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Worlds of Homogeneous Artifacts

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Abstract We present a formal first-order theory of artificial objects, i.e., objects made out of a finite number of parts and subject to assembling and dismantling processes. These processes are absolutely reversible. The theory is an extension of the theory of finite sets with urelements. The notions of transformation and identity are defined and studied on the assumption that the objects are *homogeneous*, that is to say, all their atomic parts are of equal ontological importance. Particular emphasis is given to the behavior of classes of artifacts in time. We call such classes satisfying certain preservation conditions *worlds*. Various results concerning the existence, extension, and completeness of worlds are proved.

1 Introduction This paper is a continuation of Tzouvaras [5]. In that paper we discussed a concept of identity for artifacts based on the notion of "significant part." We considered objects all the parts of which are equally important only as a limit case. Let us call *homogeneous* objects which consist of parts of the same ontological value. True, such artifacts are rare and marginal among those commonly constructed and used by humans. However, one *may* treat an artifact as homogeneous for various reasons, as was the case for example in the Theseus's puzzle (see Lowe [3]).

On the other hand the models of homogeneous objects may offer some help in understanding the behavior of material things as totalities of elementary particles-constituents. Many of these constituents (appropriately grouped according to their size, or spin) seem in fact to be equivalent so we might think of an atom or molecule of matter as a mere concentration of a definite number of copies from few sorts of elementary bricks. Of course we also talk about *structure*. However structure could be an *outcome* of the number of bricks being present or of their nature, and not an arrangement imposed from without. For example the structure of a machine has already been engraved upon the particular shape and size of its atomic parts, and once these parts are available so is the machine's structure. An atom of carbon can fit with four hydrogen atoms (not three or five) or two oxygen atoms, and this entails structure. Similar phenomena may hold for the most ultimate bricks of matter.

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Our intuition, however, will be normally fed by what happens in the familiar world of artifacts made by humans. There, if we assume that the objects are homogeneous in their constituents, we easily see that the continuity principle does not hold in general, thus in such cases we have to take into account the amount of the parts substituted with respect to the total amount of the object's parts and accept a principle like the following:

Definition 1.1 (Restricted Continuity Principle (RCP)) If a homogeneous object is formed out of a large number of parts and we replace a comparatively small number of them by new ones, then the identity of the object in question is preserved. Otherwise it changes.

Now the only rigorous way to make the terms "small" and "large" precise is to use a nonstandard set of natural numbers for counting, and identify "small" with "standard" and "large" with "nonstandard." Then RCP entails an identity relation $\doteq$. Using this we study artifact identities and their transformations in time. In particular we consider totalities of objects which co-exist and evolve in a certain period of time. Such totalities, fulfilling certain compatibility conditions are called *worlds*. Various results are proved concerning worlds, specifically their extension, completion etc. The results are based on a model-theoretic constraint assumed to be satisfied by the "real world" where the worlds of artifacts are embedded. Namely the condition that the real world is ω_1-saturated. This is a richness condition. It means that our world realizes whatever it *could* do.

Artificial objects (or *artifacts*) form a category of entities that would naturally attract mathematical investigation and abstract, formal treatment. As being products of human intelligence, they present, at least in the most typical cases, a clear *structure* as well as irreducible *constituents*—i.e., what mathematical reasoning starts with. Of course this does not suffice. We need much more in order to have some interesting fragments of a mathematical theory of artifacts. In fact, the key properties of the latter, which put them in sharp contrast with what we call *natural* objects (trees, rocks, animals etc.), could be summarized as follows.

1. They are designed for some specific use or purpose. In a lot of cases the design imitates, up to a certain limit, the form and structure of a natural object bearing some analogous function (e.g., house-cave, airplane-bird etc).
2. They have absolutely specified boundaries and independence in space and time relative to other entities.
3. They consist of parts which fit one to the other, and the parts consist of parts again, until we reach parts which are *atoms*. In other words they are bottomed (or well-founded) structures with respect to a "fitting relation."
4. From the disconnected parts one can recover exactly the same object, i.e., assembling and dismantling are fully reversible processes.

In contrast, natural objects lack all these properties, i.e., they satisfy the contraries of (1–4). Thus, (1) they are *random* formations of material evolution; (2) most frequently they do *not* have clear boundaries (seas, mountains); (3) in one sense they do *not* consist of parts but rather they are "wholes," out of which one might distinguish just *pieces*; in another sense, one might talk about "parts" here as well, but it is by no means obvious that the parthood relation is this time bottomed; and (4) as a consequence of the last remark, clearly no reversible decomposition can occur—only irreversible decay.

Moreover, as follows from property (2), each artifact can be reduced to a finite number of *atoms* or *atomic objects*. These can exist either as independent objects or as *parts* of complex objects, called in that case *atomic parts*. If we, now, assume that the atoms are unchanging—and this is a plausible assumption if we confine ourselves in a reasonable period of time—then the only changes an artifact may suffer are the following.

a. *dismantling* into some of its parts;
b. *replacement* of one or more of its parts by similar ones; and
c. *reassembling* of some or all of the parts into other parts or the initial object.

The above constitute three elementary *operations* in the universe of artifacts. Notice that the third operation contains implicitly an idea that often skips our attention. Namely, that an artifact *can* be considered to exist even when disassembled in its parts. Whenever a soldier, for instance, disassembles his rifle to clean it, clearly he does not think that the rifle ceases to exist. Rather he thinks it passes temporarily to another state. However it can remain at that state arbitrarily long. The parts of the rifle may be put in a case and stored, or some parts may be sent to the gunsmith for repair. Meanwhile, of course, the rifle exists—unless we have decided to use its parts as spare parts for other similar rifles. On the other hand, we *can* equally well consider that dismantling interrupts existence. Thus we have two alternative options concerning dismantling which lead to two slightly different versions of the theory. Here we shall consider the less complicated version, namely the one according to which dismantling leads to temporal ceasing of existence (and hence change of identity).

