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The Logic of Non-contingency

I. L. HUMBERSTONE

Abstract We consider the modal logic of non-contingency in a general setting, without making special assumptions about the accessibility relation. The basic logic in this setting is axiomatized, and some of its extensions are discussed, with special attention to the expressive weakness of the language whose sole modal primitive is non-contingency (or equivalently, contingency), by comparison with the usual language based on necessity (or equivalently, possibility).

1 Introduction We investigate the general logic of the singulary operator Δ with the intended reading of ΔA (A any formula) being "it is non-contingent whether A." Δ was introduced in Montgomery and Routley [5], along with an inverted Δ operator (which we shall not use) for expressing "it is contingent whether A."[1] In more detail, we take the language of truth-functional logic, with, for definiteness, $\bot$, (zero-ary) and $\rightarrow$ (binary) as primitive boolean connectives, the other boolean connectives introduced as by any standard definitions, as well as singulary Δ, and work with the language based on a set $\mathbb{P}$ of countably many propositional variables (sentence letters) $p_1, \ldots, p_n, \ldots$ (We write 'p', 'q' for 'p_1', 'p_2'.) Models are triples (W, R, V) with $W \neq \varnothing$, $R \subseteq W \times W$, $V : \mathbb{P} \longrightarrow \mathcal{P}(W)$, with an inductive definition of the truth of a formula A at a point $x \in W$ in a model $\mathcal{M}$ of this form (notation: $\mathcal{M} \models_x A$) as follows.

$$\mathcal{M} \models_x p_i \quad \text{iff} \quad x \in V(p_i) \tag{1.1}$$

$$\mathcal{M} \not\models_x \bot;\ \mathcal{M} \models_x A \rightarrow B \quad \text{iff} \quad \mathcal{M} \not\models_x A \text{ or } \mathcal{M} \models_x B \tag{1.2}$$

$$\mathcal{M} \models_x \Delta A \quad \text{iff} \quad \forall y, z \in R(x),\ \text{if } \mathcal{M} \models_y A \text{ then } \mathcal{M} \models_z A \tag{1.3}$$

In (1.3) $R(x)$ denotes $\{w \in W \mid xRw\}$: the set of points "accessible" to x. Alternative formulations of (1.3) which it is sometimes convenient to work with include the following.

$$\mathcal{M} \models_x \Delta A \quad \text{iff} \quad \text{for all } y, z \in R(x),\ \mathcal{M} \models_y A \text{ iff } \mathcal{M} \models_z A \tag{1.4}$$

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$$\mathcal{M} \models_x \Delta A \quad \text{iff} \quad \text{either } \mathcal{M} \models_y A \text{ for all } y \in R(x) \text{ or } \mathcal{M} \not\models_y A \text{ for all } y \in R(x) \tag{1.5}$$

If the first disjunct of the right-hand side of (1.5) holds, we speak of A as necessary at x (in $\mathcal{M}$), and if the second disjunct holds, of A as impossible at x. (Note that these cases are not mutually exclusive, since $R(x)$ may be empty.) Of course in the usual language of modal logic, one would write $\Box A$ and $\Box\neg A$ (or $\neg\Diamond A$), respectively, for these cases, so that ΔA becomes definable in that language as the disjunction of these representations.

Conversely, the definability of $\Box$ in terms of Δ and the boolean connectives is explored in Cresswell [2]. Specifically, Cresswell there considers normal modal logics with $\Box$ as primitive, and asks whether $\Box p$ is provably equivalent to some formula built up from p with the boolean connectives and Δ (taking ΔA as abbreviating $\Box A \vee \Box\neg A$). He in effect partitions the class of normal modal logics into three subclasses:

Class 1. Systems S with $\vdash_S \Box\bot$,
Class 2. Systems S with $\vdash_S \neg\Box\bot$,
Class 3. Systems S such that $\nvdash_S \Box\bot$ and $\nvdash_S \neg\Box\bot$.

With respect to this partition the results found are as follows. $\Box$ is definable in the sole normal system in Class 1 (the "*Verum*" or "Absurd" system), even without the aid of Δ, since here we have Δp (like $\Box p$ itself) provably equivalent to $p \to p$. In no system in Class 3, is $\Box$ definable ([2], Theorem 3). In some systems in Class 2 (= normal extensions of the system **KD**), $\Box$ is definable, whereas in others it is not. The logics originally investigated in [5] are all in this class, and in fact all extend **KT**, for which the definition of $\Box A$ as $\Delta A \wedge A$ is available. (See also Mortensen [4] and the references therein to other papers by Montgomery and Routley, in which nonnormal as well as normal systems are considered, all of them sharing the **T** schema $\Box A \to A$.) Cresswell also gives an example ([2], Theorem 9) of a logic in Class 2 not extending **KT** in which $\Box$ is definable. (Nomenclature for normal modal logics here and below follows Chellas [1], q.v. also for any other terminology not explained here; note that we use the terms 'logic' and 'system' interchangeably.) It would be interesting to have a general characterization of the normal modal logics in which $\Box$ is definable in terms of Δ; in view of Cresswell's observations, what is required is an analysis of the situation in Class 2.

We should note, in view of the fact that we are considering arbitrary (classes of) models, as well as arbitrary normal modal logics, that the reading of 'ΔA' as "it is non-contingent whether A" is not universally appropriate, for the same reason as it is not always appropriate to read '$\Box$' as "necessarily" (or '$\Diamond$' as "possibly"). Such readings are particularly ill-suited, if taken literally, to systems without the **T** schema and their Δ-based analogues. (Accordingly, von Wright [10], p. 61, spoke of its being (morally) *indifferent* whether A, to express the deontic analogue of contingency.) We retain them only for the sake of convenience. Likewise with the use of 'necessary' and 'impossible' introduced after (1.5) above.

Before proceeding, we need to adapt some familiar semantic terminology. Let a *non-contingency logic* be any set of formulas in the language described in our opening paragraph, which includes all truth-functional tautologies and is closed under Uniform Substitution (of arbitrary formulas for propositional variables) and Modus Ponens (equivalently: under tautological consequence). If $\mathbb{C}$ is a class of models, in which formulas are interpreted as by (1.1) – (1.3), an axiomatically presented proof-system for non-contingency logic is *sound* with respect to $\mathbb{C}$ if every formula provable in the system is true at every point in every model in $\mathbb{C}$, complete with respect to $\mathbb{C}$ if every formula true at every point in every model in $\mathbb{C}$ is provable in the system, and is *determined by* $\mathbb{C}$ if it is both sound and complete with respect to $\mathbb{C}$. Given a normal modal logic S, we understand by $(S)_\Delta$ the set of theorems of S containing the boolean connectives and Δ (with ΔA taken as abbreviating $\Box A \vee \Box\neg A$). This same collection of formulas is then a non-contingency logic (in fact what we shall later call an *nc*-normal such logic) when Δ is regarded as a primitive connective in its own right. Clearly, whenever S is determined by $\mathbb{C}$, $(S)_\Delta$ is also determined by $\mathbb{C}$. The converse is false, since one may have $(S_1)_\Delta = (S_2)_\Delta$ even when $S_1 \neq S_2$: we shall see several illustrations of this possibility in Section 4section.4. It arises because of differences between S_1 and S_2 which do not emerge when attention is restricted to their respective fragments in which the only occurrences of $\Box$ are in subformulas of the form $\Box A \vee \Box\neg A$.

