the following property:

$$\nu^*(s) = \{\nu(s)\}.$$

Thus any refuted in **K** equation is refuted also in $\mathbf{K}^{\mathbf{set}}$: $\nu(s) \neq \nu(t) \Rightarrow \nu^*(s) \neq \nu^*(t)$.

Definition 2.4 An identity $s = t$ is **K**-*linear* if there exist a substitution σ and an equation $u = v$ such that $s = u^\sigma$, $t = v^\sigma$, $u = v$ is a **K**-identity, and both u and v are linear. An identity $s = t$ is *balanced*, if $\mathit{Var}(s) = \mathit{Var}(t)$.

Definition 2.5 An inequality $s \neq t$ is *satisfiable in a class* **K** of lattices if there is a lattice $\mathbf{L} \in \mathbf{K}$ and a valuation ν into **L** such that $\nu(s) \neq \nu(t)$. A family of inequalities $\{s_i \neq t_i\}_{i \in I}$ is *simultaneously satisfiable* in **K** if there exist a lattice $\mathbf{L} \in \mathbf{K}$ and a valuation ν into **L** such that $\nu(s_i) \neq \nu(t_i)$ for all $i \in I$.

Theorem 2.6 *Let* **K** *be a class of lattices such that any finite family of satisfiable in* **K***-inequalities is simultaneously satisfiable in* **K***. Then the following two conditions are equivalent:*

1. $s = t$ *is a* $\mathbf{K}^{\mathbf{set}}$ *identity;*
2. $s = t$ *is balanced and* **K***-linear.*

Proof: Let $s = t$ be a balanced and **K**-linear identity. Note that, due to the fact that $s = t$ is balanced, when evaluated in an expansion $\mathbf{L}^{\mathbf{set}}$ s and t get values $\top$ in exactly the same instances. Thus in order to check if $s = t$ is an identity in $\mathbf{L}^{\mathbf{set}}$, it is sufficient to consider only valuations which do not have $\top$ in their range.

With the above restriction we have the following representation of the values of terms, when $s_1(v_1, \ldots, v_n)$ and $t_1(v_1, \ldots, v_n)$ are linear:

$$\begin{aligned} s_1(X_1, \ldots, X_n) &= \{s_1(a_1, \ldots, a_n) : a_i \in X_i\}, \\ t_1(X_1, \ldots, X_n) &= \{t_1(a_1, \ldots, a_n) : a_i \in X_i\}. \end{aligned}$$

Let now $s_1 = t_1$ be that **K**-identity of which $s = t$ is a substitutional instance. Since $s_1 = t_1$ is an identity in **L**, the above two sets are equal, so s_1 and t_1 get the same values in $\mathbf{L}^{\mathbf{set}}$, but then (as substitutional instances) s and t also get the same values. Thus $s = t$ is indeed a $\mathbf{K}^{\mathbf{set}}$-identity.

In the opposite direction we reason by contraposition: if $s = t$ is either not **K**-linear or not balanced, then it is not a $\mathbf{K}^{\mathbf{set}}$-identity. It is immediately clear that a non-balanced equation cannot be an identity in $\mathbf{L}^{\mathbf{set}}$. Thus we are left with the case of $s = t$ being not **K**-linear, which means that any $s_1 = t_1$, which is linear and of which $s = t$ is a substitutional instance, is not a **K**-identity, i.e., $s_1 \neq t_1$ is satisfiable in **K**. In particular, if we consider the term s^* obtained from $s(v_1, \ldots, v_k)$ by assigning to consecutive occurrences of a variable v_i different new variables $v_{i1}, v_{i2}, \ldots, v_{im_i}$, we get a family of satisfiable in **K** inequalities:

$$\{s^*(v_{11}, \ldots, v_{km_k}) \neq t^* : t^* \in \mathcal{T}\}.$$

Here $\mathcal{T}$ is the family of all linear terms t^* obtained from t by the procedure of replacing different occurrences of a variable v_i by different variables v_{ij} in all possible combinations. Clearly $\mathcal{T}$ is finite and the above family of inequalities has to be simultaneously satisfiable in **K**, so in some lattice $\mathbf{L} \in \mathbf{K}$ we have for some elements a_{ij}:

$$s^*(a_{11}, \ldots, a_{km}) \neq t^*(.., a_{ij}, \ldots).$$

Let $X_1 = \{a_{11}, \ldots, a_{1m_1}\}$, ..., $X_i = \{a_{i1}, \ldots, a_{im_i}\}$, etc. The claim $s(X_1, \ldots, X_k) \neq t(X_1, \ldots, X_k)$ follows from the observation that $s^*(a_{11}, \ldots, a_{km}) \in s(X_1, \ldots, X_k)$, but $s * (a_{11}, \ldots, a_{km_k}) \notin t(X_1, \ldots, X_k)$.

Remark 2.7 The property from Definition 2.5 is possessed by a variety of classes of lattices, e.g., any class of Boolean algebras containing arbitrary large finite Boolean algebras or some infinite ones, any class of pseudo-Boolean algebras with the same restrictions, etc.

Remark 2.8 A useful counterexample to a more liberal formulation of the above theorem is the class of all linear orders **Lin**, for which, e.g., $X \wedge X = X$ and $X \vee X = X$ are $\mathbf{Lin}^{\mathbf{set}}$-identities (since in a linear order $a \wedge b$ and $a \vee b$ equal either a or b).

2.3 External operations Since the set expansion of a lattice is built up from sets of elements, it is only natural to introduce also the set-theoretical operations:

$$\begin{aligned} X \oplus Y &= X \cap Y, \\ X \otimes Y &= X \cup Y, \end{aligned}$$

their infinitary versions $\sum$ and $\prod$, as well as the relation

$$X \leq_k Y \text{ iff } X \supseteq Y.$$

In $\mathbf{L}^{\mathbf{set}}$ it is also possible to consider the complement of a set X:

$$X^c = L \backslash X.$$

With such additional operations (called *external* to contrast them with the previous class of operations) set expansions turn into truth value spaces similar to the algebra **4** of Belnap and to bilattices (cf. [16]): $\leq_k$ represents ordering by degree of knowledge (the smaller the set, the more we know about the "real" truth value), $\oplus$ and $\otimes$ give us different combinations of information about this possible truth value: "accept anything" joining of the information in $\oplus$, and "consensus" reduction to the information common to both sets in $\otimes$. The analogy is not perfect though, as will be seen below. Note that the constant $\top$ represents a kind of nonsignificance value capable of destroying any internal statement (thus the information leading to $\top$ is not simply contradictory, but rather *senseless*).

Remark 2.9 Continuing this discussion of the intuitions behind sets of truth values let us point out that the notion of *sets of truth values as generalized truth values* has its origin in a simple observation: "X is the generalized truth value of A" means nothing more than "all we know at present is $||A|| \in X$." Maximal possible knowledge corresponds to singletons, defective (contradictory, nonsensical) knowledge leads to an empty set of possible truth values. On this path we are immediately confronted with the problem: how are the sets of possible values X, Y, Z *given*? For example they can be thought as given *directly* (enumerated, etc.) or they can be *represented* by certain conditions defining the sets (these conditions are usually restrictions on the possible truth values). Now the question arises as to the language in which such conditions are formulated, how are they verified, etc. Although quite important, especially in applications, we leave their detailed analysis aside due to lack of space. We can think of the information concerning the Xs as of a family of restrictions (primitive restrictions) on the elements of the sets. The *consensus* approach would combine two families of restrictions in such a way that any restriction that does not appear in both families would be dropped, so we would be left with only the restrictions common to X and Y. Now this guarantees that we get $X \cup Y$ as a result. A similar argumentation for the intersection though is not so conclusive; perhaps this is the cause of the problems with $\oplus$ in the algebraic treatment of set expansions.

2.4 External properties of set expansions Let us continue with further facts about set expansions. The internal operations are k-monotone, e.g.:

$$X \leq_k X', Y \leq_k Y' \text{ imply } X \vee Y \leq_k X' \vee Y', \text{ etc.}$$

Proposition 2.10 *With respect to the introduced external operations $\mathbf{L}^{\mathbf{set}}$ is a co-atomic, complete, and completely distributive Boolean algebra (the algebra of all subsets of* $\mathbf{L}$ *with inverse inclusion) with least element* $\bot$ *and greatest element* $\top$*. In particular the following laws hold:*

$$
\begin{aligned}
(\sum\{X_i : i \in I\}) \otimes Y &= \sum\{X_i \otimes Y : i \in I\}, \\
(\prod\{X_i : i \in I\}) \oplus Y &= \prod\{X_i \oplus Y : i \in I\}, etc.
\end{aligned}
$$

The relations between external and internal operations are quite complicated, e.g., the following hold only as inequalities with respect to the k-order:

$$
\begin{aligned}
X \wedge X \leq_k X, &\qquad X \vee X \leq_k X, \\
X \wedge (X \vee Y) \leq_k X, &\qquad X \vee (X \wedge Y) \leq_k X, \\
(X \wedge Y) \vee (X \wedge Z) \leq_k X \wedge (Y \vee Z), &\qquad (X \vee Y) \wedge (X \vee Z) \leq_k X \vee (Y \wedge Z),
\end{aligned}
$$

etc.

Lemma 2.11 *In the set expansion of any bounded lattice we have:*

$$
\begin{aligned}
X \wedge (Y \otimes Z) &= (X \wedge Y) \otimes (X \wedge Z), \\
X \vee (Y \otimes Z) &= (X \vee Y) \otimes (X \vee Z),
\end{aligned}
$$

and in general for any linear internal term $s(\ldots, X, \ldots)$*:*

$$s(\ldots, Y \otimes Z, \ldots) = s(\ldots, Y, \ldots) \otimes s(\ldots, Z, \ldots);$$

with an infinitary version:

$$s(\ldots, \prod\{X_i : i \in I\}, \ldots) = \prod\{s(\ldots, X_i, \ldots) : i \in I\}.$$

As for $\oplus$ - we can claim only that

$$
\begin{aligned}
(X \vee Y) \oplus (X \vee Z) &\leq_k X \vee (Y \oplus Z), \\
(X \wedge Y) \oplus (X \wedge Z) &\leq_k X \wedge (Y \oplus Z), \\
s(\ldots, Y, \ldots) \otimes s(\ldots, Z, \ldots) &\leq_k s(\ldots, Y \otimes Z, \ldots).
\end{aligned}
$$

For example the triple $X = 1$, $Y = 0$, $Z = 1$ shows the failure of an inequality opposite to the first one: $Y \oplus Z = \top$, so $X \vee (Y \oplus Z) = \top$ while $(X \vee Y) \oplus (X \vee Z) = 1$.

Example 2.12 The simplest set expansion is $\mathbf{2}^{\mathbf{set}}$: it has four elements, all of them signature constants: $0, 1, \bot$ and $\top$. With respect to $\oplus$, $\otimes$, and the complementation c $\mathbf{2}^{\mathbf{set}}$ is a Boolean algebra, though with respect to $\vee$, $\wedge$ it is not even a lattice since $\top$ is a *nonsignificance* value. $\mathbf{2}^{\mathbf{set}}$ can be considered with its internal negation $\neg$ (inherited from the Boolean algebra $\mathbf{2}$), it also has a sort of *conflation* operation (cf. Fitting [8], [9] for the definition and the properties of conflation in a bilattice setting) $-$, related to c and the internal negation $\neg$ by $-X = (\neg X)^c$ (by the way, in this structure this equals $\neg(X^c)$). In this way $-1 = 1$, $-0 = 0$, $-\bot = \top$, $-\top = \bot$ (just as a conflation should act), moreover we have $X \leq_k Y$ implies $-Y \leq_k -X$.