Let us examine these options more closely in connection with the following question: can the same object be part of two distinct objects? For assembled objects the answer is unquestionably "no." The same wheel, for instance, cannot be fitted simultaneously to two distinct cars. However the matter becomes more delicate when considering dismantled artifacts. To illustrate this by an example, suppose we decompose your car A into its chassis x and the engine y. Suppose that a friend of yours has a similar car A' (same brand, same model) which we dismantle too into its chassis x' and engine y'. Having before us the parts x, y, x', y' we ask: how many and what cars are there at present? There are three possible answers.

i. There are four cars, namely: $\{x, y\}$, $\{x, y'\}$, $\{x', y\}$, $\{x', y'\}$.
ii. There is no car at all. Instead there are two chassis x, x' and two engines y, y'.
iii. There are two cars, namely *either* the cars $\{x, y\}$ and $\{x', y'\}$ or the cars $\{x, y'\}$ and $\{x', y\}$. It is up to us to decide which two really exist.

Answer (i) must be ruled out since we are interested in objects which are actually and not potentially existing. Answer (ii) can be accepted. It amounts to the idea that once an artifact is dismantled, it ceases to exist and, in turn, the parts start existing. If we assemble back, the initial object comes into being again while the parts cease to exist. Finally (iii) can also be accepted. According to this, the artifacts continue to exist after dismantling. However, human decision as to what part belongs to what object now plays a major role.

Both answers (ii) and (iii), however, share the idea that an object participating in the formation of another more complex object is *consumed*, and therefore cannot be used more than once. Thus the artifacts, according to this view, appear and disappear as convertible rearrangements of a multitude of ultimate elements, the atoms. The atoms (like those of Democritus) are imperishable and ontologically equivalent. Each artifact is a temporal assembly of a finite number of them combined in a certain way. For as long as an atom belongs to an assembly it is consumed by it. If and when the assembly disperses, the atom becomes free again. This is, intuitively, the general picture of the world of artifacts we shall attempt to formalize in the subsequent sections.

2 Sketch of the formal framework and notation We shall be rather loose in the formal treatment of the notions involved and we shall be constantly appealing to the reader's intuition. We assume familiarity with basic set-theoretic and model-theoretic notions.

Our metatheory will be $\mathrm{ZF_{fin}}$ (i.e, the ordinary ZF axioms with the infinity axiom replaced by its negation) enhanced a) with urelements and b) with "classes," i.e., arbitrary sets containing finite sets and urelements. Let $\mathrm{ZFU_{fin}}$ be the extension of $\mathrm{ZF_{fin}}$ by adding urelements and let $\overline{\mathrm{ZFU}}_{\mathrm{fin}}$ be the second order extension of the latter by adding full comprehension yielding classes. It follows that we shall have three sorts of elements, namely i) (finite) *sets*, for which we shall use the letters $u, v, w, \ldots$, ii) *urelements* which are nonsets and for which we reserve the letters $x, y, z, \ldots$, and iii) *classes* which we denote by uppercase letters $X, Y, S, \ldots$. The urelements are intended to represent the "artifacts" (or just "objects") that we are going to formalize. As for their number, we may assume for simplicity that there are as many as the natural numbers themselves. The finite sets will represent the "real" collections of the world, for example, the set of all parts of a machine. The classes represent the "abstract" collections that exceed practical manipulation in real time. Such are for instance the class S of all sets, A of all objects, and N of all natural numbers.

A model of $\overline{\mathrm{ZFU}}_{\mathrm{fin}}$ will obviously have the form $\mathfrak{M} = (M, E, \mathcal{M})$, where M is a two-sorted universe, containing sets and objects (i.e., $M = M_0 \cup M_1$ where M_0 contains the "sets" of $\mathfrak{M}$, M_1 contains the "objects" and $M_0 \cap M_1 = \varnothing$) and $\mathcal{M}$ is a collection of classes closed under comprehension. Our "real world," however, will be not an arbitrary model of the above theory but one with the following two properties (for richness and simplicity):

1. E will be the standard membership relation and $\mathfrak{M}$ will contain as many classes as possible, i.e., $\mathcal{M} = P(M)$.
2. $\mathfrak{M}$ will be an ω_1-saturated model of $\mathrm{ZFU_{fin}}$, that is to say, a model M such that if $t = \{\varphi_i(x) : i \in I\}$ is a set of formulas of the language of $\mathrm{ZFU_{fin}}$, with at most countably many extra parameters from M, consistent with the theory of $\mathfrak{M}$, then t is realized in $\mathfrak{M}$. (For the definition of saturation see Chang [1], p. 214.)

A way to obtain such models is by means of suitable ultrapowers. Any such model is, clearly, uncountable and nonstandard, i.e., the class of natural numbers $N^{\mathfrak{M}}$ of $\mathfrak{M}$ is a proper extension of the standard model ω of arithmetic. We shall refer to the numbers of ω as "standard," and denote them by $m, n, k, \ldots$, while for the elements of $N - \omega$, as well as for arbitrary elements of N, we shall use the letters $a, b, \ldots$. If $a \in N - \omega$, we shall also write $a > \omega$.

A countable set $X \subseteq M$ is said to be *coded* in $\mathfrak{M}$ if there is a function $f \in M$ such that $X = \{f(n) : n \in \omega\}$. A useful characterization of ω_1-saturation, that will be used repeatedly in the sequel, has been given in Pabion [4]. $\mathfrak{M}$ is ω_1-saturated iff the following condition holds.