Now if, in some normal modal logic S, $\Box$ is definable in terms of the boolean connectives and Δ, then an axiomatization of S can be converted, using the resulting interdefinability of Δ and $\Box$, into an axiomatization of $(S)_\Delta$, by a method indicated in Hiż [3]. Montgomery and Routley [5] (from which—see the references in [4]—the above "$(S)_\Delta$" notation is adapted) in effect further processes the cumbersome results of this direct method into a much more elegant axiomatization, arriving, in the case of **KT**, for example, at an axiomatization of $(\mathbf{KT})_\Delta$ with the following rule,

$$\frac{A}{\Delta A}, \tag{1.6}$$

added to a basis for truth-functional logic, together with axiom schemata,[2]

$$\Delta A \leftrightarrow \Delta\neg A \tag{1.7}$$

$$A \rightarrow (\Delta(A \rightarrow B) \rightarrow (\Delta A \rightarrow \Delta B)). \tag{1.8}$$

They further show that we obtain non-contingency formulations of **S4** and **S5** by adding the schemata (1.9), (1.10), respectively.

$$\Delta A \rightarrow \Delta\Delta A \tag{1.9}$$

$$\Delta\Delta A \tag{1.10}$$

Elegant as these results are, they leave untouched the general case, in which we do not have $\Box$ definable in terms of Δ. (As Segerberg [7], p. 128, remarks of the Montgomery-Routley enterprise, the definition of $\Box A$ as $\Delta A \wedge A$ "accord(s) with intuitive preconceptions" only in extensions of **KT**; see the parenthetical comment at the end of the proof of Theorem 1 in Cresswell [2] for a sharpening of this remark.) At its most general, the case that needs to be considered is that of the non-contingency logic determined by the class of *all* models—the analogue for Δ of the system **K**, the smallest normal modal logic, for $\Box$ (in the notation introduced above, this is the system $(\mathbf{K})_\Delta$). We shall axiomatize this system in Section 3section.3 below, with some remarks on extensions (as well as on the non-contingency analogue of modal definability) deferred to Section 4section.4. In Section 2section.2, we make some preparatory remarks.

2 Some principles for non-contingency We use the term 'principle' here to cover rules as well as (potential) axiom schemata. The schema (1.7) of the Montgomery-Routley axiomatization mentioned in the preceding section does not owe its validity to any restriction of attention to models with reflexive accessibility relations, so we shall need to prove all instances of this schema in any non-contingency logic complete with respect to the class of all models. The left-to-right direction is particularly interesting as an illustration of something more general.

$$\Delta A \to \Delta \neg A \tag{2.1}$$

The more general principle involved here is

$$(\Delta A_1 \wedge \ldots \wedge \Delta A_n) \to \Delta \natural(A_1, \ldots, A_n) \tag{2.2}$$

in which $\natural$ is any (n-ary, $n > 0$) boolean mode of composition. (That is, $\natural(p_1, \ldots, p_n)$ is a formula built from at most the displayed variables using only boolean connectives, and $\natural(A_1, \ldots, A_n)$ is the result of substituting A_i for p_i in this formula.) (2.1) is the special case in which $n = 1$ and $\natural(A)$ is $\neg A$. No instance of (2.2) can be false at any point x in a model, since if the antecedent is true at x, then all elements of $R(x)$ assign the same truth value to A_1, the same truth value to $A_2, \ldots,$ the same truth value to A_n, in which case all elements of $R(x)$ must assign the same truth value to $\natural(A_1, \ldots, A_n)$, since the latter is whatever value is dictated by the truth function associated (in the obvious way) with $\natural$ when this is applied to the shared truth values of $A_1, \ldots, A_n$.

The Montgomery-Routley rule (1.6), analogous to the familiar rule of Necessitation, is also closely related to our general compositional principle (2.2), in the case where $n = 0$. The conjunctive antecedent vanishes in this case, and we are left with the consequent. So when we take $\natural$ as $\top (= \neg\bot$, or, in fully primitive notation, $\bot \to \bot)$, we have,

$$\Delta\top \tag{2.3}$$

as a special case of (2.2). So all we should need, for deriving (1.6) from (2.2) would be the rule of "congruentiality" (substitutivity of provable equivalents) for Δ:

$$(\Delta\text{Cong}) \qquad \frac{A \leftrightarrow B}{\Delta A \leftrightarrow \Delta B}.$$

Then from $\vdash A$ we can pass to $\vdash \Delta A$ using (Δ Cong), Modus Ponens, and (2.3). Alternatively, if we start with the following schema considered by Mortensen (see [4], and also [5]),

$$\Delta(A \leftrightarrow B) \to (\Delta A \leftrightarrow \Delta B), \tag{2.4}$$

then from (the provability of) a premise for (ΔCong), we can prefix a "Δ" by (1.6), and then detach the consequent by appeal to (2.4), obtaining the corresponding conclusion of (ΔCong). Going in the reverse direction, we may derive (2.4) from

(ΔCong) and (2.2) by using the following special case of the latter principle:

$$(\Delta A \wedge \Delta B) \to \Delta(A \leftrightarrow B). \tag{2.5}$$

More specifically still, we have

$$(\Delta(A \leftrightarrow B) \wedge \Delta A) \to \Delta((A \leftrightarrow B) \leftrightarrow A). \tag{2.6}$$

(ΔCong) and some purely truth-functional manipulations then give (2.7) and (2.8):

$$(\Delta(A \leftrightarrow B) \wedge \Delta A) \to \Delta B \tag{2.7}$$

$$\Delta(A \leftrightarrow B) \to (\Delta A \to \Delta B) \tag{2.8}$$

By a parallel derivation, we obtain a schema like (2.8) with A and B interchanged in the consequent, and the conjunction of these two is of course equivalent to (2.4).

For the axiomatization we shall show in the following section to be complete (with respect to the class of all models), however, we proceed somewhat differently and set aside (2.2) along with its special cases (2.1), (2.3), (2.5,6). We will there work with the converse of (2.1), which we call $(\Delta\neg)$, as well as a collection of rules, only one of which will be discussed here, to be called $(\text{NCR})_1$, the letters standing for "Noncontingency Rule'. We will derive (ΔCong) from this rule shortly and, in the following section, derive (2.2) from the collection of rules just mentioned as generalizing $(\text{NCR})_1$.