2.5 Singletons and other curiosities in $\mathbf{L}^{\mathbf{set}}$ In this subsection we present several examples of interesting expressive possibilities in $\mathbf{L}^{\mathbf{set}}$. Let us start with the observation that some relevant properties of subsets of L can be guaranteed by simple equations in $\mathbf{L}^{\mathbf{set}}$, e.g., $X \vee \perp = X$ defines all subsets of L that are *upward closed*; $X \wedge \perp = X$ defines all subsets of L that are *downward closed*; $X \vee X = X$ defines all $\vee$-closed subsets of L, i.e., subsets with the property $a, b \in X \Rightarrow a \vee b \in X$; $X \wedge X = X$ defines all $\wedge$-closed subsets, and in general $o(X, \dots, X) = X$ defines the sets that are closed with respect to the operation o.

The meaning of the "bilattice projection operations" (cf. Ginsberg [16] or Fitting [9]) $(x)_0$ and $(x)_1$ may also be of interest, so we mention that $(X)_0 = X \wedge \perp$ is the *downward cone* of X; $(X)_1 = X \vee \perp$ is the *upward cone* of X. The combinations

$$\begin{aligned}(X)_0 \oplus (X)_1 &= \{y : \exists x_0, x_1 \in X(x_0 \leq y \leq x_1)\} \text{ and}\\ (X)_0 \otimes (X)_1 &= \{y : \exists x_0, x_1 \in X(y \leq x_0 \text{ and } x_1 \leq y)\}\end{aligned}$$

give us the so-called *convex hull* and *cylinder* of X, respectively.

Using these observations one can give a definition of filters in $\mathbf{L}$ as the solutions of a simple system of equations:

$$\begin{aligned}X \vee \perp &= X \text{ (i.e., } (X)_1 = X)\\ X \wedge X &= X.\end{aligned}$$

For ideals there is a dual system:

$$\begin{aligned}X \wedge \perp &= X \text{ (i.e., } (X)_0 = X)\\ X \vee X &= X\end{aligned}$$

We can define singletons in $\mathbf{L}^{\mathbf{set}}$ as atoms in the lattice of set-theoretic operations, or, equivalently, as co-atoms in the $\leq_k$ order, where,

$$X \text{ is a co-atom iff } X \neq \top \text{ and } \forall Y(X \leq_k Y \Rightarrow X = Y \text{ or } Y = \top).$$

Singletons have the following characteristic properties:

$$\begin{gathered}\prod\{X_i : i \in I\} \leq_k \{x\} \Rightarrow \exists i X_i \leq_k \{x\};\\ X = \prod\{\{x\} : X \leq_k \{x\}\}.\end{gathered}$$

2.6 Homomorphisms of set expansions General algebraic considerations suggest the importance of studying homomorphisms of set expansions and in particular homomorphisms into the smallest such algebra $\mathbf{2}^{\mathbf{set}}$. As turns out, if homomorphisms $f : \mathbf{L}^{\mathbf{set}} \to \mathbf{M}^{\mathbf{set}}$ should respect the infinitary k-operations, i.e., $f(\sum\{X_i : i \in I\}) = \sum\{f(X_i) : i \in I\}$ and $f(\prod\{X_i : i \in I\}) = \prod\{f(X_i) : i \in I\}$, then the rigid structure of the set expansions leaves no room for variety, as witnessed by the following facts.

1. *Homomorphisms isolate* $\top$. For a homomorphism $f : \mathbf{L}^{\mathbf{set}} \to \mathbf{M}^{\mathbf{set}}$ and a singleton $\{x\}$ in $\mathbf{L}^{\mathbf{set}}$ one has $\{x\} \vee 1 = 1$, so $f(\{x\}) \vee 1 = 1$, which implies that $f(\{x\}) \neq \top$. Now using the representation $X = \prod\{\{x\} : x \in X\}$ we get $f(X) = \prod\{f(\{x\}) : x \in X\}$. Thus if $X \neq \top$, then $f(X) \neq \top$.

2. *Source singletons are mapped into special partitions of the target singletons.* If $x \neq y$, then in $\mathbf{L}^{\mathbf{set}}$ $\{x\} \oplus \{y\} = \top$, so $f(\{x\}) \oplus f(\{y\}) = \top$. Thus different singletons are mapped into disjoint elements of $\mathbf{M}^{\mathbf{set}}$. Moreover from $\prod\{\{x\} : x \in L\} = \bot$, we get that $\prod\{f(\{x\}) : x \in L\} = \bot$, so the restriction of f over the singletons generates a special partition of M: let $P_x = f(\{x\})$, then the family $\{P_x : x \in L\}$ is a partition of M of a special kind, i.e., besides $x \neq y \Rightarrow P_x \cap Py = \varnothing$ and $\cup\{P_x : x \in L\} = M$, we have also $P_x \vee P_y = P_{x \vee y}$; $P_x \wedge P_y = P_{x \wedge y}$, and in general $P_{o(x,\ldots,y)} = o(P_x, \ldots, P_y)$. Applied to homomorphisms of $\mathbf{2}^{\mathbf{set}}$ this has the following effect: *if* $\mathbf{L}$ *is different from* $\mathbf{2}$, *then there are no homomorphisms* $f : \mathbf{2}^{\mathbf{set}} \to \mathbf{L}^{\mathbf{set}}$.
3. *Epimorphisms are isomorphisms.* Epimorphisms of set expansions map singletons onto singletons: if $f(X) = \{z\}$ for a $z \in M$, then having, for any $x \in X$, $X \leq_k \{x\}$ and consequently $\{z\} \leq_k f(\{x\})$, by the properties of co-atoms in $\mathbf{M}^{\mathbf{set}}$, $\{z\} = f(\{x\})$, but that, in view of (2) above, implies $X = \{x\}$. Therefore there exist no epimorphisms of set-expansions except isomorphisms. Applied to homomorphism into $\mathbf{2}^{\mathbf{set}}$, which are all epimorphisms (since all elements of $\mathbf{2}^{\mathbf{set}}$ are signature constants), the latter fact yields the following corollary: *if* $\mathbf{L}$ *is different from* $\mathbf{2}$, *then there are no homomorphisms* $f : \mathbf{L}^{\mathbf{set}} \to \mathbf{2}^{\mathbf{set}}$.

2.7 The external modalities If it is assumed that $\mathbf{L}$ is complete, then one can introduce in $\mathbf{L}^{\mathbf{set}}$ a pair of unary operations: $\blacksquare X = \{\inf X\}$; $\blacklozenge X = \{\sup X\}$, with the following elementary properties.

$$\begin{array}{cccc} \blacksquare\blacksquare X = \blacklozenge\blacksquare X = \blacksquare X & & \blacklozenge\blacklozenge X = \blacksquare\blacklozenge X = \blacklozenge X & \\ \blacksquare 1 = 1 & \blacksquare 0 = 0 & \blacklozenge 1 = 1 & \blacklozenge 0 = 0 \\ \blacksquare \top = 1 & \blacksquare \bot = 0 & \blacklozenge \top = 0 & \blacklozenge \bot = 1 \end{array}$$

These can be easily checked as well as the next proposition.

Proposition 2.13 *In the set expansion of a complete lattice* $\mathbf{L}$ *the following identities hold:*

$$\blacksquare(X \otimes Y) = \blacksquare X \wedge \blacksquare Y, \quad \blacklozenge(X \otimes Y) = \blacklozenge X \vee \blacklozenge Y,$$

and their infinitary versions

$$\begin{array}{rcl} \blacksquare \prod\{X_i : i \in I\} & = & \wedge\{\blacksquare X_i : i \in I\}; \\ \blacklozenge \prod\{X_i : i \in I\} & = & \vee\{\blacklozenge X_i : i \in I\}. \end{array}$$

For $X, Y \neq \top$ *we have also:*

$$\nu(X \wedge Y) = \blacksquare X \wedge \blacksquare Y, \quad \blacklozenge(X \vee Y) = \blacklozenge X \vee \blacklozenge Y,$$

and the infinitary (for $X_i \neq \top$*)*

$$\begin{array}{rcl} \blacksquare \wedge \{X_i : i \in I\} & = & \wedge\{\blacksquare X_i : i \in I\}; \\ \blacklozenge \vee \{X_i : i \in I\} & = & \vee\{\blacklozenge X_i : i \in I\}. \end{array}$$

If $\mathbf{L}$ *is in addition completely distributive, then for* $X, Y \neq \top$

$$\blacksquare(X \vee Y) = \blacksquare X \vee \blacksquare Y, \quad \blacklozenge(X \wedge Y) = \blacklozenge X \wedge \blacklozenge Y,$$

and the infinitary laws (for $X_i \neq \top$):

$$\blacksquare \vee \{X_i : i \in I\} = \vee\{\blacksquare X_i : i \in I\};$$
$$\blacklozenge \wedge \{X_i : i \in I\} = \wedge\{\blacklozenge X_i : i \in I\}.$$

If there is a de Morgan negation $\neg$ among the internal operations, then

$$\neg\blacksquare X = \blacklozenge\neg X \text{ and } \neg\blacklozenge X = \blacksquare\neg X.$$

Having $\blacksquare$ and $\blacklozenge$ the notion "X is a singleton" can be expressed as $\blacksquare X = X$ (which is equivalent to $\blacklozenge X = X$).

Proposition 2.14 *The set S of singletons in $\mathbf{L}^{\mathbf{set}}$ can be characterized as follows:*

1. $S = \{X : \blacksquare X = X\} = \{X : \blacklozenge X = X\}$
2. *S is the set of co-atoms in $\mathbf{L}^{\mathbf{set}}$.*

Corollary 2.15

1. *By the above we have in $\mathbf{L}^{\mathbf{set}}$: $\blacksquare X = X$ iff $X \neq \top$ and $\forall Y(X \leq_k Y \Rightarrow X = Y$ or $Y = \top)$.*
2. *With the internal operations $\wedge, \vee$ restricted to it S becomes a lattice isomorphic to $\mathbf{L}$.*
3. *If $s = t$ is an $\mathbf{L}$-identity, then $s' = t'$ is an $\mathbf{L}^{\mathbf{set}}$-identity,where s' is s^σ, t' is t^σ and the subsitution σ assigns to a variable X the term $\blacksquare X$.*

Remark 2.16 Unfortunately complementation and $\oplus$ behave erratically with respect to the external modalities. This could be explained by the intuition behind external modalities: both operators reflect the implicit information contained in a generalized truth value X, $\blacklozenge X$ representing the greatest "possible" ordinary truth value consistent with the infromation in X, whereas $\blacksquare X$ gives the "necessary" ordinary truth value implicit in X. When we join the information in X and Y according to the rule "accept anything" (getting their intersection) we in fact may overgenerate restrictions and thus overgenerate information, leading to a senseless truth value.