(Sat) *Every countable subset of M is coded in $\mathfrak{M}$.*

Henceforth we shall fix an ω_1-saturated model $\mathfrak{M}$ and we shall call it "real world." For simplicity we can also require for it to have cardinality ω_1. It follows then that all uncountable subsets of M are equipotent. For instance all intervals $[0, a]$ of $N^{\mathfrak{M}}$, for nonstandard a, are uncountable (as easily shown), hence equipotent to N itself. Since M is fixed we shall write simply N, A, S instead of $N^{\mathfrak{M}}$, $A^{\mathfrak{M}}$, $S^{\mathfrak{M}}$ respectively. Clearly these classes are equipotent.

3 The language and axioms of artifacts From now on we shall be working in $\mathfrak{M}$. The variables $x, y, z, \ldots$ should be understood as ranging over *object-states* rather than objects. That is to say, when we write $x \neq y$, we do not mean that x, y are necessarily distinct objects (as identities), but, rather, distinct states of the same identity. The precise definition of identity will be given in the next section, but upon referee's suggestion to clear up the confusion between "object" and "object state," I must try to clarify here the point. Let us say from the beginning that identity is an equivalence relation $x \doteq y$ on the class of object-states, expressing the fact "x, y have the same identity." Now disturbances in equality and identity are exclusively due to replacements of parts of an object by other spare ones or to simple reconstructions of the existing parts. If x, x' are the states of an object in two consecutive moments and no transformation has been executed, then of course $x = x'$. But if x' is, say, the result of dismantling and assembling back x (or dismantling, replacing a part and reassembling), then $x \doteq x'$, though $x \neq x'$. On the other hand, there will be objects with no proper parts, called "atomic" (see Definition 3.4 below). These objects, clearly, can undergo no transformation at all, therefore the relation $\doteq$ for them coincides with $=$. Thus we can have in mind that for atoms the notions of "object" and "object-state" practically coincide, while for complex objects they do not. Nevertheless, we shall keep saying "the object x" instead of the cumbersome "the object-state x", believing that after the above explanations no confusion will arise.

A fundamental relation among artifacts is the *fitness relation*. This is a binary relation F, where xFy means that the objects "x, y fit together and can produce a new object z." A clock, a pencil, a machine-gun is each formed out of parts which fit one to the other in a unique manner. Fitness, however, expresses a possibility rather than a state of affairs. The parts of a clock certainly fit all the time but at a certain time these very parts may be dispersed here and there. The state of affairs is expressed by the assembling operation $\square$. This is a binary partial operation on the class of objects with $\mathrm{dom}(\square) = F$. That is, $x \square y$ is defined if and only if xFy.

Let us extend the language $L = \{\in\}$ to

$$L_1 = L \cup \{F, \square\}.$$

We identify the symbols $F, \Box$ with corresponding entities of the world $\mathfrak{M}$ which are to satisfy the axioms stated below.

(O_1) Fitness-assembling relation: $xFy \iff (\exists z)(x \Box y = z)$.
(O_2) Commutativity of $\Box$ (hence of F): $x \Box y = y \Box x$.

The equality of artifacts is a somewhat delicate matter. Remembering that x stands for an object-state, in order that $x = y$, they should be completely identical, i.e., not only composed of the same parts but also in the same way. To illustrate, suppose z is a car, x is its front left wheel, y is z minus x, x' is z's front right wheel and y' is z minus x'. Then the assemblies $x \Box y, x' \Box y'$ represent, of course, the same car z. Can we write $x \Box y = x' \Box y'$? The answer, I think, should be "no," if we want to be consistent with the meaning of variables. The above assemblies are distinct states of the same identity (that of the car in question). Therefore, though

$$x \Box y \doteq x' \Box y',$$

they are distinct in general. The above discussion leads to the following axiom.

(O_3) Uniqueness of the decomposition: $x \Box y = x' \Box y' \Rightarrow \{x, y\} = \{x', y'\}$.

The preceding equality principle refers already to the part-structure of objects: if x, y are equal, then they have the same parts. (The converse, however, is not true.)

Definition 3.1 x is an *immediate part* of y, in symbols $x <_0 y$, if for some z, $y = x \Box z$. x is a *proper part* of y, in symbols $x < y$, if

$$(\exists a \in N)(\exists f)(\text{dom } f = a \;\&\; f(0) = y \;\&\; f(a-1) = x \;\& \\ \& \; (\forall b < a-1)(f(b+1) <_0 f(b)).$$

Finally, x is a *part* of y, $x \leq y$, if $x < y$ or $x = y$.

In words, $x < y$ if we can reach x following a *set-path* f of length a (standard or nonstandard), departing from y and going down through steps of immediate parthood.

A basic intuition, concerning existence of artifacts, was the idea that the parts used at some moment for the formation of a certain object x are consumed and, therefore, cannot be used at the same time for the formation of another object y. It follows that the objects x, y do not co-exist and, *a fortiori*, cannot fit. This can be formulated as follows.

(O_4) Overlapping objects do not fit: $(\exists z)(z \leq x \;\&\; z \leq y) \Rightarrow \neg(xFy)$.

Remark 3.2 If we write $x \between y$ for the relation $(\exists z)(z \leq x \;\&\; z \leq y)$, then $x \between y$ is precisely Nelson Goodman's "overlapping relation." This relation is taken as primitive in his [2], while the parthood $x \leq y$ is defined to be the relation $(\forall z)(z \between x \Rightarrow z \between y)$.

O_4 has rather strong and good consequences. For instance the following holds.

Proposition 3.3

1. $\neg(xFx)$.
2. $x \Box y \neq x$.
3. $\neg(x < x)$.

4. *If $x < y$, then there is a unique path f, as defined in Definition 3.1, going down from y to x.*

Proof: (1) $x \leq x$, therefore $x \lozenge x$. By O_4, $\neg(xFx)$. (2) Suppose $x \square y = x$. Then $y <_0 x$, hence $x \lozenge y$. Consequently $\neg(xFy)$, contrary to the assumption.