$(\Delta\neg)$ $\qquad \Delta\neg A \to \Delta A$

$(\text{NCR})_1$ $\qquad \dfrac{A \to B_0 \qquad\qquad \neg A \to B_1}{\Delta A \to (\Delta B_0 \vee \Delta B_1)}.$

We check that $(\text{NCR})_1$ preserves truth throughout any model. Suppose the premises for an application of $(\text{NCR})_1$ are true throughout a model $\mathcal{M} = (W, R, V)$ and that the antecedent of the conclusion is true at $x \in W$. Then, at x, A is either necessary or impossible. If A is necessary, then, since the left premise is true throughout $\mathcal{M}$, B_0 is necessary, so that ΔB_0 is true at x. If A is impossible, then $\neg A$ is necessary, so, the right premise being true throughout $\mathcal{M}$, B_1 is necessary, making ΔB_1 true at x. So either way, the disjunctive consequent $\Delta B_0 \vee \Delta B_1$ is true at x.

For those who find the detour through the necessity of $\neg A$ in the preceding argument distasteful, we note a more symmetrical $\neg$-free version of $(\text{NCR})_1$.

$$\frac{A \to B \qquad\qquad C \to A}{\Delta A \to (\Delta B \vee \Delta C)} \tag{2.9}$$

This rule is interderivable with $(\text{NCR})_1$, given $(\Delta\neg)$. Its import can be seen more readily by contraposing the conclusion, which then says that if B and C are contingent (as of some point in some model), then so is A. (2.9) thus amounts to a convexity principle for contingency: anything (here A) "inferentially sandwiched" between contingencies (B, C) is itself contingent. (It would not, of course, be correct to claim that whatever followed from—or that whatever implied—something contingent was itself contingent.) We shall continue to work with $(\text{NCR})_1$, however, for continuity with material in the following section.

The replacement principle (ΔCong) mentioned above is derivable from $(\text{NCR})_1$ and $(\Delta\neg)$, since if $\vdash A \leftrightarrow B$, we have $\vdash A \rightarrow B$ and $\vdash \neg A \rightarrow \neg B$, so by $(\text{NCR})_1$, taking B as B_0 and $\neg B$ as B_1, we conclude that $\vdash \Delta A \rightarrow (\Delta B \vee \Delta\neg B)$, whence by $(\Delta\neg)$ and truth-functional logic, $\vdash \Delta A \rightarrow \Delta B$. Appealing to ($\Delta$Cong), we may derive the converse of $(\Delta\neg)$, by beginning with an instance of the latter principle, $\Delta\neg\neg A \rightarrow \Delta\neg A$, and then replacing $\neg\neg A$ by A in accordance with (ΔCong), which delivers (2.1) above. (Of course, we could equally well reverse this argument, and derive $(\Delta\neg)$ from (2.1).)

By way of preparation for the following section, let us see what becomes of an attempt to show that the smallest non-contingency logic containing all instances of $(\Delta\neg)$ and closed under the rule $(\text{NCR})_1$ is complete with respect to the class of all models. (We know already that the corresponding claim of soundness would be correct.) As usual, we build a canonical model $\mathcal{M}$ for the system, where $\mathcal{M} = (W, R, V)$ and W is the set of all sets of formulas maximal consistent with respect to the present logic. V is defined in the usual way: $V(p_i) = \{x \in W \mid p_i \in x\}$, whereas R is defined in terms of a certain function λ we proceed to describe. The intuitive idea is that for $x \in W$, $\lambda(x)$ is the set of formulas which are necessary at x. We think of $\lambda(x)$ as a "labeling" of all formulas A such that $\Delta A \in x$, labeling each such formula as *Necessary* (recorded by putting A into $\lambda(x)$) or else as *Impossible* (putting $\neg A$ into $\lambda(x)$). Since necessity is not only not a primitive connective of our language but, as remarked in Section 1, is in the general case—and therefore in particular, for the current minimal system—not definable, we cannot take the usual course at this point (putting $\lambda(x) = \{A \mid \Box A \in x\}$) and must exercise greater ingenuity.

Provisionally, we might entertain the following definition of λ. For $x \in W$, put

$$\lambda(x) = \{A \mid \Delta A \in x \text{ and } \forall B \text{ such that } \vdash A \rightarrow B, \Delta B \in x\}.$$

The idea of the entry condition on A, that only such A (with $\Delta A \in x$) should be labeled as Necessary if all their consequences are non-contingent, is that although the class of non-contingencies is not closed under logical implication (so that a Monotony Rule analogous to (ΔCong) but with $\leftrightarrow$ replaced by $\rightarrow$) would destroy the soundness of our logic), those non-contingencies which qualify as such because they, rather than their negations, are necessary and have only non-contingent consequences, since those consequences are themselves necessary.

The above definition of λ allows us to show (by an argument the interested reader can easily provide) the following.

$$\text{For any formula } A,\ \text{any } x \in W, \Delta A \in x \text{ iff } A \in \lambda(x) \text{ or } \neg A \in \lambda(x), \tag{2.10}$$

and we should like to define in terms of λ the accessibility relation R of the canonical model by saying: for $x, y \in W$, xRy if and only if $\lambda(x) \subseteq y$. The strategy would then be to show (cf. (1.5)) that for all formulas A, and all $x \in W$,

$$\Delta A \in x \iff \forall y(xRy \Rightarrow A \in y) \text{ or } \forall y(xRy \Rightarrow A \notin y), \tag{2.11}$$

as is needed for the case of Δ in the inductive proof of the claim that membership and truth coincide in the canonical model. (We assume familiarity with the Scott-Makinson canonical model technique here; the claim mentioned is the analogue of

Theorem 5.7 of [1].) The $\Rightarrow$ direction follows from the $\Rightarrow$ direction of (2.10). But for $\Leftarrow$, we need that if $\Delta A \notin x$ then $\lambda(x) \cup \{A\}$ and $\lambda(x) \cup \{\neg A\}$ are consistent. Nothing has been said to guarantee this. Take the latter case. We should need to show we cannot have A following (according to the present logic) from $\lambda(x)$, to show which we would presumably need to know that, for any B following from $\lambda(x)$ we have $\Delta B \in x$, in order to get a contradiction from the fact that $\Delta A \notin x$. Our rule $(\mathrm{NCR})_1$ does not guarantee this, and the developments of the following section can be seen as supplementing the basis consisting of this rule together with $(\Delta\neg)$ so that this part of the argument will go through.