2.8 Info-algebras Here we try algebraically to define the "useful" part of the set expansions of lattices of truth values. Perhaps it is already clear from the results cited above that the operation $\oplus$ (together with the constant $\top$) is the cause of most of the discrepancies between set algebras and bilattices. One way out of this is to drop $\oplus$ and c from the signature (but keep $\top$ as some interesting applications involve such nonsensical values). A similar connsrtruction, in a restricted setting, was studied by Vakarelov [27].

Definition 2.17 An info-algebra $\mathbf{A}$ of a given internal signature $(\wedge, \vee, \ldots, 0, 1)$ is a partially ordered set $\langle A, \leq_k\rangle$ which is a *complete lower semi-lattice*. The least and greatest elements of $\mathbf{A}$ are $\bot$ and $\top$ respectively. The join operation is denoted by $\prod$ (its finitary version by $\otimes$). The algebra has the following properties.

A1. Internal structure:

(a) internal operations are monotone with respect to $\leq_k$;

(b) $s(\ldots, \top, \ldots) = \top$ for internal terms;

(c) $s(\ldots, \prod\{X_i : i \in I\}, \ldots) = \prod\{s(\ldots, X_i, \ldots) : i \in I\}$, for linear terms.

A2. Singletons:

(a) the set of singletons $a, b, c \ldots$ of $\mathbf{A}$: $\mathbf{L_A}$ is a distributive lattice with respect to the restrictions of $\wedge$ and $\vee$ (and it is bounded by 0 and 1);

(b) $\prod\{X_i : i \in I\} \leq_k a \Rightarrow \exists i X_i \leq_k a$;

(c) $X = \prod\{a : X \leq_k a\}$.

Lemma 2.18 *For an internal operation $o(X_1, \ldots, X_n)$ and a singleton a:*

$$o(X_1, \ldots, X_n) \leq_k a \Rightarrow \exists b_1, \ldots, b_n \in L_{\mathbf{A}}(X_i \leq_k b_i(i = 1, \ldots, n)$$

and

$$o(b_1, \ldots, b_n) = a).$$

Proof: By A2.c $o(X_1, \ldots, X_n) = o(\prod\{a : X_1 \leq_k a\}, \ldots, \prod\{a : X_n \leq_k a\})$. Applying A1.c we get $o(X_1, \ldots, X_n) = \prod\{o(b_1, \ldots, b_n) : X_1 \leq_k b_1, \ldots, X_n \leq_k b_n\}$. Now if $o(X_1, \ldots, X_n) \leq_k a$, then $\prod\{o(b_1, \ldots, b_n) : X_1 \leq_k b_1, \ldots, X_n \leq_k b_n\} \leq_k a$, and using axiom A2.b we get the desired result.

It is not difficult to see that $\mathbf{L^{set}}$ is an info-algebra. Conversely, any info-algebra $\mathbf{A}$ can be represented as a set expansion (with the restricted signature), namely as $\mathbf{L_A^{set}}$, as the next lemma points out.

Lemma 2.19 *Let $\mathbf{A}$ be an info-algebra, then $\mathbf{A} \cong \mathbf{L_A^{set}}$ (as info-algebras).*

Proof: The map $f : A \to \varrho(L_{\mathbf{A}})$ defined as $f(X) = \{a : X \leq_k a\}$ is an isomorphism of $\mathbf{A}$ and $\mathbf{L_A^{set}}$:

$$\begin{aligned} f(o(X_1, \ldots, X_n)) &= \{a : o(X_1, \ldots, X_n) \leq_k a\} \\ &= \{a : \exists b_1, \ldots, b_n(X_i \leq_k b_i(i = 1, \ldots, \nu) \text{ and} \\ &\quad\ o(b_1, \ldots, b_n) = a)\} \\ &= \{a : \exists b_1, \ldots, b_n(bi \in f(X_i) \text{ and } o(b_1, \ldots, b_n) = a\} \\ &= o(f(X_1), \ldots, f(X_n)); \end{aligned}$$

$$\begin{aligned} f(\prod\{X_i : i \in I\}) &= \{a : \prod\{X_i : i \in I\} \leq_k a\} = \{a : \exists i X_i \leq_k a\} \\ &= \prod\{\{a : X_i \leq_k a\} : i \in I\} = \prod\{f(X_i) : i \in I\}. \end{aligned}$$

The rest of conditions are checked as usual.

2.9 Homomorphisms of info-algebras An interesting special case of the info-algebra homomorphims are the homomorphisms of info-algebras into $\mathbf{2^{set}}$ which exist in abundant numbers in contrast with the full set expansions.

Lemma 2.20 *Let $f : \mathbf{A} \to \mathbf{2^{set}}$ be a homomorphism. The subset F of $L_{\mathbf{A}}$ defined by $F = \{a : f(a) \in D_1\}$, where $D_1 = \{\top, 1\}$ is a prime filter in $L_{\mathbf{A}}$.*

Proof: Checking that F possesses the properties of a prime filter:

1. If $a \in F$ and $a \leq b$, then $b \in F$: $a \in F$ means that $f(a) \in D_1$, $a \leq b$ implies that $a \vee b = b$. Consider $a \vee b = a \vee b = b$. Apply f and get $f(a) \vee f(b) = f(b)$. In $\mathbf{2^{set}}$ the equation $a \vee b = b$ together with $a \in D_1$ guarantees that $b \in D_1$, therefore $f(b) \in D_1$.
2. $a \wedge b \in F$ iff $a \in F$ and $b \in F$: if $a \wedge b \in F$, then clearly by 1 $a, b \in F$. Assume now that $a \in F$ and $b \in F$, which is equivalent to $f(a) \in D_1$ and $f(b) \in D_1$, that gives $f(a) \wedge f(b) \in D_1$ (by the laws of $\mathbf{2^{set}}$) and, since f is a homomorphism, $f(a \wedge b) \in D_1$, so $a \wedge b \in F$.
3. $a \vee b \in F$ iff $a \in F$ or $b \in F$: clearly, if $a \in F$ or $b \in F$, then $a \vee b \in F$. In the opposite direction: if $a \vee b \in D_1$, then $f(a) \vee f(a) \in D_1$, but in $\mathbf{2^{set}}$ this guarantees that either $f(a) \in D_1$ or $f(b) \in D_1$, i.e., $a \in F$ or $b \in F$.

Lemma 2.21 *Let* **A** *be an info-algebra of the minimal internal signature* $(\wedge, \vee, 0, 1)$. *Then the mapping* $f : A \to \mathbf{2^{set}}$ *defined from a prime filter* F *in* $L_{\mathbf{A}}$ *by first stipulating for the singletons:*

$$f(a) = \begin{cases} 1 & \textit{if } a \in F \\ 0, & \textit{otherwise} \end{cases}$$

and then for arbitrary elements of $\mathbf{L^{set}}$: $f(X) = \prod\{f(a) : X \leq_k a\}$, *is a homomorphism.*

Proof: Clearly the constants $0, 1, \perp, \top$ are mapped correctly by f: $f(1) = 1$ and $f(0) = 0$ by the fact that F is a proper filter, $f(\perp) = \perp$, since $1 \otimes 0 = \perp$ and $f(\top) = \top$ trivially. Let us also note that for $a, b \in L$,

$$\begin{aligned} f(a \wedge b) &= f(a) \wedge f(b); \\ f(a \vee b) &= f(a) \vee f(b), \end{aligned}$$

since F is prime. To check if $f(X \wedge Y) = f(X) \wedge f(Y)$ reason as follows:

$$\begin{aligned} f(X \wedge Y) &= \prod\{f(c) : X \wedge Y \leq_k c\} \\ &= \prod\{f(\{a \wedge b\}) : X \leq_k a, Y \leq_k b\} \\ &= \prod\{f(a) \wedge f(b) : X \leq_k a, Y \leq_k b\} \\ &= \prod\{f(a) : X \leq_k a\} \wedge \prod\{f(b) : Y \leq_k b\} \\ &= f(X) \wedge f(Y). \end{aligned}$$

Similarly we can establish that $f(X \vee Y) = f(X) \vee f(Y)$. As for the operation $\otimes$, we proceed as follows:

$$\begin{aligned} f(X \otimes Y) &= \prod\{f(c) : X \otimes Y \leq_k c\} \\ &= \prod\{f(a) : X \leq_k a\} \otimes \prod\{f(b) : Y \leq_k b\} \\ &= f(X) \otimes f(Y). \end{aligned}$$

Now we can formulate an effect of the above construction important for the logical developments below.

Lemma 2.22 *If $X \notin D_1$ $(= \{1, \top\})$ in a minimal info-algebra* $\mathbf{A}$*, then there is a homomorphism* $f : \mathbf{A} \to \mathbf{2^{set}}$ *such that* $f(X) \notin D_1$ *in* $\mathbf{2^{set}}$.

Proof: Since $X \notin D_1$, there is an a with $X \leq_k a$ and such that $a \neq 1$, and we can find a filter F in $\mathrm{L}_{\mathbf{A}}$ which omits a. The homomorphism $f : \mathbf{A} \to \mathbf{2^{set}}$ associated with this filter maps X either on 0 (if $X \cap F = \varnothing$) or on $\perp$ (when $X \cap F \neq \varnothing$). Anyway, $f(X) \notin D_1$.

It would be a natural next step to formulate a representation theorem for info-algebras as sub-algebras of the set expansions of frame lattices. Along the lines of the classical approach to the representation of lattices as sublattices of set based ones, one needs to define first the notion of $\mathbf{2^{set}}$ model on a frame $\mathcal{F} = \langle W, \ldots \rangle$ as a mapping $f : W \times Fml \to \mathbf{2^{set}}$ where for each $w \in W$, $f(w, A) \in \mathrm{Hom}(\mathcal{L}, \mathbf{2^{set}})$ for the minimal signature. $|A|_1$ denotes the set $\{w : f(w, A) = 1\}$, $|A|_0, |A|_\perp, |A|_\top$ have similar meanings. A singleton a in the frame $\mathcal{F}$ is a partition of W, i.e., $|a|_1 = W \backslash |a|_0$, in other words singletons are exact truth values. We write $A \leq_\varphi B$ if $\forall w(\varphi(w, A) \leq_k \varphi(w, B))$. Now let $||A|| = \{a : a$ is a singleton in $\mathcal{F}$ and $A \leq \varphi a\}$. The idea is to establish that $||.||$ is a member of $\mathrm{Hom}(\mathcal{L}, (\mathbf{A}(\mathcal{F}))^{\mathbf{set}})$. Disappointingly enough this is true only for the internal part of the language.

Lemma 2.23 *The mapping $||.||$ belongs to the family Hom$(\mathcal{L}_0, (\mathbf{A}(\mathcal{F}))^{\mathbf{set}})$, where $\mathcal{L}_0$ is the language without $\otimes$.*

Proof: To begin with, note that $|A|_\top \neq \varnothing$ iff $||A|| = \varnothing$. That takes care of the situation when there are occurrences of $\top$ in the valuations of A or B. Observe also that $|A \wedge B|_1 = |A|_1 \cap |B|_1$, $|A \wedge B|_0 = |A|_0 \cup |B|_0$, $|A \vee B|_1 = |A|_1 \cup |B|_1$, and $|A \vee B|_0 = |A|_0 \cap |B|_0$. Using these we can prove, e.g., $||A \wedge B|| = ||A|| \wedge ||B||$: $||A \wedge B|| = \{c : A \wedge B \leq_\varphi c\}$, but $A \wedge B \leq_\varphi c$ means that $|c|_1 \supseteq |A|_1 \cap |B|_1$ and $|c|_0 \supseteq |A|_0 \cup |B|_0$. Let now a, b be the singletons with $|a|_1 = c \cup |A|_1$ and $|b|_1 = c \cup |B|_1$. It is easy to see that $A \leq_\varphi a$ and $B \leq_\varphi b$, and that $a \cap b = c$. Thus $||A|| \wedge ||B|| \supseteq ||A \wedge B||$. To justify the opposite inclusion note that if $A \leq_\varphi a$ and $B \leq_\varphi b$, then $A \wedge B \leq_\varphi a \wedge b$.