(3) Let $x < x$. Then there is a path f leading from x to x, i.e., $f(0) = f(a-1) = x$, for some $a \in N$. If $a = 1$ then $x <_0 x$, which contradicts ii). Suppose $a > 1$. Let $x' = f(a-2)$. Then $x <_0 x'$, hence $x' = x \square y$ for some y. Now yFx and $y < x' < x$. That is, yFx and $y \lozenge x$, which contradicts O_4.

(4) Let $x < y$ and suppose f, g are two paths leading from y to x. Then $f(0) = g(0) = y$. Hence there is a least number b such that $f(b) = g(b) = z$ and $f(b+1) = z_1 \neq z_2 = g(b+1)$. Then, clearly, $z_1 <_0 z$ and $z_2 <_0 z$. Since $z_1 \neq z_2$, it follows from O_3 that $z = z_1 \square z_2$. But then $z_1 F z_2$, while $x < z_1$ and $x < z_2$. This contradicts again O_4. $\square$

We see that the axioms O_3, O_4 imply a *binary-tree* structure for each object, such that all of its nodes are distinct. The nodes of the tree are exactly the parts of the object with respect to a particular assembling. Another assembling (at another time) may yield another tree-structure for the same object-identity.

What is still open is whether the tree representing an artifact is well-founded, or contains non well-founded paths. This is a foundation question about the parthood relation $<$, quite analogous to that about $\in$ in the universe of sets. I do not think there can be any serious argument against the necessity of a foundation axiom for $<$. To state it we need a hierarchy of objects similar to the cumulative hierarchy of sets.

Definition 3.4 x is an *atomic object* or an *atom*, if it does not have immediate parts (therefore no proper parts at all). We write *Atom* for the class of atoms. In symbols:

$$\text{Atom} = \{x : (\forall y)(\forall z)(x \neq y \square z)\}.$$

Definition 3.5 The class C of *constructs* is defined inductively as follows.

$$\begin{aligned} C_0 &= \text{Atom}, \\ C_{a+1} &= C_a \cup \{x \square y : xFy \ \& \ x, y \in C_a\}, \\ C &= \bigcup \{C_a : a \in N\}. \end{aligned}$$

Obviously C is a definable class and is the analogue of the cumulative universe of sets. It contains the objects constructed via $\square$ from atoms along any number of steps. The following is straightforward.

Proposition 3.6 *$x \in C$ iff $(\exists a \in N)(\exists f)\{\text{dom } f = a \ \& \ f(0) = x \ \& \ (\forall i < a)(f(i) \in \text{Atom} \vee (\exists j, k < i)(f(i) = f(j) \square f(k))\}$.*

The elements of C are objects the corresponding trees of which are well founded, i.e., all their paths are sets having atoms as terminal nodes. The next axiom says that all artifacts are of this kind.

(O_5) Axiom of foundation: $A = C$

By O_5 every artifact appears at some stage of the object hierarchy. Let

$$r(x) = \text{least}\{a : x \in C_a\}$$

be the *rank* of x. Equivalently,

$$r(x) = a \iff x \in C_a - C_{a-1}.$$

Put also for every x,

$$\begin{aligned} \Pi(x) &= \{y : y \leq x\}, \\ \Pi_0(x) &= \Pi(x) \cap \text{Atom}. \end{aligned}$$

Clearly $\Pi(x)$, $\Pi_0(x)$ are definable classes. The following fact gives another formulation of axiom O_5.

Proposition 3.7 *$A = C$ iff for every x, $\Pi(x)$ is a set.*

Proof: Let $A = C$. Then we easily see by induction on the rank of x that $\Pi(x)$ is a set for every $x \in A$. Conversely, assume that all $\Pi(x)$ are sets. We show that any $x \in A$ is a construct. By the properties of the model $\mathfrak{M}$, there is a definable enumeration of the set $\Pi(x)$. Fixing such an enumeration, every subset of $\Pi(x)$ has a least element. Define the function G as follows: $G(0) = x$, $G(1) =$ the least immediate part of x (w.r.t the fixed enumeration), $G(2) =$ the other immediate part of x, and so on. Clearly G enumerates all the nodes of the tree-representation of x. Since $\Pi(x)$ is a set and $\text{rng}\, G \subseteq \Pi(x)$, it follows that G is a set, hence it defines a construct according to Proposition 3.6. $\square$

The axiom of foundation tells us how $<$ behaves downward. The upward behavior is also open, since the operation $\square$ is partial. Therefore the question "are there maximal elements with respect to $<$?" arises naturally.

One would probably tend to say "yes" because we are surrounded by artifacts which seem to be "final," in the sense that they do not participate toward the construction of other more complex objects. However, I think no object can be decided to be in principle final. It suffices to have in mind the trivial possibility to add to any object x a "cover" or a "case" that fits exclusively to it. So we shall postulate that there are no maximal objects.

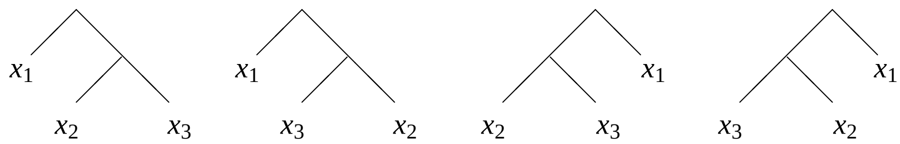
(O_6) No-maximal-object axiom: $(\forall x)(\exists y)(x < y)$

This axiom will be helpful in proving the existence (or the possibility of existence; these two things are equivalent in a saturated model) of objects with a nonstandard number of parts as well as in the definition of replacement in Sections 4 and 5 below.

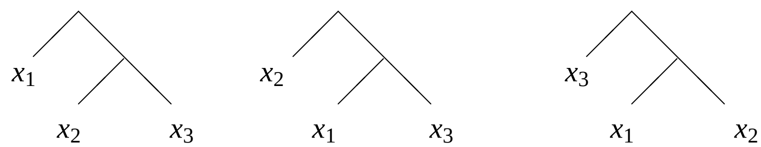
4 Full binary trees. We have already seen that every construct can be represented by a tree whose nodes correspond to the parts. To be specific these are *full binary trees* (f.b.t).