It is clear anyway that $(\mathrm{NCR})_1$ and $(\Delta\neg)$ cannot constitute the whole non-truth-functional part of a complete basis for the general logic of Δ, so that the hitch just noted is not merely a difficulty for the particular completeness-proof strategy attempted. For (2.3), by way of example, is certainly not provable on this basis, even though true at every point in every model. To see this, interpret 'Δ' as expressing the constant false (1-ary) truth-function rather than in accordance with (1.3). All instances of $(\Delta\neg)$ are true at every point in every model, with this change to the truth-definition, whereas the rule $(\mathrm{NCR})_1$ preserves this property; since no formula of the form ΔA has the property thus preserved, no such formula is provable.

3 *Axiomatizing the general logic of non-contingency* We respond to the difficulties encountered in the would-be completeness proof of the preceding section by providing a more general non-contingency rule than $(\mathrm{NCR})_1$. To be more precise, we offer a whole battery of such rules, namely rules $(\mathrm{NCR})_k$ for each $k \in \omega$, of which our old rule is indeed the $k = 1$ case. For the general case, we schematically denote by $s(A_1, \ldots, A_k)$ a "state-description" in the A_i: a k-termed conjunction, that is, each of whose conjuncts is either A_i or else $\neg A_i$ $(i = 1, \ldots, k)$. Enumerate these 2^k state descriptions by subscripting the "s" with $1, \ldots, 2^k$. Then, for each k, the rule we need is the following.

$$(\mathrm{NCR})_k \qquad \frac{s_1(A_1, \ldots, A_k) \to B_1 \ldots s_{2^k}(A_1, \ldots, A_k) \to B_{2^k}}{(\Delta A_1 \wedge \ldots \wedge \Delta A_k) \to (\Delta B_1 \vee \ldots \vee \Delta B_{2^k})}.$$

For the same reasons as in the special case already considered $(k = 1)$, we have, for any $k \in \omega$, that the rule $(\mathrm{NCR})_k$ preserves the property of being true throughout an arbitrary model.[3] Thus the smallest non-contingency logic containing all instances of $(\Delta\neg)$ and closed under each of the rules $(\mathrm{NCR})_k$, a logic we shall call **NC**, is sound with respect to the class of all models. It is this logic we will be showing to be determined by that class of models, but before passing to the "completeness" half of this claim, we make a couple of remarks about the rules $(\mathrm{NCR})_k$.

The first remark concerns the case of $k = 0$. As always, we have 2^k premises, so in the present instance we are dealing with a 1-premise rule. Since there are no A_i, the antecedent of the premise vanishes, with the effect that rule licenses transitions from B_1 to ΔB_1. So here we have the Necessitation-like rule ("Noncontingentization," more accurately) called (1.6) above, and whose underivability from $(\mathrm{NCR})_1$ together with $(\Delta\neg)$ is evident from the unprovability on that basis (remarked on at the end of the preceding section) of (2.3).

Secondly, we consider the fate of the compositionality principles (2.2) in **NC**: $(\Delta A_1 \wedge \ldots \wedge \Delta A_n) \to \Delta\natural(A_1, \ldots, A_n)$. Recall that $\natural(A_1, \ldots, A_n)$ here is any boolean compound of $A_1, \ldots, A_n$. (Here we assume $n \geq 1$.) Thus for each state-description s_i $(1 \leq i \leq 2^n)$ in $A_1, \ldots, A_n$, we have either,

$$\vdash_{\mathbf{NC}} s_i(A_1, \ldots, A_n) \to \natural(A_1, \ldots, A_n),$$

or else,

$$\vdash_{\mathbf{NC}} s_i(A_1, \ldots, A_n) \to \neg\natural(A_1, \ldots, A_n).$$

The rule $(\mathrm{NCR})_n$ therefore delivers the conclusion whose antecedent is the conjunction $\Delta A_1 \wedge \ldots \wedge \Delta A_n$, (which is the antecedent of (2.2)) and whose consequent is a disjunction of various occurrences of $\Delta\natural(A_1, \ldots, A_n)$ and $\Delta\neg\natural(A_1, \ldots, A_n)$. By $(\Delta\neg)$ and truth-functional logic, this disjunction collapses to the desired consequent of (2.2): $\Delta\natural(A_1, \ldots, A_n)$.

We now turn to the completeness proof for **NC**. It will be convenient to use the abbreviation $S \vdash_{\mathbf{NC}} A$, where S is any set of formulas, to mean that for some conjunction B (in some arbitrary order) of finitely many elements of S, we have $\vdash_{\mathbf{NC}} B \to A$. As in the preceding section, we define the canonical model for the present system, $\mathcal{M}_{\mathbf{NC}}$, to be $(W_{\mathbf{NC}}, R_{\mathbf{NC}}, V_{\mathbf{NC}})$, understanding $W_{\mathbf{NC}}$ and $V_{\mathbf{NC}}$ as W and V were in that discussion, except now taking (maximal) consistency with respect to **NC** rather than with respect to the weaker logic there in play. And we shall define $xR_{\mathbf{NC}}y$ to hold precisely when $\lambda(x) \in y$, with λ yet to be defined. (Actually, "defined" is the wrong word, given the nonconstructive nature of the proof which follows: "shown to exist" would be more accurate.) Recall the idea that $\lambda(x)$ is to be a "labeling" of the formulas A_i for which $\Delta A_i \in x$: we label a formula with the identity prefix (with nothing at all, that is) to mark it as necessary according to x, and with a $\neg$, to mark it as impossible according to x.

We work up to the general labeling by noting the existence of suitable finite labelings of sets of formulas $A_1, \ldots, A_n$ with $\Delta A_1, \ldots, \Delta A_k \in x \in W_{\mathbf{NC}}$. "Suitable" here means that we want our partial labeling—call it $\lambda_k(x)$—to satisfy the condition (2.10) of the preceding section, as far as these formulas are concerned,

(α) $\Delta A \in x$ iff $A \in \lambda_k(x)$, for $A \in \{A_1, \ldots, A_k\}$,

as well as the condition we noted the need for in that section:

(β) For any formula B, if $\lambda_k(x) \vdash_{\mathbf{NC}} B$, then $\Delta B \in x$.

We can choose as the desired $\lambda(x)$ the set of conjuncts of some state-description $s_i(A_1, \ldots, A_k)$. Clearly any of these satisfies (α). If no such state-description provides a set of conjuncts satisfying (β), then for each $s_i(A_1, \ldots, A_k)$, there exists B_i with $\vdash_{\mathbf{NC}} s_i(A_1, \ldots, A_k) \to B_i$ and $\Delta B_i \notin x$. But all these $s_i(A_1, \ldots, A_k) \to B_i$ (with $1 \leq i \leq 2^k$) make up the premises for an application of $(\mathrm{NCR})_k$, whose conclusion then gives us a contradiction, since its antecedent is in x (as each $\Delta A_j \in x$), but its consequent is not (since no $\Delta B_i \in x$). Therefore some $\lambda_k(x)$ satisfying (β) does indeed exist.