The case with $\vee$ is left to the reader.

2.10 Link to supervaluations The above construction can be cast in a slightly different form in order to reveal its kinship to a very well-known idea in many-valued logic: the notion of *supervaluations*. Call a mapping $\psi : W \times Fml \to \mathbf{2^{set}}$ a supervaluation for φ if:

1. $\varphi \leq_k \psi$;
2. ψ is *exact*, i.e., for variables p, $\psi(w, p) \in \{0, 1\}$.

Let $||A||_s = \{|A|_1^\psi$: ψ is a supervaluation for $\varphi\}$. The claim that $||A||_s = ||A||$ for $A \in Fml(\mathcal{L}_0)$ is easily justified by the fact that $\{|A|_1^\psi, |A|_0^\psi\}$ is a singleton.

The difference from the tradition of van Fraasen lies in the way supervaluations are used: whereas customarily the family of supervaluations for a given φ is converted into a valuation $\bar{\varphi}$, which in our case would look as follows: $\bar{\varphi}(w, A) = \top$ if there are no supervaluations; and in the presence of supervaluations,

$$\bar{\varphi}(w, A) = \begin{cases} 1 & \text{if } \forall y(y(w, A) = 1) \\ 0 & \text{if } \forall y(y(w, A) = 0) \\ \perp & \text{otherwise.} \end{cases}$$

Our usage avoids such a conversion and keeps the family of supervaluations as a new generalized truth value.

2.11 External info-algebras An info-algebra **A** is called *external* if two unary operations $\blacksquare$ and $\blacklozenge$ can be defined in **A** satisfying:

1. $\blacksquare\Pi\{X_i : i \in I\} = \wedge\{\blacksquare X_i : i \in I\}$;
2. $\blacklozenge\prod\{X_i : i \in I\} = \vee\{\blacklozenge X_i : i \in I\}$;
3. $\blacksquare a = \blacklozenge a = a$, for singletons.

All relevant properties of the external modalities in set expansions (as documented in Proposition 2.13) can be derived from the above definition (which presupposes that $\mathbf{L_A}$ is a complete lattice).

It should have become clear by now that admitting arbitrary sets of lattice elements as generalized truth values, insisting at the same time that this move is caused by incompleteness of information, uncertainty of data, or vagueness of predicates, etc., is somewhat inconsistent: to define an *arbitrary* set X requires very detailed information about the individual members of X, which seems implausible in circumstances when one *lacks* the relevant knowledge. Thus restricted classes of such generalized truth values seem to be a more realistic way of modeling imperfect epistemic situations.

As an example of restrictions that arise from specific imperfections of data let us consider a frame $\mathcal{F} = \langle W, \ldots \rangle$ where the available knowledge permits us to discern different possible worlds only up to certain equivalence relation $\approx$, so the only subsets of W one "can be aware of" are unions of equivalence classes $[w] = \{w' : w \approx w'\}$ with respect to the *indiscernibility* relation $\approx$ (called *rough* sets, cf. Pawlak's introductory paper [20]). For a set $V \subseteq W$ denote by V_1 the set $\{w : [w] \subseteq V\}$ and let $V_0 = \{w : [w] \cap V \neq \varnothing\}$; these are respectively the biggest rough set inside V and the biggest rough set including V. An observer having the above limitations can know the "real" truth value $||A||$ only to contain $||A||_1$ and to be contained in $||A||_0$, so any set between these two bounds would be a possible "real" truth value for him, i.e., the generalized interpretation of A would be $\{U : ||A||_1 \subseteq U \subseteq ||A||_0\}$. Note that in such a setting the existence of the operations $\blacksquare$ and $\blacklozenge$ is a natural consequence: $\blacksquare\{U : ||A||_1 \subseteq U \subseteq ||A||_0\} = ||A||_1$ and $\blacklozenge\{U : ||A||_1 \subseteq U \subseteq ||A||_0\} = ||A||_0$.

3 Logical aspects of set expansions and info-algebras In this section we explore the possibility of treating some of the algebras introduced above in the traditional fashion of algebraic logic: as logical matrices semantically defining logical systems, i.e., as a generalization of the truth-table method used in classical logic.

3.1 Generalized matrices Let us recall some definitions and some basic facts from the theory of propositional logics (cf. [4], [29]), restricted to our current needs, e.g., we presuppose only a finite number of finitary logical connectives.

A propositional language $\mathcal{L}$ over an infinite (countable in our case) set of variables $Var(\mathcal{L})$ is an absolutely free algebra of some signature (with the above restrictions), freely generated by $Var(\mathcal{L})$. We assume that the operations include conjunction and disjunction, the two constants 0, 1, and eventually other constants and operations o_i. The elements of this algebra are called *formulas* and form a set $Fml(\mathcal{L})$, so $\mathcal{L} = \langle Fml(\mathcal{L}), \wedge, \vee, \ldots, 0, 1\rangle$. Reference to $\mathcal{L}$ will be dropped from now on whenever possible.

A mapping C: $\wp(Fml) \to \wp(Fml)$ is a *consequence operation*, if the following conditions are satisfied for all subsets Γ, Δ of Fml:

1. $\Gamma \subseteq C(\Gamma)$;
2. $C(\Gamma) = C(C(\Gamma))$;
3. $\Gamma \subseteq \Delta$ implies $C(\Gamma) \subseteq C(\Delta)$.

A consequence operation C is *compact* if for every Γ:

$$C(\Gamma) = \cup\{C(\Delta) : \Delta \subseteq \Gamma \text{ and } \Delta \text{ is finite}\}.$$

A *generalized matrix* $\mathcal{M} = \langle \mathbf{A}, D, H\rangle$ for $\mathcal{L}$ is a triple where:

1. $\mathbf{A}$ is an algebra similar to the language $\mathcal{L}$: the truth-value space;
2. D is subset of the truth-value space: the elements of D are the *designated* truth values;
3. $H \subseteq \mathrm{Hom}(\mathcal{L}, \mathbf{A})$: its elements are called *admissible valuations*.

A matrix $\mathcal{M}$ is called standard when $H = \mathrm{Hom}(\mathcal{L}, \mathbf{A})$. Every class of matrices $\mathbf{K}$ determines a consequence operation $C_{\mathbf{K}}$:

$$A \in C_{\mathbf{K}}(\Gamma) \text{ iff } \forall\, \mathcal{M} = \langle \mathbf{A}, D, H\rangle \in \mathbf{K}\, \forall h \in H \text{ (if } h[\Gamma] \subseteq D, \text{ then } h(A) \in D).$$

For singleton classes $\{\mathcal{M}\}$ we write simply $C_{\mathcal{M}}$. As a rule instead of $A \in C_{\mathbf{K}}(\Gamma)$ we write $\Gamma \models_{\mathbf{K}} \mathbf{A}$, or just $\Gamma \models \mathbf{A}$ when there is no danger of confusion.

A *propositional logic* $\mathbf{S}$ is a pair $\langle \mathcal{L}, C\rangle$ where C is a consequence operation in the language $\mathcal{L}$ of $\mathbf{S}$. A class $\mathbf{K}$ is called a semantics for the logic $\langle \mathcal{L}, C\rangle$, if $C = C_{\mathbf{K}}$. *Finitely-approximable logics* $\mathbf{S}$ are characterized by a class of finite matrices, i.e., there is a class $\mathbf{K}$ of finite matrices which is a semantics for $\mathbf{S}$. *Finite* logics (or *finite-valued* logics) are determined by a finite class of finite matrices.

Below we present a variety of logical systems arising from the truth spaces considered in the previous section in a unified framework by proving theorems of the following kind: *the system* $\mathbf{S}$ *is characterized by a class of algebras (set expansions, info-algebras), viewed as generalized logical matrices.*

Within this framework one is confronted with several *choices* which determine the logical system:

- the choice of language $\mathcal{L}$, i.e., what operations should be considered *logical*, as opposed to others that are computational in character;
- the choice of D, the distinguished truth values;
- the choice of the class H, the admissible valuations, i.e., what types of homomorphisms are considered relevant.

With respect to the first mentioned choice there are several approaches. The *liberal* approach is to consider *all* operations available in the investigated class as equally logical, so the propositional language is to have the same signature as the class itself, i.e., all operations have corresponding logical connectives. We are going to give several examples of this approach, e.g., the logic of info-algebras, etc.

A second, more *restrictive*, approach is to put down criteria by which an operation can be judged to be logical or not. Let us formulate a few criteria as an example.

1. A logical operation should preserve acquired information about truth values. Assuming this, we arrive at the requirement of k-*monotonicity*. Such a criterion excludes for example the external modalities as candidates for logical operations.
2. One may insist on *conservativity* of a logical operation in the sense that when applied to exact truth values (i.e., singletons) it should yield exact values. This criterion excludes the k-operations $\oplus$ and $\otimes$ but admits the external modalities.
3. An even less restrictive requirement is to demand the operations to *preserve consistency*, i.e., when applied to consistent truth values the operation should give consistent truth values; the operation $\oplus$ is excluded in this case.

The problem with the set of distinguished truth values is in fact a version of the more general question of what conditions should be met in order to recognize a statement as "true." A first thing that comes to mind is that the intended meaning of the elements of set expansions implies that $1 \in D$, so the simplest decision is $D = \{1\}$, i.e., to recognize as valid only such inferences that preserve the property of "being only true." But, besides being a not very happy choice technically, the restriction of D to $\{1\}$ is not easily justifiable.

Now one might wish to be positive, i.e., to accept a statement as true if having all the reasons to do so (in this case let the set of distinguished truth values be D_1), or he might assume a negative attitude and favor statements that are *not refutable*, i.e., if there are no reasons to reject a statement one accepts it (let the corresponding set be D_0). In set expansions D_1 and D_0 can be viewed as the sets of elements X satisfying respectively $\forall x \in X(x = 1)$ and $\exists x \in X(x = 1)$. Unfortunately the former definition leads to some technical complications and destroys the duality between the two choices. A better option is $D_0 = \{X : \sup X = 1\}$, dual to $\{X : \inf X = 1\}(= D_1)$.

Restrictions concerning the set of admitted homomorphisms can include such requirements as the following.

- H is the set of all consistent valuations, i.e., functions whose range contains only consistent elements, or
- H is the set of valuations into info-algebras that contain only finite sets in their ranges, etc.

Let us fix some terminology: if a logic is defined by a class of generalized logical matrices with no restrictions on the admissible homomorphisms we speak of a *standard* system, if H is restricted to the homomorphisms with consistent values, then we use the term *consistent* logics; if it contains only finite sets as values (this in the case of set expansions), then the logic is *finitary*. If the set of distinguished truth values is D_1, we speak of *positive* logics; if it is D_0, we say that the logic is *negative*. If the signature of the language and the algebras of the class coincide, we speak of a *full* logic, otherwise we use different adjectives showing which operations in the algebras have language connectives as counterparts.