Definition 4.1 A *full binary tree* is a partially ordered set which is a rooted tree such that each one of its nodes has two or zero children. There is no distinction between a right child and a left child. Moreover we assume that each node is fully determined by its children. Therefore the only nodes of the tree that should be labeled are the terminal ones.

According to the preceding definition the following trees are all identical.



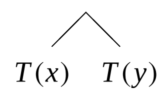
Thus, the only distinct trees with terminal nodes x_1, x_2, x_3, are the following.



Given, now, an object x we define the f.b.t. $T(x)$ corresponding to x as follows.

$$T(x) \quad = \quad \dot{x} \qquad \text{if } x \text{ is an atom}$$

$T(x \,\square\, y) \quad =$



It follows from O_3 that there is a unique f.b.t. corresponding to the object-state x. Different decompositions, however, of the same object may produce different trees.

Example 4.2 Consider the toy-constructs y_1, y_2, x, x' shown in Figure 1.

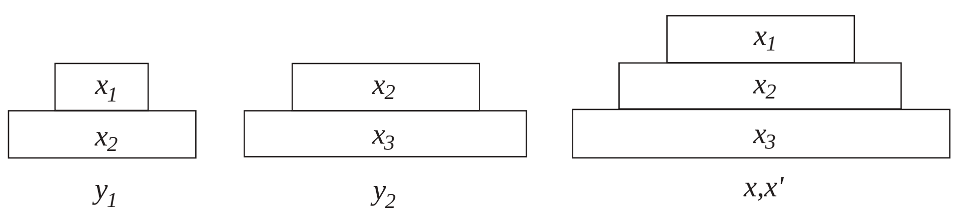
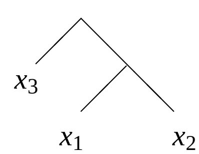


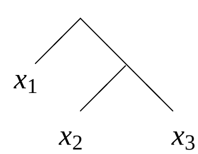
Figure 1: Toy-constructs

Then $y_1 = x_1 \,\square\, x_2$, $y_2 = x_2 \,\square\, x_3$, while the third drawing may depict either the state $x = (x_1 \,\square\, x_2) \,\square\, x_3 = y_1 \,\square\, x_3$, or the state $x' = x_1 \,\square\, (x_2 \,\square\, x_3) = x_1 \,\square\, y_2$. Therefore

$T(x) =$

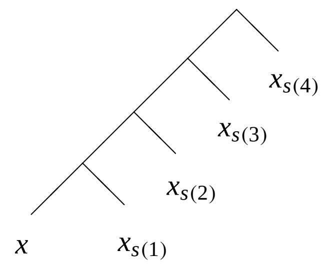


$T(x') =$



Example 4.3 Let y be a table, formed by the top x and four legs x_i, $i = 1, \dots, 4$.

Then, to the various states of y there correspond the trees:



where s is a permutation of $(1, 2, 3, 4)$. We have, thus, 24 trees representing the various states of y.

Definition 4.4 Let T be a f.b.t. The length of a path $(y_1, \ldots, y_n)$ of T is defined to be $n - 1$. The height of T is

$$h(T) = \max\{\text{lengths of maximal paths of } T\}.$$

Also let

$$\begin{aligned} p(T) &= \text{the number of nodes of } T, \\ a(T) &= \text{the number of terminal nodes of } T. \end{aligned}$$

Proposition 4.5

1. *For any f.b.t. T,*

$$\begin{aligned} a(T) &= a \Rightarrow p(T) = 2a - 1 \textit{ for } a > 0, \\ a(T) &= a \iff \log_2 a < h(T) < a - 1. \end{aligned}$$

2. *For any object x,*

$$h(T(x)) = r(x) \textit{ and } a(T(x)) = |\Pi_0(x)|.$$

Therefore if $|\Pi_0(x)| = a$, then $\log_2 a < r(x) < a - 1$.

Proof: This is left as an easy exercise. □

Corollary 4.6

1. *Let T be a f.b.t. Then,*

$$a(T) \in \omega \iff p(T) \in \omega \iff h(T) \in \omega.$$

2. *For any object x,*

$$|\Pi_0(x)| \in \omega \iff |\Pi(x)| \in \omega| \iff r(x) \in \omega.$$

Proof: Immediate from the preceding proposition and the fact that ω is closed under all arithmetic operations. □

Given a set $u \subseteq \text{Atom}$ let us put

$$\tau(u) = \{T : T \text{ is a f.b.t. whose set of terminal nodes is u}\}.$$

Then the following proposition holds.

Proposition 4.7 *If $u \subseteq \text{Atom}$ and $|u| = a$, $a > 2$, then, $|\tau(u)| = 1 \cdot 3 \cdot 5 \cdots (2a - 3)$.*

Proof: By induction on a. For $a = 2$ clearly there is a single tree with terminal nodes $\{x_1, x_2\}$. Suppose the claim holds for u such that $|u| = a - 1$ and let $u \subseteq \text{Atom}$ with $|u| = a$. Let $u = \{x_1, \ldots, x_a\}$. Put $u' = \{x_1, \ldots, x_{a-1}\}$. By the induction hypothesis, $|\tau(u')| = 1 \cdot 3 \cdot (2a - 5)$. For any $T \in \tau(u')$, T has $2a - 3$ nodes according to Proposition 4.5.1) and, consequently, $2a - 3$ subtrees, since each node is the root of a subtree and vice versa. Now we easily verify that any tree in τ arises from some tree in $\tau(u')$ if we insert in a subtree of the latter the extra node x_a. (To insert in the subtree T' the x_a means to replace T' by another whose root branches into T' and the node x_a). Then different subtrees produce different trees of $\tau(u)$. Thus $|\tau(u)| = |\tau(u')| \cdot (2a - 3)$, and the claim is proved. □

We can now make the important distinction between objects with a small and objects with a large number of parts.