The next problem is: how do we infer the existence of a suitable $\lambda(x)$ which works for all A such that $\Delta A \in x$, rather than, as $\lambda_k(x)$ does, just for $A_1, \ldots, A_k$ with $\Delta A_1, \ldots, \Delta A_k \in x$? For this purpose we use a variant on König's Lemma which we shall call the Word Lemma: its derivation from König's Lemma is deferred to an appendix. The setting for this result is formal language theory, in which we consider sets of words on some alphabet, including infinite words, or ω-words, in the terminology of Salomaa [6]; we use the phrase "initial segment" for what is often expressed by the term "(proper) prefix."

Lemma 3.1 (Word Lemma) *Suppose acceptability is some property of words on a finite alphabet such that any initial segment of an acceptable word is acceptable and any infinite word is acceptable whenever all of its initial segments are. Then if there are acceptable words of each finite length, there is an acceptable infinite word.*

Take now as our finite alphabet the set 0,1, and, for an arbitrary $x \in W_{\mathbf{NC}}$, consider the sets $\Lambda_0(x), \Lambda_1(x), \ldots, \Lambda_n(x), \ldots$ such that $\Lambda_k(x)$ comprises all candidates for the role of $\lambda_k(x)$: that is, each contains, for the formulas $\Delta A_1, \ldots, \Delta A_k \in x$, either A_i or else $\neg A_i$ $(= (\alpha))$, and satisfies the condition (β) above, that for any formula B, such that $\lambda_k(x) \vdash_{\mathbf{NC}} B$, we have $\Delta B \in x$. We have seen, using $(\mathrm{NCR})_k$, that for every k, there is at least one such candidate $\lambda_k(x)$, so the sets $\Lambda_k(x)$ are all nonempty. Now continuing in terms of our fixed enumeration $A_1, \ldots, A_n, \ldots$ of all the formulas A_i for which $\Delta A_i \in x$, let us call a word u, finite or infinite, whose i^{th} letter ($i \leq$ length of u) is u_i $(\in \{0, 1\})$ *acceptable* when, for the set

$$S(u) = \{C \mid C = A_i \text{ if } u_i = 1, \text{ and } C = \neg A \text{ if } u = 0\}$$

we have: $S(u) \vdash_{\mathbf{NC}} B$ implies $\Delta B \in x$, for all formulas B. This is a notion of acceptability which meets the conditions of the Word Lemma. The first condition was that any initial segment of an acceptable word is acceptable. Suppose t is an initial segment of u but that t is not acceptable. Thus we have $S(t) \vdash_{\mathbf{NC}} B$ for some B for which $\Delta B \notin x$. Since t is an initial segment of u, $S(t) \subseteq S(u)$, and so ("monotonicity of $\vdash_{\mathbf{NC}}$") we have $S(u) \vdash_{\mathbf{NC}} B$, meaning that u is not acceptable. The second condition was that if every initial segment of an infinite word u is acceptable, then u is acceptable. To check this, suppose that u is not acceptable, so that $S(u) \vdash_{\mathbf{NC}} B$ for some B with $\Delta B \notin x$. Then ("finitariness of $\vdash_{\mathbf{NC}}$") for some finite subset S_0 of $S(u)$, we have $S_0 \vdash_{\mathbf{NC}} B$; but then S_0 is a subset of some $S(t)$, t an initial segment of u, so $S(t) \vdash_{\mathbf{NC}} B$. Thus not every initial segment of u is acceptable. Thus the Word Lemma assures us that if there are acceptable words, on the present understanding of acceptability, of each finite length, there is an acceptable infinite word. And, each of the Λ_k being nonempty, there are acceptable words of every finite length. So there is an infinite acceptable word. Let u be such a word, and define $\lambda(x)$ as $S(u)$: this set now contains either A or $\neg A$ for each formula A such that $\Delta A \in x$ and meets the condition called (β) in the finite case above: B follows from the set only if $\Delta B \in x$, for all formulas B. For later reference, we formulate what the Word Lemma has proved for us.

Lemma 3.2 (Existence Lemma) *Where $A_1, \ldots, A_n, \ldots$ is an enumeration of all the formulas A_i such that $\Delta A_i \in x \in W_{\mathbf{NC}}$, there is a set $\lambda(x)$ with (α) each element of $\lambda(x)$ is either A_i or $\neg A_i$, and (β) for any formula B, if $\lambda(x) \vdash_{\mathbf{NC}} B$, then $\Delta B \in x$.*

Thus we can use $\lambda(x)$, for each $x \in W_{\mathbf{NC}}$, to define $xR_{\mathbf{NC}}y$ by: $y \supseteq \lambda(x)$. In more detail, we have the following analogue of the "Fundamental Theorem" of normal ($\Box$-based) modal logic for our non-contingency logics.

Theorem 3.3 *For any formula A and any $x \in W_{\mathbf{NC}}$: $\mathcal{M}_{\mathbf{NC}} \models_x A$ if and only if $A \in x$.*

Proof: By induction on the construction of A. The only novelty here is the case in which A is ΔB for some formula B, for which case it suffices (with the help of the inductive hypothesis) to show,

$$\Delta B \in x \Leftrightarrow \forall y \in W_{\mathbf{NC}}(xR_{\mathbf{NC}}y \Rightarrow B \in y) \text{or } \forall y \in W_{\mathbf{NC}}(xR_{\mathbf{NC}}y \Rightarrow B \notin y).$$

The $\Rightarrow$ direction follows from (α) of the Existence Lemma, since if $\Delta B \in x$, then either B or $\neg B$ belongs to $\lambda(x)$ and hence to all $y \supseteq \lambda(x)$. For the $\Leftarrow$ direction, we must show that on the assumption that $\Delta B \notin x$, $\lambda(x) \cup \{\neg B\}$ and $\lambda(x) \cup \{B\}$ are both **NC**-consistent, so that each has a maximal extension to falsify the respective disjuncts on the right above. Taking the first case first, suppose that $\lambda(x) \cup \{\neg B\}$ is not **NC**-consistent. Thus $\lambda(x) \vdash_{\mathbf{NC}} B$. So by ($\beta$) of the Existence Lemma, $\Delta B \in x$, contradicting our assumption. For the second case: if $\lambda(x) \cup \{B\}$ is not consistent, we have $\lambda(x) \vdash_{\mathbf{NC}} \neg B$, so again $\Delta\neg B \in x$. But by ($\Delta\neg$), $\Delta B \in x$, again contradicting our assumption.

Corollary 3.4 **NC** *is determined by the class of all models.*

Proof: We have already seen that **NC** is sound with respect to the class of all models. It is complete with respect to this class since if $\nvdash_{\mathbf{NC}} A$ then the **NC**-consistent set $\{\neg A\}$ is included in some $x \in W_{\mathbf{NC}}$, with (for consistency) $A \in x$, so by the Theorem $\mathcal{M}_{\mathbf{NC}} \not\models_x A$.