For the presentation of this and subsequent logical systems we choose *sequential style* calculi. For our purposes it is sufficient to adopt the view that sequents are of the form $\Gamma \vdash A$, where Γ is a finite set of formulas and A is a formula. Thus our systems are inherently intuitionistic (having the restriction of single formula in the right-hand side).

3.2 The standard positive logic of all info-algebras Our basic system will be formulated in a language which includes the operations $\wedge, \vee, \otimes$, and has *no propositional constants*. We restrict the language in this way with simplicity of presentation in mind (the addition of the constants does not change the results but complicates the system of rules and the proofs). We consider the info-algebras as standard matrices with the set $D_1 = \{\top, 1\}$ as the set of distinguished truth values. Let us point out that this semantics has the following property: there are no tautologies in the language, i.e., for no formula A, $\varnothing \models A$ (e.g., for any A one can always find a valuation h with $h(A) = \bot$).

Call a formula *internal* if $\otimes$ does not occur in it. A set Γ is internal if all its members are internal formulas. The systems will have as basic sequents expressions of the form $\Gamma \vdash A$, where $A \in \Gamma$.

3.2.1 Rules With each connective of $\mathcal{L}$ two types of rules are associated, governing association of formulas in the left-hand and the right hand side respectively:

$$(\wedge\vdash)^+ \quad \frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C} \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$(\vdash\wedge) \quad \frac{\Gamma \vdash A;\ \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$(\vee\vdash) \quad \frac{\Gamma, A \vdash C;\ \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}$$

$$(\vdash\vee) \quad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

$$(\otimes\vdash) \quad \frac{\Gamma, A \vdash C}{\Gamma, A \otimes B \vdash C} \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$(\vdash\otimes) \quad \frac{\Gamma \vdash A;\ \Gamma \vdash B}{\Gamma \vdash A \otimes B}$$

Note that conjunction poses a problem: although $(\vdash\wedge)$ is acceptable, the rules $(\wedge\vdash)$ are not correct without any restriction when interpreted in info-algebras (because of the possibility to assign $A \wedge B$ a value in D_1 keeping the value of A outside D_1, as for example is the case with the valuation h defined for p and q as $h(p) = \top$ and $h(q) = 0$, so $h(p \wedge q) \in D_1$ but $h(q) \notin D_1$). Therefore we have to put up with weaker rules $(\wedge\vdash)^+$, where the plus sign marks the restriction on the type and variables of the formulas which appear in the bottom sequents: *A is internal and $Var(A) \supseteq Var(B)$*, for the left rule and *B is internal and $Var(B) \supseteq Var(A)$*, for the right one. Such a weakening calls for additional compensatory rules. In the first place we need the *cut rule*:

$$\text{(Cut)} \quad \frac{\Gamma \vdash A;\ \Gamma, A \vdash B}{\Gamma \vdash B}$$

but also the following *distributivity* rules:

$$(\wedge\vee\vdash) \quad \frac{\Gamma, (A \wedge C) \vee (B \wedge C) \vdash D}{\Gamma, (A \vee B) \wedge C \vdash D}$$

$$(\vdash\wedge\vee) \quad \frac{\Gamma \vdash (A \wedge C) \vee (B \wedge C)}{\Gamma \vdash (A \vee B) \wedge C}$$

$$(\wedge\otimes\vdash) \quad \frac{\Gamma, (A \wedge C) \otimes (B \wedge C) \vdash D}{\Gamma, (A \otimes B) \wedge C \vdash D}$$

$$(\vdash\wedge\otimes) \quad \frac{\Gamma \vdash (A \wedge C) \otimes (B \wedge C)}{\Gamma \vdash (A \otimes B) \wedge C}$$

$$(\wedge\wedge\vdash) \quad \frac{\Gamma, (A \wedge B) \wedge C \vdash D}{\Gamma, (A \wedge C) \wedge B \vdash D}$$

$$(\vdash\wedge\wedge) \quad \frac{\Gamma \vdash (A \wedge C) \wedge (B \wedge C)}{\Gamma \vdash (A \wedge B) \wedge C} \quad .$$

A sequent is provable if it can be derived from basic sequents by means of the rules.

Lemma 3.1 *The resulting system is correct: if* $\Gamma \vdash A$ *is provable, then* $\Gamma \models A$.

Proof: The proof is by straightforward checking. The only more exciting rules to treat are the additional distributivity rules, since most of them do not correspond to identities in the info-algebras. Let us do as an example the $(\wedge\vee)$-rules. For $(\wedge\vee\vdash)$ it is sufficient to establish that $(A \vee B) \wedge C \models (A \wedge C) \vee (B \wedge C)$. Take a valuation h such that $h((A \vee B) \wedge C)) \in D_1$. If the value is $\top$, then at least one of $h(A)$, $h(B)$ or $h(C)$ is $\top$, but then $h((A \wedge C) \vee (B \wedge C)) = \top$, too. For the other possibility, let the value be 1. This means that $(a \vee b) \wedge c = 1$ for any $a \in h(A), b \in h(B), c \in h(C)$, and implies $c = 1$ and $a \vee b = 1$. But then for a typical member of $h((A \wedge C) \vee (B \wedge C))$—$(a \wedge c_1) \vee (b \wedge c_2)$—we have $(a \wedge c_1) \vee (b \wedge c_2) = a \vee b = 1$. For $(\vdash \wedge\vee)$ we have to check whether $(A \wedge C) \vee (B \wedge C) \models (A \vee B) \wedge C$, but this is easy: skipping the case of occurrence of $\top$, we consider a typical member of the left-hand side $(a \wedge c_1) \vee (b \wedge c_2) = 1$, and moreover $(a \wedge c) \vee (b \wedge c) = 1$ for any a, b, c from the appropriate sets. Applying the distributive law in the underlying lattice we obtain $(a \vee b) \wedge c = 1$ for a typical member of the right-hand side.

Let Γ now be an arbitrary set of formulas. We define $[\Gamma]$ as $\{B$: for some finite subset Γ_0 of Γ the sequent $\Gamma_0 \vdash B$ is provable$\}$. Call Δ a theory if $[\Delta] = \Delta$. The set of all formulas *Fml* is an example of a theory: the trivial theory. A nontrivial theory Δ is *prime* if $A \vee B \in \Delta \Rightarrow A \in \Delta$ or $B \in \Delta$.

Remark 3.2 A slightly more sophisticated counterexample involving $A = p \otimes q$ and $B = p$ for which $A \wedge B \not\models A$ (as witnessed by the assignment $h(p) = \top$, $h(q) = 0$) shows that the restriction to internal formulas in $(\wedge \vdash)^+$ is indeed necessary.

The next technical lemma is left entirely to the diligent reader (its proof relies on such provable sequents as $A \otimes B \vdash A$; $A \otimes B \vdash B$; $A, B \vdash A \otimes B$; $A \wedge (B \otimes C) \vdash (A \wedge B) \otimes (A \wedge C)$; $(A \wedge B) \otimes (A \wedge C) \vdash A \wedge (B \otimes C)$, etc.).

Lemma 3.3 *For any* Γ *there exists an internal* Γ' *such that* $[\Gamma'] = [\Gamma]$.

The lemma shows that questions of the type "Is $\Gamma \vdash$ A provable?" can be reduced to the same questions about internal formulas and sets of internal formulas.

Here is another list of provable sequents to be used in the lemma that follows:

$$\begin{array}{rcl}
A \wedge B & \vdash & A \vee B, \\
A \wedge B & \vdash & B \wedge A, \\
A \wedge (B \wedge C) & \vdash & (A \wedge B) \wedge C, \\
(A \wedge B) \wedge C & \vdash & A \wedge (B \wedge C), \\
((A \wedge C) \wedge (B \wedge C)) \wedge D & \vdash & ((A \wedge B) \wedge C) \wedge D), \\
((A \wedge B) \wedge C) \wedge D & \vdash & ((A \wedge C) \wedge B) \wedge D, \\
((A \wedge C) \vee (B \wedge C)) \wedge D & \vdash & ((A \vee B) \wedge C) \wedge D, \\
((A \vee B) \wedge C) \wedge D & \vdash & ((A \wedge C) \vee (B \wedge C)) \wedge D.
\end{array}$$

The space permits one proof as an example.

$$\dfrac{\dfrac{\dfrac{A \wedge B \vdash A \wedge B}{A \wedge B \vdash (A \wedge B) \vee (B \wedge B)}\,(\vdash \vee)}{A \wedge B \vdash (A \vee B) \wedge B}\,(\vdash \wedge\vee) \qquad \dfrac{A \vee B \vdash A \vee B}{(A \vee B) \wedge B \vdash A \vee B}\,(\wedge \vdash)^{+}}{A \wedge B \vdash A \vee B}\,(\mathrm{Cut})$$

Unfortunately the derivations known to the author depend crucially on applications of the cut rule, so the cut elimination property of the above system is an open problem.

Lemma 3.4 *For internal formulas $B_1, \ldots, B_n$, A, if $B_1, \ldots, B_n \vdash A$ is provable, then $B_1 \wedge C, \ldots, B_n \wedge C \vdash A \wedge C$ is also provable for any C.*

Proof: By induction on the height of the derivation tree. The case of axioms is clear, so we need to check the induction step, proving that if the top sequent in an application of a rule satisfies the above property, then the bottom sequent also satisfies this property. The notorious fate of such proofs notwithstanding, we present just a sample of the simplest cases. In general the derivations use (Cut) and depend on the provable sequents shown above.

Let $B_1, \ldots, B_n \vdash A$ be obtained by an application of $(\wedge \vdash)^{+}$, i.e., $B_n = B \wedge D$, $Var(B) \supseteq Var(D)$ and

$$\dfrac{B_1, \ldots, B \vdash A}{B_1, \ldots, B_n \vdash A}\,(\wedge \vdash)^{+}.$$

By the induction hypothesis: $B_1 \wedge C, \ldots, B \wedge C \vdash A \wedge C$ is provable. But then the following is a proof of what is needed.

$$\dfrac{\dfrac{B_1 \wedge C, \ldots, B \wedge C \vdash A \wedge C}{B_1 \wedge C, \ldots, (B \wedge C) \wedge D \vdash A \wedge C}\,(\wedge \vdash)^{+}}{B1 \wedge C, \ldots, (B \wedge D) \wedge C \vdash A \wedge C}\,(\wedge\wedge \vdash).$$

Let us also consider the case when the last applied rule is $(\vdash \wedge)$. Now $A = A_1 \wedge A_2$, and the application is:

$$\dfrac{B_1, \ldots, B_n \vdash A_1; \;\; B_1, \ldots, B_n \vdash A_2}{B_1, \ldots, B_n \vdash A_1 \wedge A_2}.$$

By the induction hypothesis: $B_1 \wedge C, \ldots, B_n \wedge C \vdash A_1 \wedge C$ and $B_1 \wedge C, \ldots, B_n \wedge C \vdash A_2 \wedge C$ are provable. But then

$$\cfrac{\cfrac{B_1 \wedge C, \ldots, B_n \wedge C \vdash A_1 \wedge C; \;\; B_1 \wedge C, \ldots, B_n \wedge C \vdash A_2 \wedge C}{B_1 \wedge C, \ldots, B_n \wedge C \vdash (A_1 \wedge C) \wedge (A_2 \wedge C)}\,(\vdash \wedge\wedge),}{B_1 \wedge C, \ldots, B_n \wedge C \vdash (A_1 \wedge A_2) \wedge C}\,(\vdash \wedge)$$

and we are done.