Definition 4.8 An object x is said to be *simple* if $|\Pi_0(x)| \in \omega$ and *intricate* if $|\Pi_0(x)| > \omega$.

Equivalently, x is simple iff the f.b.t. representing x is *standard*, i.e., all the magnitudes mentioned in Definition 4.4 are standard.

All atoms are, of course, simple objects. How do we know that intricate objects exist after all? This is a consequence of axiom O_6 and real world's saturation.

Proposition 4.9 *For any object x there is an intricate object y such that $x < y$.*

Proof: Here we use for the first time the saturation of the model $\mathfrak{M}$. Given x, we can find by axiom O_6 a sequence (x_n), $n \in \omega$, of objects such that $x = x_0 < x_1 \cdots < x_n \cdots$. By ω_1-saturation the sequence is coded, i.e., there is a function $f \in \mathfrak{M}$ such that $f(n) = x_n$ for all $n \in \omega$. Since $<$ is definable in $\mathfrak{M}$, if we put

$$u = \{b \in (\text{dom } f) \cap N : (\forall c < b)(f(c) < f(c+1)\},$$

then, clearly, u is a bounded initial set-segment of N, thus $u = [0, a]$, and clearly, a is nonstandard. Therefore $x_0 < \cdots < x_a$, hence $x < x_a$ and x_a is intricate since it contains nonstandard-many parts. □

5 Copies and replacements The industrial mass production of artifacts consists in manufacturing large quantities of objects which are quite similar to one another. That is, industrialized artifacts are produced in equivalence classes such that, if some member x of the class I fits with some member y of the class II, then every member of I fits with every member of II. We call the members of each of those classes *spare* or *replicas* or *copies*.

Two spare objects are assumed to be absolutely interchangeable. They fulfill exactly the same purposes and satisfy the same needs. We shall formalize the property of being spare by defining the relation $\cong$ of "isomorphism." Intuitively, $x \cong y$ if x can replace y wherever y appears as a part. Here seems to arise a difficulty concerning the objects which are "final," i.e., they do not appear as parts of other objects. However, axiom O_6 helps to overcome the difficulty by asserting that there are no objects final in principle.

Obviously $\cong$ is an equivalence relation that can be defined in terms of $\Box$ and F. However the details of the definition are tedious, so we shall omit it. The net outcome of such a formal analysis would be that if $y < x,\ y \cong y'$ and

$$x = (\ldots((y \Box z_0) \Box z_1)\ldots) \Box z_{a-1},$$

then there is a (unique) x' such that

$$x' = (\ldots((y' \Box z_0) \Box z_1)\ldots) \Box z_{a-1}.$$

We denote this object x' by

$$x[y'/y].$$

The latter is the object resulting from x by replacing the part y by the copy y'. We may agree that $x[y'/y]$ makes sense even if $\neg(y \cong y')$. In this case we put simply

$$x[y'/y] = x.$$

The main facts concerning replacements are the following and can be proved using the strict definition of $\cong$.

Proposition 5.1

1. $x < y \;\&\; x \cong x' \Rightarrow y \cong y[x'/x]$.
2. $x < y < z \;\&\; x \cong x' \Rightarrow z[x'/x] = z[y[x'/x]/y]$.
3. $x \cong x' \Rightarrow \neg(x < x')$.

Proof: Let us prove only (3). Suppose $x \cong x'$ and $x < x'$. Then, clearly, there is a $y < x'$ such that xFy. But since $x \cong x'$ and y fits x, it follows that y fits x' too. This, however, contradicts axiom O_4. $\Box$

The existence of copies is what makes transformations of artifacts possible. In a world where everything would be authentic there would be no change of artifacts and, consequently, no identity problem at all. On the other hand, the existence of copies in general is, in practice, reduced to the existence of copies for atoms. If an abundance of spare atoms is available—for each particular atom—then it is clear that for any particular artifact x there is an abundance of artifacts x' such that $x' \cong x$. Since this fact does not follow from the so far accepted principles, it is necessary to include it as an additional axiom.

(O_7) Abundance of spare atoms: $(\forall x \in \mathrm{Atom})(\forall a \in N)(\exists u \subseteq \mathrm{Atom})(|u| = a \;\&\; (\forall x' \in u)(x' \cong x))$

This completes the discussion of the axioms about artifacts. The system $O_1 - O_7$ captures, we hope, some essential aspects of their behavior and, in the context of $\mathrm{ZFU_{fin}}$ constitutes what we shall call the Formal Theory of Artifacts, or FTA.

6 The relation of identity for homogeneous objects We have repeatedly referred so far to the identity relation $\doteq$ among object-states. Here we shall define it explicitly. Remember that we are dealing with homogeneous objects, i.e., those whose parts cannot be distinguished in "important" and "unimportant" ones. We just possess objects which are either simple (with a standard number of parts) or intricate.

The idea is that two simple objects cannot have the same identity, unless they are formed precisely out of the same atoms (probably assembled in different ways); on the other hand, two intricate objects are of the same identity iff they differ in a standard number of atomic parts only.

Definition 6.1 x, y are *of the same identity*, in symbols $x \doteq y$, if either they are simple and $\Pi_0(x) = \Pi_0(y)$, or they are intricate and $|\Pi_0(x) \triangle \Pi_0(y)| \in \omega$ (where $\triangle$ denotes symmetric difference). The equivalence class of x is called the *identity* of x and is denoted $Id(x)$.

Clearly, if $x \doteq y$, then x, y are both simple or both intricate, therefore we may talk about *simple* and *intricate identities*. What kind of objects does $Id(x)$ contain?