In the notation introduced in Section 1, this Corollary tells us that $\mathbf{K}_\Delta = \mathbf{NC}$. The completeness half of Corollary 3.4 is more general. Calling a non-contingency logic *nc-normal* if it includes all instances of ($\Delta\neg$) and is closed under the rules $(\text{NCR})_k$ for all $k \in \omega$, we have proved a special case of a general fact about *nc*-normal logics S, whose canonical models $\mathcal{M}_S$ are defined as $\mathcal{M}_{\mathbf{NC}}$ was, except that W_S comprises the sets of formulas maximal consistent with respect to S.

Corollary 3.5 *If S is any nc-normal logic, S is determined by* $\mathcal{M}_S$.

In the following section we shall consider a few of these stronger *nc*-normal logics. We close the present section with a question which naturally arises in view of the use, alongside the truth-functional basis and the schema ($\Delta\neg$), of the infinitely many rules $(\text{NCR})_k$ to axiomatize the non-contingency logic determined by the class of all models. Does **NC** have an axiomatization using only finitely many rules and schemata? (As mentioned in Section 1, Montgomery and Routley [5] show *inter alia* that the non-contingency logic determined by the class of all models with reflexive accessibility relations has an axiomatization with this finiteness property.)

4 Some extensions of the basic system In Section 1, we adapted the model-oriented terminology of conventional modal logic to the language of non-contingency logic. To avoid various circumlocutions, it will be helpful here to make a similar adaptation of the frame-oriented terminology. Recall that a *frame* is a pair (W, R) with $W \neq \varnothing$ and $R \subseteq W \times W$, and that a model (W, R, V) is said to be a model on the frame (W, R). A formula is *valid* on a frame (W, R), which we notate by writing $(W, R) \models A$ when for every model $\mathcal{M}$ on (W, R), we have $\mathcal{M} \models_x A$ for every $x \in W$. A non-contingency logic is sound (complete) with respect to a class of frames when all (only) formulas provable in the logic are valid on each frame in the class, and is determined by a class of frames when both sound and complete with respect to that class. Thus Corollary 3.4 can be rephrased as: **NC** is determined by the class of all frames. Note that the non-contingency logic determined by any class of frames is *nc*-normal, just as the modal logic determined by any class of frames is normal. (In the latter case, it is well known—see van Benthem and Humberstone [9]—that not every normal modal logic is determined by some class of frames; although there is presumably nothing to prevent a similar manifestation of "Kripke-incompleteness" for non-contingency logics, the author does not have an example of an *nc*-normal logic determined by no class of frames.) Finally, a class of frames is *nc-defined* by a set S of formulas (not necessarily a non-contingency logic) when the frames in the class are all and only those on which all formulas in S are valid, and is *nc-definable* when there is some set of formulas which *nc*-defines it. (This is the Δ-based analogue of the usual box-based notion of modal definability, as in [9].)

Let us begin by considering five normal modal logics determined by various subclasses $\{(W, R) \mid \forall x \in W. |R(x)| < 1\}$. (Here $|X|$ is the cardinality of the set X.) We have $\mathbf{KD_c}$, determined by the class of all such frames; **KD**!, determined by the class of all (W, R) in which $|R(x)| = 1$ for all $x \in W$; $\mathbf{KT_c}$, determined by the class of all (W, R) in which $xRy \Rightarrow x = y$ for all $x, y \in W$; **KT**!, determined by the class of all (W, R) in which $R(x) = \{x\}$ for all $x \in W$; and finally the *Verum* system, determined by the class of all (W, R) in which $R(x) = \varnothing$, for all $x \in W$. In all these cases, the logics cited are not just determined by the classes of frames mentioned, but modally define those classes of frames. We consider *nc*-definability in a moment, after first enquiring into the non-contingency logics these various classes of frames determine. The plural turns out to be inappropriate, since a single non-contingency logic is determined by all five classes.

The logic in question is the smallest non-contingency logic containing all instances of the schema ΔA. Notice that this automatically qualifies as an *nc*-normal logic: just as the smallest modal logic containing all instances of $\Box A$—the *Verum* system—qualifies as a normal modal logic. Let us call this the $\textit{Verum}_{nc}$ system, in fact, because, as in the case of $\Box$, here Δ can be thought of as expressing the constant-true truth-function. Soundness with respect to the most comprehensive class of all frames mentioned above,

$$\{(W, R) \mid \forall x \in W. |R(x)| \leq 1\}, \tag{4.1}$$

is clear, since ΔA can only be false at a point in a model if that point bears R to at least two distinct points (one to verify and one to falsify A), in which case the frame of the model lies outside our class.

The extreme simplicity of the $\textit{Verum}_{nc}$ system allows us to bypass the canonical model construction of the preceding section to obtain our five completeness results. We use falsifying models (for nontheorems) in which $|W| = 1$. Where $W = \{x\}$, put R_0 for $\varnothing$ and R_1 for $\{\langle x, x\rangle\}$. Where C is not provable in the $\textit{Verum}_{nc}$ system, let x be a maximal consistent (with respect to the system) superset of $\{\neg C\}$. Put $V(p_i) = \{x\}$ if $p_i \in x$, and $= \varnothing$ otherwise. Then for each of the models $\mathcal{M}_0 = (W, R_0, V)$ and $\mathcal{M}_1 = (W, R_1, V)$ we have,

$$\mathcal{M}_i \models_x A \Leftrightarrow A \in x, \text{ for all formulas } A. \tag{4.2}$$

For, in the non-boolean case of interest, the case of $A = \Delta B$, the left and right sides of (4.2) are both invariably true: the right because ΔB is provable, and the left because $|R_i(x)| \leq 1$, for $i = 0, 1$. Thus the *Verum$_{nc}$* system is complete (as well as sound) with respect to the class of all (W, R) satisfying $R(x) = \{x\}$, by taking R_1 as R, and also therefore the class of frames satisfying the weaker conditions (1) $|R(x)| = 1$; (2) $xRy \Rightarrow x = y$; (3) $|R(x)| \leq 1$ (all $x, y \in W$). Finally, taking R_0 as R, we get completeness with respect to the class of frames in which $R(x) = \varnothing$ for all frame elements x. We can re-express this multiple determination conclusion in terms of the non-contingency fragments of the normal modal logics listed above: our *Verum$_{nc}$* system is $(S)_\Delta$ where S is any of: the *Verum* system (= **KVer**, where **Ver** is the formula $\Box\bot$, or the schema $\Box A$), **KD**$_c$, **KD!**, **KT**$_c$, **KT!**.