One more rule as the last applied one: $(\wedge\wedge \vdash)$. In this case $B_n = (B_1 \wedge B_2) \wedge B_3$ and by the induction hypothesis $B_1 \wedge C, \ldots, ((B1 \wedge B3) \wedge B2) \wedge C \vdash A1 \wedge C$ is provable. The following derivation gets the desired result:

$$\frac{\begin{array}{c} B_1 \wedge C, \ldots, ((B_1 \wedge B_3) \wedge B_2) \wedge C \vdash A_1 \wedge C; \\ (B_1 \wedge B_2) \wedge B_3) \wedge C \vdash (B_1 \wedge B_3) \wedge B_2) \wedge C \quad \text{(see above)} \end{array}}{B_1 \wedge C, \ldots, ((B_1 \wedge B_2) \wedge B_3) \wedge C \vdash A_1 \wedge C}\,(\text{Cut})$$

The rest of the cases are left to the reader.

Theorem 3.5 *If* $\Gamma \models A$, *then* $A \in [\Gamma]$.

Proof: We follow an already familiar path with some minor deviations: assuming without loss of generality that A and Γ are internal and $A \notin \Gamma$, we find a maximal theory Δ among the theories that extend Γ and omit A. Such a theory need not be prime, but it still does the job because it turns out to be *relatively* prime, namely with respect to the class of internal formulas built up from the propositional variables occurring in A.

Call the variables of A *significant*. A *significant* formula B is such that $Var(B) \subseteq Var(A)$.

Lemma 3.6 *For significant internal formulas B and C:*

1. $B \wedge C \in \Delta$ *iff* $B \in \Delta$ *and* $C \in \Delta$.

For any formulas B and C:

2. $B \vee C \in \Delta$ *iff* $B \in \Delta$ *or* $C \in \Delta$
3. $B \otimes C \in \Delta$ *iff* $B \in \Delta$ *and* $C \in \Delta$.

Proof: The establishing of (2) and (3) is routine. Let us check (1), which differs from the standard case. If $B \in \Delta$ and $C \in \Delta$, then by $(\vdash \wedge)$ $B \wedge C \in \Delta$. In the opposite direction: if $B \wedge C \in \Delta$, then by one of the listed provable sequents $B \vee C \in \Delta$ and so either B or C belong to Δ. Let $B \in \Delta$. Now, $C \notin \Delta$ means that $A \in [\Delta, C]$, so for some internal formulas $D_1, \ldots, D_m \in \Delta$,

$$D_1, \ldots, D_m, C \vdash A \text{ is provable.}$$

Applying Lemma 3.4 we get that $D_1 \wedge B, \ldots, D_m \wedge B$, $C \wedge B \vdash A \wedge B$ is also provable. All formulas of the left-hand side are from Δ. Thus $A \wedge B \in \Delta$. Since B is significant, $A \wedge B \vdash A$ is provable, so $A \in \Delta$—a contradiction with the assumptions on Δ. Therefore $C \in \Delta$, too.

Having a theory Δ with the above properties, we can define a function $h : \mathit{Var} \to \mathbf{2^{set}}$:

$$h(p) = \begin{cases} 1, & \text{if } p \text{ is significant and } p \in \Delta \\ 0, & \text{if } p \text{ is significant and } p \notin \Delta \\ \top, & \text{if } p \text{ is not significant.} \end{cases}$$

Note first that the extension of h to a homomorphism $L \to \mathbf{2^{set}}$ maps all internal insignificant formulas to $\top$. For significant ones it can be established by induction that

$$h(B) = 1 \text{ iff } B \in \Delta.$$

To conclude the proof of the completeness theorem we note that A is significant and so $h(A) \notin D_1$, whereas all members of Γ are mapped onto an element of D_1.

Remark 3.7 We have found a simple algebra adequate for the system: the logic of all info-algebras coincides with the logic of $\mathbf{2^{set}}$. In this way it turns out to be in fact a finite logic, consequently decidable.

Remark 3.8 The admission of the constants $0, 1, \top, \bot$ to the language though not changing the rules will necessitate changes in the notion of a basic sequent. Axioms will have to include also $\Gamma \vdash A$ where one of the following holds:

1. A is an *internal* formula with an occurrence of $\top$;
2. $A = 1$;
3. $0, \bot \in \Gamma$;
4. $A = B \wedge C$ and $\{0 \wedge B, 0 \wedge C, \bot \wedge B, \bot \wedge C\} \cap \Gamma \neq \varnothing$.

3.3 The general negative case The full standard *negative* logic of info-algebras, defined by choosing as distinguished truth values $D_0 = \{X : 1 \in X\}$ poses the first major setback to our program: this logic is *not* anything like a dual to the above system. Let us start with the observation that the info-algebra $\mathbf{2^{set}}$ is not adequate for this particular consequence relation any more: for example $A \vee A \models A$ is in $\mathbf{2^{set}}$ but not in the set expansion of the four element Boolean algebra (with elements $a, b \neq 0, 1$), as shown by the valuation h for which $h(A) = \{a, b\} \notin D_0$ when $h(A \vee A) = \{a, b, 1\} \in D_0$. This and similar counterexamples demonstrate the incorrectness of several rules, e.g., $(\vdash \vee)$ or the distributivity rules. Although the logic can be axiomatized and shown to be finitely approximable (but not finite) we leave that matter to another paper. The problem lies in the origin of the "nice" properties of the elements of D_1 and D_0: in the former case $\inf X = 1$ is equivalent to $X \in D_1$, while in the latter $\sup X = 1$ (the "real" dual) is weaker than $X \in D_0$. Below we consider several systems with the weaker condition on D_0.

3.4 The positive logic of info-algebras with negation Consider now the class of info-algebras with underlying de Morgan lattices. Its full positive standard logic is an extension of the system in the previous subsection.

- the language has an additional connective, $\neg$;
- the notion of basic sequent is augmented to incorporate a restricted version of the *Duns Scot law*: $\Gamma \vdash A$ is an axiom if A is internal and also $B \wedge \neg B \in \Gamma$, for some formula B with $\mathit{Var}(B) \subseteq \mathit{Var}(A)$;

- new rules concerning $\neg$ are added:

$$(\neg\wedge\vdash)\quad \frac{\Gamma, \neg A \vdash C;\ \ \Gamma, \neg B \vdash C}{\Gamma, \neg(A \wedge B) \vdash C}$$

$$(\vdash\neg\wedge)\quad \frac{\Gamma \vdash \neg A}{\Gamma \vdash \neg(A \wedge B)} \qquad \frac{\Gamma \vdash \neg B}{\Gamma \vdash \neg(A \wedge B)}$$

$$(\neg\vee\vdash)\quad \frac{\Gamma, \neg A \wedge \neg B \vdash C}{\Gamma, \neg(A \vee B) \vdash C}$$

$$(\vdash\neg\vee)\quad \frac{\Gamma \vdash \neg A;\ \ \Gamma \vdash \neg B}{\Gamma \vdash \neg(A \vee B)}$$

$$(\neg\otimes\vdash)\quad \frac{\Gamma, \neg A \vdash C}{\Gamma, \neg(A \otimes B) \vdash C} \qquad \frac{\Gamma, \neg B \vdash C}{\Gamma, \neg(A \otimes B) \vdash C}$$

$$(\vdash\neg\otimes)\quad \frac{\Gamma \vdash \neg A; \Gamma \vdash \neg B}{\Gamma \vdash \neg(A \otimes B)}$$

$$(\neg\neg\vdash)\quad \frac{\Gamma, A \vdash C}{\Gamma, \neg\neg A \vdash C}$$

$$(\vdash\neg\neg)\quad \frac{\Gamma \vdash A}{\Gamma \vdash \neg\neg A}$$

Lemma 3.9 *The resulting system is correct.*

Proof: The correctness of the new rules is obvious. As for the new axiom: if $h(B \wedge \neg B) \in D_1$, then $h(B) = \top$ and therefore for some variable p, $h(p) = \top$, because, if $h(B)$ is nonempty, then it has members $a \wedge \neg a$ which cannot be 1. Since $Var(B) \subseteq Var(A)$ and A is internal, $h(A) = \top$, also.

Lemma 3.10 *For any Γ there exists an internal Γ' such that $[\Gamma'] = [\Gamma]$.*

Lemma 3.11 *If $B_1, \ldots, B_n \vdash A$ is provable, then $B_1 \wedge C, \ldots, B_n \wedge C \vdash A \wedge C$ is also provable (under the same conditions as above).*

Proof: The induction step now requires checking of the added rules. Let us do an example in which the last applied rule is $(\neg\vee\vdash)$. In this case $B_n = \neg(B \vee D)$ and by the induction hypothesis $B_1 \wedge C, \ldots, (\neg B \wedge \neg D) \wedge C \vdash A \wedge C$. The following sequent is provable: $\neg(B \vee D) \wedge C \vdash (\neg B \wedge \neg D) \wedge C$. Applying (Cut) we get $B_1 \wedge C, \ldots, \neg(B \vee D) \wedge C \vdash A \wedge C$, the desired result.

All other details are left to the reader.

Theorem 3.12 *If $\Gamma \vdash A$, then $A \in [\Gamma]$.*

Proof: Once again we can assume without loss of generality that we deal exclusively with internal formulas. Let A in particular be an internal formula such that $A \notin [\Gamma]$. Just as in the proof of Theorem 3.5 we can extend Γ to a maximal theory Δ, for which $\Gamma \subseteq \Delta$ and $A \notin \Delta$. This theory turns out to be relatively prime with respect to internal significant formulas, i.e., to formulas B with $Var(B) \subseteq Var(A)$.

Note now that in this case: p is a significant variable implies $p \wedge \neg p \notin \Delta$, because $p \wedge \neg p \in \Delta$ would mean that A is also from Δ (recall that $p \in Var(A)$ and A is internal). Otherwise we have to add to the properties of relatively prime theories (from Lemma 3.6) some clauses concerning the negation (B, C significant):

4. $\neg(B \wedge C) \in \Delta$ iff $\neg B \in \Delta$ or $\neg C \in \Delta$ (B, C internal);
5. $\neg(B \vee C) \in \Delta$ iff $\neg B \in \Delta$ and $\neg C \in \Delta$;
6. $\neg(B \otimes C) \in \Delta$ iff $\neg B \in \Delta$ and $\neg C \in \Delta$ (in fact not needed in the proof);
7. $\neg\neg B \in \Delta$ iff $B \in \Delta$.