Let us say that two objects x, x' are *restructures* of one another if $\Pi_0(x) = \Pi_0(x')$. The objects x, x', for instance, of Figure 2, where $x = (x_1 \square x_2) \square x_3$ and $x' = x_1 \square (x_2 \square x_3)$, are restructures of one another. If x is simple, then $Id(x)$ contains precisely the restructures of x. By Proposition 4.7 $|Id(x)| \in \omega$, hence $Id(x)$ is a set. If x is intricate, then $Id(x)$ contains, besides the restructures, the objects resulting from x after some standard number of replacements of atomic parts by spare ones has been committed on x. Since, by axiom O_7, there is a proper class of replicas for each atom, it follows that $Id(x)$ includes arbitrarily large sets. (Notice, however, that $Id(x)$ is *not* definable.) Thus, we have shown the following.

Proposition 6.2 *If x is simple, then* $\mathrm{Id}(x)$ *is a set and if* $|\Pi_0(x)| = n \in \omega$ *and* $n > 2$, *then* $|\mathrm{Id}(x)| \leq 1 \cdot 3 \cdot 5 \cdots (2n-3)$. *If* $|\Pi_0(x)| \leq 2$, *then* $\mathrm{Id}(x) = \{x\}$. *If x is intricate, then* $\mathrm{Id}(x)$ *is a (nondefinable) proper class including sets of arbitrarily large cardinality.*

Our intuition is that elements of an identity class (i.e., objects of the same identity) could not coexist, as being states of the given object corresponding to distinct times. *A fortiori*, objects of the same identity should not fit. This indeed can be easily deduced by means of O_4.

Proposition 6.3 *If* $x \doteq y$ *then* $\neg(xFy)$.

Proof: If $x \doteq y$, then $\Pi_0(x) \cap \Pi_0(y) \neq \varnothing$. Hence x, y overlap and, by O_4, $\neg(xFy)$. $\square$

7 Continuous transformations The notion of "continuous transformation," involved heavily for example in the ship-of-Theseus puzzle, can now be rigorously defined. Each step of the transformation consists in replacing a certain atom y by a spare one y'. If $y' = y$, then the step leads to a mere restructure of the initial object.

Of course, the process of replacing comprises three distinct phases: (a) the dismantling of the object up to the point we reach the replaceable part; (b) the exchanging of y with y'; and (c) the reassembling. However, we may abbreviate the process considering these three phases as a single step.

Definition 7.1 A *continuous transformation*, or simply a *transformation*, is a set-function f such that:

1. $\text{dom } f = [a, b]$ for some interval $[a, b] \subseteq N$,
2. $\text{rng } f \subseteq A$ and
3. $(\forall i \in \text{dom } f)(f(i+1) = f(i)[y'/y])$, where y is an atom.

Thus, time-units are discrete, represented by natural numbers. The interval $[a, b]$ is the duration of the transformation f. The definition covers all kinds of an artifact change, namely:

a. If $\neg(y < f(i))$ or $\neg(y' \cong y$, then $f(i+1) = f(i)$ and we have no change at all.
b. If $y < f(i)$ and $y' = y$, then $f(i+1)$ is a restructure of $f(i)$, therefore $f(i+1) \doteq f(i)$.
c. If $y < f(i)$, $y' \cong y$, $y' \neq y$, then $\Pi_0(f(i+1)) \neq \Pi_0(f(i))$. If $f(i)$ is simple, $f(i+1) \not\doteq f(i)$, while $f(i+1) \doteq f(i)$ if $f(i)$ is intricate.

From now on until the end of this section the letters $f, g, h \ldots$ will denote transformations.

It is clear that the notion of identity of the preceding section has been chosen to the effect that the Restricted Continuity Principle (RCP) be true in the present formalization. Namely, the following holds.

RCP: If x is an intricate object and f is a transformation such that $f(a) = x$ for some $a \in \text{dom } f$, then for every $n \in \omega$, $f(a+n) \doteq x$.

Transformations can be thought of as motions in the universe of object-states. Those preserving identity keep traveling inside a single identity $Id(x)$. Others cross various identities and either return to previous ones, or leave them forever. Figure 2 shows examples of such motions. Clearly, every transformation crosses either only simple or only intricate identities.

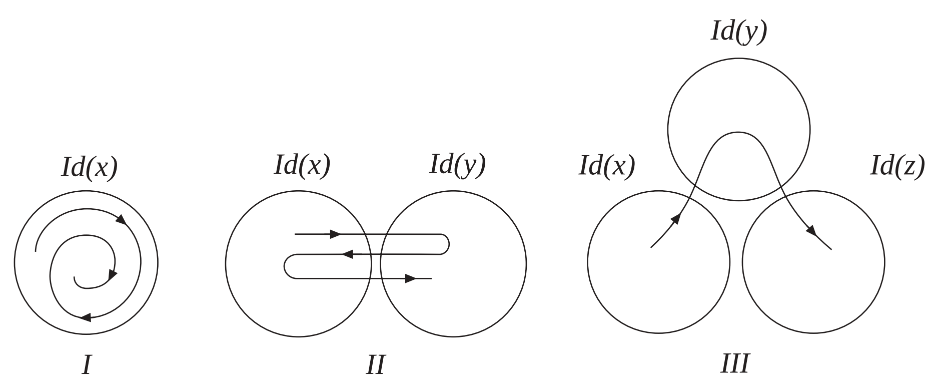


Figure 2: Transformations

Another way to illustrate transformations is using a two-axis system, one for time and one for object-states, as in Figure 3. Here f starts with object x and transforms it into y of different identity. We assume that the elements of $Id(x)$ are crowded around and close to x, so that any identity-preserving transformation of x travels parallel to the time-axis.