Multiplicity of determining classes of frames for a single logic is of course a measure of expressive weakness, since it reflects the unavailability of formulas valid on all frames in one determining class but not in another. This phenomenon is familiar from conventional modal logic, so the point of interest here is the decrease in expressive power as we pass from the language of $\Box$ to the language of Δ. The only one of the five classes of frames described above that is *nc*-definable is the largest, given by (4.1). It is *nc*-defined by the class of theorems of the *Verum$_{nc}$* system, or alternatively, by the class of all formulas of the form ΔA, or again, simply by $\{\Delta p\}$. We have already observed that all formulas in any of these classes are valid on any frame satisfying the condition in (4.1); conversely, suppose we have a frame not satisfying this condition: since there is a point x with more than one R-successor, we can arrange V to put one but not the other of these successors into $V(p)$, falsifying Δp at x in the resulting model. (It follows, for the same reasons as in $\Box$-based modal logic, that where several classes of frames determine the same logic, at most one of those classes is *nc*-definable.)

We have introduced our discussion of *nc*-normal logics with a look at the top end of the lattice of such logics (with zero = **NC**, unit = the inconsistent system): the *Verum$_{nc}$* system is the sole Post-complete, *nc*-normal logic (lattice-theoretically: the only co-atom).[4] But an important moral may be drawn from our consideration of that case, with repercussions further down amongst weaker systems with greater interest for deontic, alethic, etc., applications. The relation between the frames (W, R_0) and (W, R_1) above is a paradigm of a relationship we shall describe by calling one frame an "R-reduction" of another. Informally, thinking of frames pictorially represented with arrows from one point to another indicating the holding of the accessibility relation, we "reduce" a frame by discarding some or all arrows which go from points x to points y in cases where no arrows from x go to points other than y. More precisely: (W, R^-) is an *R-reduction* of (W, R) just in case $R^- \cup \{\langle x, y\rangle \in W \times W \mid R(x) = \{y\}\} = R$. We then have the a simple observation given as Part (i) of the following Lemma. For Part (ii) we need an additional piece of terminology: we define frames to be *reduction-related* if both are R-reductions of some common frame (equivalently: if they stand in the ancestral of the union of the R-reduction relation with its converse).

Lemma 4.1 (Reduction Lemma) *(i) If (W, R^-) is an R-reduction of (W, R), then for all formulas A, $(W, R^-) \models A$ if and only if $(W, R) \models A$. (ii) Frames which are reduction-related validate the same formulas.*

Proof: (i) follows from the fact that for any models $\mathcal{M}^- = (W, R^-, V)$, $\mathcal{M} = (W, R, V)$ on the given frames, we have $\mathcal{M}^- \models_x A$ if and only if $\mathcal{M} \models A$, for all $x \in W$, all formulas A. This is established by induction on the construction of A. (ii) is an immediate consequence of (i).

It follows from the Reduction Lemma that any *nc*-definable class of frames must be closed under reduction-relatedness. (A question arises as to whether we have a converse: are all modally definable classes of frames which are in addition closed under reduction-relatedness, *nc*-definable?) We can obtain negative results by finding cases in which this necessary condition for *nc*-definability is not satisfied. We assemble some examples here.

Theorem 4.2 *The following classes of frames are not nc-definable: The class of (i) reflexive frames, (ii) serial frames, (iii) symmetric frames, (iv) transitive frames, (v) transitive reflexive frames.*

Proof: In all cases we observe that the relevant class of frames is not closed under reduction-relatedness, so the result follows by the Reduction Lemma. By way of example, take (iii) and (iv).

For (iii), consider the frames (W, R) and (W, R^-) where $W = \{w_1, w_2\}$ with $w_1 \neq w_2$, $R = \{\langle w_1, w_2\rangle, \langle w_2, w_1\rangle\}$, $R^- = \{\langle w_1, w_2\rangle\}$. (W, R) is symmetric whereas (W, R^-) is not, though the latter is an R-reduction of the former. The same example works for (iv), since of these two reduction-related frames only (W, R^-) is transitive.

Cashing in on the application of the Reduction Lemma to Part (ii) of the Theorem, we have a further completeness result for **NC**: in addition to being determined by the class of all frames (as we saw, in effect, in the preceding section), this logic is determined by the class of all serial frames, since we can "reflexivize the dead ends" to obtain an equivalent frame. That is, given (W, R), we pass to the serial frame of which it is an R-reduction, (W, R^+) where $R^+ = R \cup \{\langle x, x\rangle \mid x \in W, R(x) = \varnothing\}$. By the Reduction Lemma, then, any frame invalidating a nontheorem of **NC** gives us a serial frame which does the same job. In the notation of Section 1, we have $(\mathbf{K})_\Delta = (\mathbf{KD})_\Delta = \mathbf{NC}$.

Recalling that a logic is said to be Halldén-incomplete if it has some disjunction as a theorem, where neither disjunct is a theorem and the disjuncts have no propositional variables in common, we may observe a difference in respect of this property between the modal logic **K** determined by the class of all frames, and the non-contingency logic **NC** determined by the class of all frames. The former is Halldén-incomplete (see [9]), whereas the latter is Halldén-complete. The Halldén-completeness of **NC** can be deduced directly from Theorem 2 of [9], which states that any normal modal logic determined by a class of serial frames which is closed under direct products is Halldén-complete. The same reasoning establishes this result for *nc*-normal (non-contingency) logics. Since the class of all serial frames satisfies this closure condition, and we have just observed **NC** to be determined by this class, Halldén-completeness follows. Alternatively, we may take the result for **KD** and transfer it across to **NC** using the obvious fact that if a normal modal logic S is Halldén-complete, then so is the logic $(S)_\Delta$. Note that, in view of the fact that not only does $\mathbf{NC} = (\mathbf{KD})_\Delta$, but also $\mathbf{NC} = (\mathbf{K})_\Delta$, we do not have a similar transfer of Halldén-incompleteness from S to $(S)_\Delta$. The same phenomenon arises with the

Verum$_{nc}$ system, which is Halldén-complete since it is the non-contingency fragment (image under $(\ )_\Delta$) of the Halldén-complete **KT!** (the class of frames (W, R) with $R(x) = \{x\}$ for all $x \in W$ being a class of serial frames closed under direct products) even though this system is also $(\mathbf{KD}_c)_\Delta$ and $\mathbf{KD}_c$ is Halldén-incomplete (in view of examples such as: $\Box(p \wedge \neg p) \vee (\Box q \leftrightarrow \Diamond q)$.[5]

As remarked in Section 1, Mongomery and Routley [5] axiomatize the non-contingency logic determined by the class of reflexive frames using, to supplement the purely truth-functional basis, (1.6) – (1.8), in which the only ingredient not valid on every frame is (1.8), represented here by means of a characteristic instance.

$$p \rightarrow (\Delta(p \rightarrow q) \rightarrow (\Delta p \rightarrow \Delta q)) \tag{4.3}$$