From Δ we can define a mapping h by setting $h(p) = \top$ for insignificant variables and for significant ones:

$$h(p) = \begin{cases} 1, \text{ if } p \in \Delta, \ \neg p \notin \Delta \\ 0, \text{ if } p \notin \Delta, \ \neg p \in \Delta \\ \bot, \text{ if } p \notin \Delta, \ \neg p \notin \Delta. \end{cases}$$

Clearly the extension of h to a homomorphism assigns $\top$ to insignificant internal formulas B, whereas for the significant ones by induction on their complexity one can prove:

$$h(B) \in D_1 \text{ iff } B \in \Delta.$$

3.5 The positive logic of intuitionistic info-algebras Our next example will be the logic determined by the class of all info-algebras $\mathbf{L}^{\mathbf{set}}$ where $\mathbf{L}$ is a pseudo-Boolean algebra. In this case the language has an internal operation of implication $\supset$ and an internal pseudo-negation $\neg(\neg A = A \supset 0)$. The system extends the basic info-algebra logic with rules for the implication and an additional class of basic sequents similar to the case of info-algebras with negation: $\Gamma \vdash A$ is an axiom, if A is internal and also $B, \neg B \in \Gamma$, for some formula B with $Var(B) \subseteq Var(A)$. The rules for $\supset$ include:

$$(\supset\vdash)^+ \qquad \frac{\Gamma \vdash A; \ \Gamma, B \vdash C}{\Gamma, A \supset B \vdash C}$$

with the restrictions: C is internal, $Var(A) \subseteq Var(C)$ and a series of distributivity rules which compensate the absence of a suitable $(\vdash\supset)$-type rule:

$$(\supset \wedge \vdash) \qquad \frac{\Gamma, (C \supset A) \wedge (C \supset B) \vdash D}{\Gamma, C \supset (A \wedge B) \vdash D}$$

$$(\vdash\supset \wedge) \qquad \frac{\Gamma \vdash (C \supset A) \wedge (C \supset B)}{\Gamma \vdash C \supset (A \wedge B)}$$

$$(\supset \vee \vdash) \qquad \frac{\Gamma, (A \supset C) \wedge (B \supset C) \vdash D}{\Gamma, (A \vee B) \supset C \vdash D}$$

$$(\vdash\supset \vee) \qquad \frac{\Gamma \vdash (A \supset C) \wedge (B \supset C)}{\Gamma \vdash (A \vee B) \supset C}$$

$$(\supset \otimes \vdash) \qquad \frac{\Gamma, (C \supset A) \otimes (C \supset B) \vdash D}{\Gamma, C \supset (A \otimes B) \vdash D} \qquad \frac{\Gamma, (A \supset C) \otimes (B \supset C) \vdash D}{\Gamma, (A \otimes B) \supset C \vdash D}$$

$$(\vdash\supset \otimes) \qquad \frac{\Gamma \vdash (C \supset A) \otimes (C \supset B)}{\Gamma \vdash C \supset (A \otimes B)} \qquad \frac{\Gamma \vdash (A \supset C) \otimes (B \supset C)}{\Gamma \vdash (A \otimes B) \supset C}$$

$$(\supset\supset\vdash) \qquad \frac{\Gamma, (A \wedge B) \supset C \vdash D}{\Gamma, (A \supset B) \supset C \vdash D}$$

$$(\vdash\supset\supset) \qquad \frac{\Gamma \vdash (A \wedge B) \supset C}{\Gamma \vdash (A \supset B) \supset C}$$

A list of useful provable sequents would contain, e.g., B, $B \supset C \vdash B \wedge C$, $\neg(B \supset C) \vdash \neg\neg B \wedge \neg C$, $\neg\neg B \wedge \neg C \vdash \neg(B \supset C)$, $\neg(B \vee C) \vdash \neg B \wedge \neg C$, $\neg B \wedge \neg C \vdash \neg(B \vee C)$, $\neg B \vee \neg C \vdash \neg(B \wedge C)$, etc., besides sequents like $C \supset (A \otimes B) \vdash (C \supset A) \otimes (C \supset B)$, $(C \supset A) \otimes (C \supset B) \vdash C \supset (A \otimes B)$, $(A \otimes B) \supset C \vdash (A \supset C) \otimes (B \supset C)$, $(A \supset C) \otimes (B \supset C) \vdash (A \otimes B) \supset C$, etc., needed to show that just as in the previous cases one can concentrate exclusively on internal formulas when dealing with problems of derivability and semantic consequence: an analog of Lemmas 3.3 and 3.10 holds here, too. A counterpart of Lemmas 3.4 and 3.11 also holds for the present case. Thus in the proof of the completeness theorem we tread a familiar path.

Theorem 3.13 *The positive logic of intuitionistic info-algebras is complete with respect to the semantic consequence relation.*

Proof: We need the construction of relatively prime theories used in the previous two proofs. Starting from an unprovable *internal* sequent $\Gamma \vdash A$ one can find a theory Δ_0 maximal among theories containing Γ and omitting A. Δ_0 has three nice properties with respect to significant formulas B, C:

1. $B \wedge C \in \Delta$ iff $B \in \Delta$ and $C \in \Delta$;
2. $B \vee C \in \Delta$ iff $B \in \Delta$ or $C \in \Delta$;
3. if $B \in \Delta$ and $B \supset C \in \Delta$, then $C \in \Delta$.

For (3) recall that $B, B \supset C \vdash B \wedge C$ is provable, so if $B \in \Delta$ and $B \supset C \in \Delta$, then $B \wedge C \in \Delta$. For significant B and C this implies $C \in \Delta$.

Now we define a frame (in fact a generated subframe of the canonical frame) $\mathcal{F} = \langle W, \subseteq\rangle$, where $W = \{\Delta\colon \Delta_0 \subseteq \Delta$ and Δ is relatively prime with respect to the significant formulas$\}$. Thus the elements of W satisfy (1)–(3) above. The pseudo-Boolean algebra $\mathbf{A}(\mathcal{F})$ of all cones in W (with operations $a \cup b$, $a \cap b$, $a \to b$ and $\neg a = a \to \varnothing$) will be used in the spirit of Lemma 2.22, setting:

$$\varphi(\Delta, p) = \begin{cases} \top, \text{ if } p \in \Delta, \ \neg p \in \Delta \\ 1, \text{ if } p \in \Delta, \ \neg p \notin \Delta \\ 0, \text{ if } p \notin \Delta, \ \neg p \in \Delta \\ \bot, \text{ if } p \notin \Delta, \ \neg p \notin \Delta, \end{cases}$$

we extend $\varphi(D, p)$ to a member of $\mathrm{Hom}(\mathcal{L}_0, \mathbf{2}^{\mathbf{set}})$; having this φ we are able to define a mapping $||.||$ by

$$||B|| = \{a : |B|_1 \subseteq a, |B|_0 \subseteq \neg a\},$$

where $|B|_1 = \{\Delta \in W : \varphi(\Delta, B) = 1\}$, $|B|_0 = \{\Delta \in W : \varphi(\Delta, B) = 0\}$, and then to prove that when restricted to significant formulas $||.||$ is a homomorphism into $\mathbf{A}(\mathcal{F})^{\mathbf{set}}$, establishing thereby the fact that $\Gamma \not\models A$ since $\Gamma \subseteq \Delta$ for all members of W (thus for $B \in \Gamma$ one has $||B|| = \{W\} = 1$ in $\mathbf{A}(\mathcal{F})^{\mathbf{set}}$ or $||B|| = \top$) but obviously $||A|| \neq \{W\}$, since $A \notin \Delta_0$. We need to check whether:

1. $||B \wedge C|| = ||B|| \cap ||C||$;
2. $||B \vee C|| = ||B|| \cup ||C||$;
3. $||B \supset C|| = ||B|| \to ||C||$.

Leaving (1) and (2) to the reader, we treat the third equality: let us for example prove that $||B|| \to ||C|| \subseteq ||B \supset C||$, i.e., that $b \in ||B||$ and $c \in ||C|| \Rightarrow b \to c \in ||B \supset C||$. To this end we first demonstrate that $|B \supset C|_1 \subseteq b \to c$, in other words that

$$\varphi(\Delta, B \supset C) = 1 \Rightarrow \Delta \models b \to c.$$

We reason from the contrary: let $\varphi(\Delta, B \supset C) = 1$ but $\Delta \not\models b \to c$, thus $\exists \Delta' \supseteq \Delta(\Delta' \vdash b$ and $\Delta' \not\models c)$. Now $\Delta' \vdash b$ implies $\neg B \notin \Delta'$, whereas $\Delta' \not\models c$ implies $C \notin \Delta'$. Therefore we can extend $[\Delta', B, \neg C]$ to an element of $W - \Delta''$. Since $\Delta \subseteq \Delta''$ we have $B \supset C \in \Delta'$; together with $B \in \Delta''$ this yields $\{C, \neg C\} \subseteq \Delta^{''}$. This is a contradiction since C is significant.

Our second problem is $|B \supset C|_0 \subseteq \neg(b \to c)$, i.e., whether

$$\varphi(\Delta, B \supset C) = 0 \Rightarrow \Delta \vdash \neg(b \to c).$$

Reason as follows: assuming the contrary, i.e., that $\varphi(\Delta, B \supset C) = 1$ but $\Delta \not\models \neg(b \to c)$. Now we have a $\Delta' \supseteq \Delta$ such that $\Delta' \vdash b \to c$. Recalling that $\neg(B \supset C) \vdash \neg\neg B \wedge \neg C$ is provable and that $\varphi(\Delta, B \supset C) = 0$ forces $\neg(B \supset C) \in D$, it is clear that one can produce a $\Delta'' \supseteq \Delta'$ such that $B \in \Delta''$, $\neg C \in \Delta''$ which would obviously contradict the fact that $\Delta'' \vdash b \to c$. The opposite inclusion is established by similar reasoning.

3.6 Finitary logics of info-algebras We devote this subsection to the study of logics which arise when in the general algebraic scheme for the consequence operation the set of admissible valuations $\mathrm{Hom}(\mathcal{L}, \mathbf{A})$ is replaced by smaller families H of homomorphisms. As a first example we treat classes of generalized logical matrices based on info-algebras with $H = \{h : h(A)$ is a finite set for all $A\}$. Clearly any finitary map $h : Var \to \mathbf{A}$ can be extended to a unique homomorphism $h \in H$.

The positive finitary logic of all info-algebras coincides with the logic of all info-algebras (presented above). The interesting news here is the possibility of treating without complications a negative version of the logic, which is defined by the set of distinguished truth values $D^- = \{X : \sup X = 1\}$, i.e., D^- consists of the elements $\{x_1, \ldots, x_m\}$ of $\mathbf{A}$ for which $x_1 \vee \ldots \vee x_m = 1$.

For the axiomatization of the logic we need the following:

- the notion of an axiom taken unchanged from the positive case;
- we keep the rule (Cut);
- the rules for conjunction are taken without any restriction;
- $(\vee \vdash)$ is the same, but examples like $p \not\models p \vee q$ (consider $h(p) = 1, h(q) = \top$) show that $(\vdash \vee)$ is not correct and has to be altered to a weaker rule:

  $$(\vdash \vee)^- \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad\qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

  where we have familiar requirements: $Var(B) \subseteq Var(A)$ *and A is internal*, for the left rule, and $Var(A) \subseteq Var(B)$ *and B is internal* for the right;
- as should be expected the new rules for $\otimes$ are dual to the previous ones:

  $$(\otimes \vdash) \qquad \frac{\Gamma, A \vdash C; \ \Gamma, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$(\vdash \otimes) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \otimes B} \qquad\qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \otimes B};$$

- the added distributivity rules concern $\vee$ as a main connective:

$$(\vee\wedge\vdash) \quad \frac{\Gamma, (A \vee C) \wedge (B \vee C) \vdash D}{\Gamma, (A \wedge B) \vee C \vdash D}$$

$$(\vdash \vee\wedge) \quad \frac{\Gamma \vdash (A \vee C) \wedge (B \vee C)}{\Gamma \vdash (A \wedge B) \vee C}$$

$$(\vee\otimes\vdash) \quad \frac{\Gamma, (A \vee C) \otimes (B \vee C) \vdash D}{\Gamma, (A \otimes B) \vee C \vdash D}$$

$$(\vdash \vee\otimes) \quad \frac{\Gamma \vdash (A \vee C) \otimes (B \vee C)}{\Gamma \vdash (A \otimes B) \vee C}$$

$$(\vee\vee\vdash) \quad \frac{\Gamma, (A \vee B) \vee C \vdash D}{\Gamma, (A \vee C) \vee B \vdash D}$$

$$(\vdash \vee\vee) \quad \frac{\Gamma \vdash (A \vee C) \vee (B \vee C)}{\Gamma \vdash (A \vee B) \vee C}$$

One can easily check now that for each formula A there is a finite set of internal formulas $\{D_1, \ldots, D_m\}$ for which $A \vdash D_1 \otimes \ldots \otimes D_m$ and $D_1 \otimes \ldots \otimes D_m \vdash A$. This fact together with the corresponding semantic one, $A \models D_1 \otimes \ldots \otimes D_m$ and $D_1 \otimes \ldots \otimes D_m \models A$, reduces problems of provability of sequents and consequence relations to such problems in the domain of internal formulas: $\Gamma \vdash A$ iff for some i, $\Gamma \vdash D_i$, $\Gamma \vdash A$ iff for some i, $\Gamma \vdash D_i$.

Lemma 3.14 *The resulting system is correct: if $\Gamma \vdash A$ is provable, then $\Gamma \models A$.*

Proof: Standard. When checking for example the correctness of $(\vee \vdash)$ we need the proposition about homomorphisms into $\mathbf{2}^{\mathbf{set}}$ defined by prime filters F in the lattice underlying a given info-algebra and the fact that such homomorphisms map any set X with $\sup X \in F$ into D^- (because of the finiteness of X).

The proof of the completeness theorem mimics the proofs offered above: if $\Gamma \vdash A$ is not provable (assume without loss of generality that they are both internal) extend Γ to a maximal theory Δ omitting A. Call a variable p *significant*, if it occurs in an internal formula $B \in \Delta$. For Δ we have:

1. $B \wedge C \in \Delta$ iff $B \in \Delta$ and $C \in \Delta$ (for any formulas B and C);
2. $B \vee C \in \Delta$ iff $B \in \Delta$ or $C \in \Delta$ (for significant internal B and C);
3. $B \otimes C \in \Delta$ iff $B \in \Delta$ or $C \in \Delta$ (for any formulas B and C).

Although (1) and (3) are routine, (2) needs some attention. The implication from left to right depends on the unrestricted rule $(\vee \vdash)$ and is standard; the converse implication is checked as follows: assume $B \in \Delta$, then if $C \in \Delta$, we have $B \wedge C \in \Delta$ and in view of the provability of $B \wedge C \vdash B \vee C$, so we are done. If $C \notin \Delta$, then since C is significant, there is an internal formula D with $Var(C) \subseteq Var(D)$ such that $D \in \Delta$ and consequently $B \wedge D \in \Delta$. On one hand:

$$\dfrac{\dfrac{B \wedge D \vdash B \wedge D}{B \wedge D \vdash (B \wedge D) \vee C; \; (B \wedge D) \vee C \vdash (B \vee C) \wedge (D \vee C) \text{ (provable sequent)}} (\vdash \vee)^-}{B \wedge D \vdash (B \vee C) \wedge (D \vee C)} \text{(Cut)}$$

On the other hand

$$\frac{B \vee C \vdash B \vee C}{(B \vee C) \wedge (D \vee C) \vdash B \vee C} \ (\wedge \vdash)$$

and applying (Cut) again, we obtain $B \wedge D \vdash B \vee C$, which is what we need.

Defining for variables p:

$$h(p) = \begin{cases} 1, & \text{if } p \text{ is significant and } p \in \Delta \\ 0, & \text{if } p \text{ is significant and } p \notin \Delta \\ \top, & \text{otherwise,} \end{cases}$$

and extending it to a homomorphism from $\mathrm{Hom}(\mathcal{L}, \mathbf{2^{set}})$, we can see that for significant internal formulas B:

$$B \in D \text{ iff } h(B) \in D^{-}.$$

To nonsignificant formulas h assigns $\top$. All members of Γ get values which are "true." Consider our formula A: either it is significant, and then its value is 0; or it is nonsignificant, and then its value is $\top$. Anyway A is not "true" according to h. Thus we have established the following.

Theorem 3.15 *The logic is complete: if* $\Gamma \models A$, *then* $A \in [\Gamma]$.

The finitary logics of the class of *external info-algebras* coincide in fact with the finitary logics of *all* info-algebras (in a language extended with $\blacksquare$ and $\blacklozenge$) since finite sets X have always sup X and inf X. Now we can identify D_1 with $\{X : \blacksquare X = 1\}$ and D_0 with $\{X : \blacklozenge X = 1\}$. Thus the positive finitary logic extends the basic info-algebra systems with all the rules concerning $\blacksquare$ and $\blacklozenge$. Since the set of distinguished truth values is D_1 we have $\blacksquare A \models A$ and $A \models \blacksquare A$, so the corresponding rules should be:

$$(\blacksquare \vdash) \quad \frac{\Gamma, A \vdash C}{\Gamma, \blacksquare A \vdash C}$$

$$(\vdash \blacksquare) \quad \frac{\Gamma \vdash A}{\Gamma \vdash \blacksquare A.}$$

For $\blacklozenge$ we have a series of rules (parallel to the rules concerning negation):

$$(\blacklozenge\wedge \vdash) \quad \frac{\Gamma, \blacklozenge A \vdash C}{\Gamma, \blacklozenge(A \wedge B) \vdash C} \qquad \frac{\Gamma, \blacklozenge B \vdash C}{\Gamma, \blacklozenge(A \wedge B) \vdash C}$$

$$(\vdash \blacklozenge\wedge) \quad \frac{\Gamma \vdash \blacklozenge A;\ \Gamma \vdash \blacklozenge B}{\Gamma \vdash \blacklozenge(A \wedge B)}$$

$$(\blacklozenge\vee \vdash) \quad \frac{\Gamma, \blacklozenge A \vdash C;\ \Gamma, \blacklozenge B \vdash C}{\Gamma, \blacklozenge(A \vee B) \vdash C}$$

$$(\vdash \blacklozenge\vee) \quad \frac{\Gamma \vdash \blacklozenge A}{\Gamma \vdash \blacklozenge(A \vee B)} \qquad \frac{\Gamma \vdash \blacklozenge B}{\Gamma \vdash \blacklozenge(A \vee B)}$$

$$(\blacklozenge\otimes \vdash) \quad \frac{\Gamma, \blacklozenge A \vdash C;\ \Gamma, \blacklozenge B \vdash C}{\Gamma, \blacklozenge(A \otimes B) \vdash C}$$

$$(\vdash \blacklozenge\otimes) \quad \frac{\Gamma \vdash \blacklozenge A;\ \Gamma \vdash \blacklozenge B}{\Gamma \vdash \blacklozenge(A \otimes B)}$$

$$(\blacklozenge\blacklozenge \vdash) \quad \frac{\Gamma, \blacklozenge A \vdash C}{\Gamma, \blacklozenge\blacklozenge A \vdash C}$$

$$(\blacklozenge\blacksquare\vdash)\qquad \frac{\Gamma, \blacksquare A \vdash C}{\Gamma, \blacklozenge\blacksquare A \vdash C}$$

$$(\vdash\blacklozenge\blacklozenge)\qquad \frac{\Gamma \vdash \blacklozenge A}{\Gamma \vdash \blacklozenge\blacklozenge A}$$

$$(\vdash\blacklozenge\blacksquare)\qquad \frac{\Gamma \vdash \blacksquare A}{\Gamma \vdash \blacklozenge\blacksquare A.}$$

The correctness of the additions follows easily from the fact that modalized formulas cannot have $\top$ as a value. For the proof of completeness we need the machinery of relatively prime theories developed above, but fortunately there are no unexpected complications. The negative logic extends the basic finitary negative system with the k-dualized versions of the just cited rules and can be proven complete with respect to the finitary info-algebra matrices with D_0.

3.7 Consistent logics Under the term *consistent* we understand here logical systems that are defined semantically by classes of matrices with the following requirement on $H \subseteq \mathrm{Hom}(\mathcal{L}, A)$: *the values in the range of any $h \in H$ are consistent., i.e., different from* $\top$. The restriction causes changes in the language of the logics: $\top$ is dropped for obvious reasons.

In the info-algebra situation the consistent valuations validate such rules as $(\wedge\vdash)$, as well as $(\vdash\vee)$ in the negative case, without any restrictions (so in the corresponding consistent logics the distributivity rules are redundant). In view of the above remark we have the following.

Proposition 3.16 *The positive and negative consistent logic of all info-algebras coincide with the logic of* **3**. *The same is true for the case of algebras with negation.*

4 Conclusion Let us first briefly recapitulate our findings, namely the logics we have axiomatized:

- the standard positive full logics of all info-algebras, all info-algebras with negation, all set expansions of pseudo-Boolean algebras;
- the finitary logics (positive and negative) of all info-algebras, all info-algebras with negation, etc.
- the consistent versions of all the systems mentioned above.

All these logics are new, a fact due mainly to the presence of new connectives: $\otimes$, $-$, constants, etc., but even in the case of a language containing only traditional operators some systems appear in print for the first time, in particular the systems related to intuitionistic semantics.

Our aim here was to investigate in some detail the construction of set expansions as a way of treating uncertainty in logic, so let us mention some other approaches and compare very briefly the basic ideas behind them.

4.1 Probability approaches Under this title we classify attempts to represent the *uncertainty/plausibility* of knowledge and inference by assigning a probabilistic measure to statements, the so-called *probability distributions*, with the idea that the more plausible a proposition A, the greater its probability $p(A)$, etc. Many thought that a unique value is not realistic and turned to probability *intervals*, discussed by Dempster [6] among others.

Interval values take care of uncertainty pretty much in the same way as our sets, intervals being special cases of sets of real numbers. For example in Gärdenfors [10] we find the note that the two limiting probabilities—left and right ends of the corresponding interval—$p_*(A)$ and $p^*(A)$ must be interconnected with the following relation:

$$p_*(A) = 1 - p^*(\neg A),$$

which strenghtens the similarity with external modalities in set expansions. Still within the probabilistic approach not much attention is paid to the degree of knowledge ordering.

4.2 Fuzzy logic Earlier, some people working in fuzzy set theory felt uneasy with the possibility of knowing the exact numerical value which a fuzzy predicate assigns to a particular object, so among the proposals for a more quantitatively realistic picture was the idea of *interval valued fuzzy sets*: functions assigning to elements of a domain E not numbers but open intervals (a, b) of the unit interval $[0, 1]$ (cf. e.g., [1]).

In the same vein, but in another field—*Artificial Intelligence*—Sandewall [25] proposed to consider intervals of real numbers $[a, b]$ as representatives of *what we know* about the truth value of a proposition evaluated by "fuzzy" methods. He also explicitly defined the knowledge order as inverse set inclusion.

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