Since (4.3) is valid on every reflexive frame, it follows from Part (i) of Theorem 4.2 above that (4.3) is also valid on some nonreflexive frames. What class of frames is *nc*-defined by (4.3), then? The following answer, which will accordingly be the answer to the same question asked of the set of all theorems of $(\mathbf{KT})_\Delta$, is easily checked: it is the class of all (W, R) such that for all $x \in W$, either xRx or $|R(x)| \leq 1$. The completeness of the Montgomery-Routley system with respect to the class of reflexive frames, the *nc*-undefinability of that class notwithstanding, can easily be shown without appeal to the derivability of the rules $(\mathrm{NCR})_k$, since we can exploit the expressibility of A's being necessary (as $\Delta A \wedge A$), to define $\lambda(x)$ in the canonical model as $\{A \mid \Delta A \wedge A \in x\}$. Properties (α) and (β) of the Existence Lemma of the preceding section are satisfied, in the latter case because the following rule (cf. *RK* in [1], p. 19) is derivable using (1.6) and (1.8).

$$\frac{(A_1 \wedge \ldots \wedge A_n) \rightarrow B}{((\Delta A_1 \wedge A_1) \wedge \ldots \wedge (\Delta A_n \wedge A_n)) \rightarrow \Delta B} \tag{4.4}$$

From this point on the argument proceeds as for **NC** (with the additional observation that the canonical accessibility relation, holding between x and y when $\lambda(x) \subseteq y$, is reflexive). Of course, we already knew the conclusion of this argument—that the system is determined by the class of reflexive frames—from the result of [5] to the effect that the present system is indeed $(\mathbf{KT})_\Delta$.

Similar remarks apply in the case of the other systems presented in [5], since in all cases we have the $\Delta A \wedge A$ definition of $\Box$. This leaves open such questions as how to axiomatize $(\mathbf{K4})_\Delta$, or, otherwise put, the question of what the non-contingency logic determined by the class of transitive frames looks like. We cannot use (1.8) in any such axiomatization, since in view of the above remarks about (4.3), this would cost us soundness. A plausible conjecture would be that the logic we are after is the least *nc*-normal logic containing all instances of the [5] schema (1.9) $(= \Delta A \rightarrow \Delta\Delta A)$, but the author has not succeeded in adapting the argument of Section 3 to obtain this result.

Appendix Deriving the Word Lemma In this Appendix we show how the Word Lemma, appealed to in Section 3, follows from König's Lemma. We take the latter in the following form: *Any finitary tree with nodes of level n for all n* $\in \omega$ *has an infinite branch.* (For the equivalence of this formulation with customary form "Any finitary tree with infinitely many nodes has an infinite branch," as well as a proof of the latter, see Smullyan [8], p. 32, q.v. also for the terminology of ‘branch’ and ‘path’.) We repeat the Word Lemma here, including some additional numbering for back-reference.

Lemma 3.1 (Word Lemma) *Suppose acceptability is some property of words on a finite alphabet such that (i) any initial segment of an acceptable word is acceptable and (ii) any infinite word is acceptable whenever all of its initial segments are. Then if there are acceptable words of each finite length, there is an acceptable infinite word.*

To obtain the Word Lemma from König’s Lemma (as formulated above), we begin by noting that the set of all words (finite or infinite) on a finite alphabet can be thought of as the set of paths in the finitary tree with the empty word as root node and the successors of any node correspond to the letters of the alphabet. (Thus all branches in this tree are infinite.) This induces a one-one correspondence between words and paths in the tree. Finite paths are also in a one-one correspondence with nodes, since each node is the terminal node of a unique path and each finite path has a unique terminal node; thus in the finite case there is a derivative one-one correspondence between words and nodes.

We now proceed to prune the above tree on the basis of a notion of acceptability satisfying (i) and (ii) in the Word Lemma. A path is deemed to be unacceptable just in case the corresponding word is unacceptable (i.e., not acceptable). A node is unacceptable just in case the corresponding finite path (hence the corresponding finite word) is unacceptable. To prune the tree, delete all unacceptable nodes along with the subtrees they dominate. We are left with a tree whose paths are precisely the acceptable paths of the original tree, since a path in the original tree is unacceptable if (by (ii)) and only if (by (i)) some node on the path is unacceptable. It remains to show, to conclude the proof of the Word Lemma, that, on the assumption that there are acceptable words of each finite length, there is an acceptable infinite word. In terms of the pruned tree, the assumption means that this (finitary) tree contains nodes of arbitrary finite depth (or “level”). By König’s Lemma, then, this tree has an infinite branch. Since all of its paths are acceptable, such an infinite branch corresponds to an infinite acceptable word.

NOTES

1. Writers on this topic generally suggest readings in terms of its being (non-) contingent *that A*. We have replaced “that” by “whether” since talk of its being contingent that A seems appropriate only when it is true that A. A disadvantage (in respect of generality) of talk of (non-) contingency, in the *whether*-construction no less that in the *that*-construction, will be pointed out shortly.

2. In fact Montgomery and Routley use axioms and a rule of Uniform Substitution rather than schemata.

3. We indicate how the argument goes for the case $k = 2$. For an application of $(\text{NCR})_2$, the premises are $(A_1 \wedge A_2) \to B_1$, $(A_1 \wedge \neg A_2) \to B_2$, $(\neg A_1 \wedge A_2) \to B_3$ and $(\neg A_1 \wedge \neg A_2) \to B_4$, and the conclusion is:

$$(\Delta A_1 \wedge \Delta A_2) \to (\Delta B_1 \vee \Delta B_2 \vee \Delta B_3 \vee \Delta B_4).$$

The main part of the justification of the claim that the conclusion is true throughout a model, on the supposition that the premises are, consists in first making a supposition. Suppose further that the antecedent of the conclusion is true at a point x (in that model). By way of example, suppose that this is because A_1 is impossible (at x) and A_2 is necessary. Now look at the third premise, supposedly true throughout the model. We conclude that B_3 is necessary (at x), and hence that the third disjunct of the consequent of the conclusion, and hence the whole of the disjunction, is true at x.

4. From this fact about the $Verum_{nc}$ system, we may conclude that no consistent *nc*-normal logic has any theorems of the form $\neg\Delta A$ (which is obvious anyway, since no consistent normal modal logic has any theorems of the form $\diamond A \wedge \diamond\neg A$).
5. We could make the same point using the fact (established above) that the $Verum_{nc}$ system is also $(\mathbf{KT}_c)_\Delta$ and $\mathbf{KT}_c$ has the "Halldén-unreasonable" disjunction $\Box(p \wedge \neg p) \vee (\Box q \leftrightarrow q)$.

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Philosophy Department
Monash University
Wellington Road
Clayton, Victoria
Australia 3168
email: LHUMBERS@arts.cc.monash.edu.au