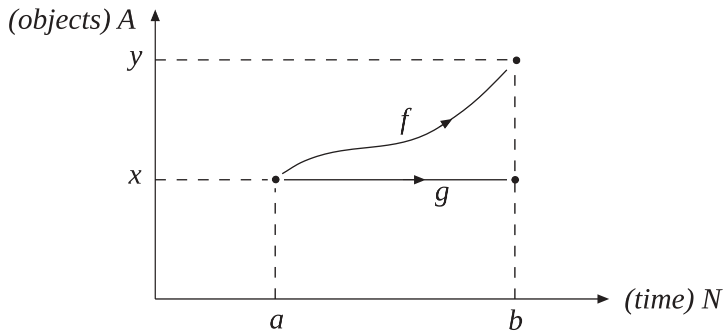


Figure 3: Transformations (2-axis)

As claimed in the introduction, transformations of artifacts are reversible. However, reversibility should be understood with respect to $\doteq$ rather than $=$.

Definition 7.2 Let f be a transformation. Then g is a reverse of f if:

1. $\mathrm{dom}\, f = [a, b]$, $\mathrm{dom}\, g = [c, d]$ with $a < b < c < d$, and
2. $f(b) \doteq g(c)$, $f(a) \doteq g(d)$.

The situation is depicted in Figure 4. Clearly a reverse transformation is by no means unique.

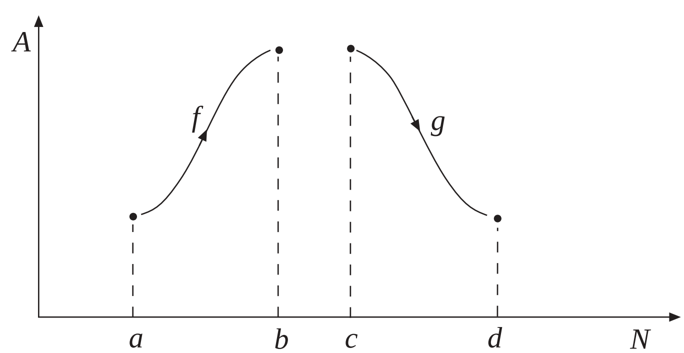


Figure 4: Reverse Transformations

We see that the relation of "being a reverse of" is not symmetric. This is due to the restriction imposed on the domains. We demand $\mathrm{dom}\, f < \mathrm{dom}\, g$ in order to capture the actual process of "return" along the arrow of time.

Such considerations lead naturally to the notions of "simultaneity" and "coexistence" for which we need a global theory of transformations. What has been said so far, on the contrary, is just of a local character.

8 Worlds For a global theory one has to consider a class of transformations obeying some rules that make them simulate actual concurrent transformations.

Let S be a class of transformations. For any $a \in N$ put:

$$\begin{aligned} \operatorname{dom} S &= \bigcup\{\operatorname{dom} f : f \in S\}, \\ \operatorname{rng} S &= \bigcup\{\operatorname{rng} f : f \in S\}, \\ S(a) &= \{f(a) : f \in S\}, \\ \Pi_0(S(a)) &= \bigcup\{\Pi_0(x) : x \in S(a)\}. \end{aligned}$$

dom S is the total "life" of S, rng S is the class of objects involved in S and $S(a)$ is the "state of S at time a." Finally $\Pi_0(S(a))$ is the totality of atoms which form the objects of S at time a. Since atoms are practically imperishable, their totality should be preserved at least for short periods of time, namely those extending within the horizon ω. Hence a first condition imposed on S should be:

$$\Pi_0(S(a+1)) = \Pi_0(S(a)), \tag{w$_1$}$$

for every $a \in \operatorname{dom} S$.

A second requirement is that two "synchronous" objects, i.e., belonging to the same $S(a)$, should not overlap. This is a consequence of the idea that an atom participating in the formation of an artifact at a given time is consumed. The condition is written as follows:

$$(\forall a)(\forall x, y \in S(a))(\Pi_0(x) \cap \Pi_0(y) = \varnothing). \tag{w$_2$}$$

$$(\textit{A fortiori}, \text{ if } x, y \in S(a), \text{ then } x \not\asymp y.)$$

Another unnatural fact would be the crossing of transformations, i.e., $f(a) = g(a)$ for some $f, g \in S$, $f \neq g$. We have to rule out situations like the ones depicted in Figure 5.

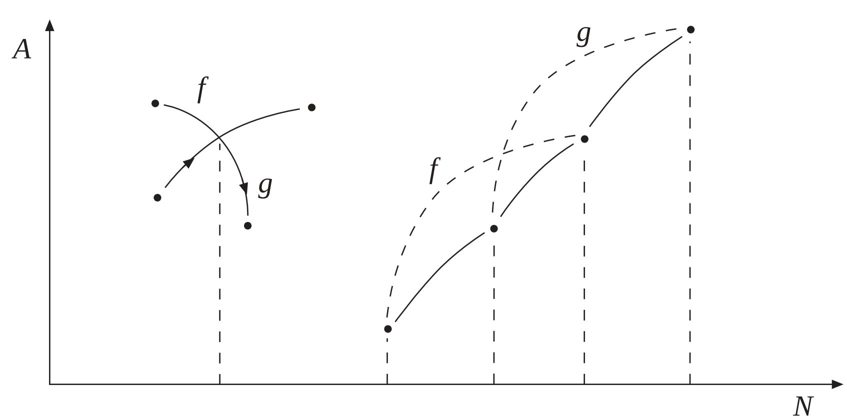


Figure 5: Crossing Transformations

Thus the graphs of distinct transformations should be disjoint, that is to say the following holds.

$$(\forall f, g \in S)(f \neq g \Rightarrow f \cap g = \varnothing). \tag{w$_3$}$$

Finally, we need a restriction on the form of the "life" of S. $\operatorname{dom} S$ should not have definable gaps, such that, for instance, $a < b \in \operatorname{dom} S$ and $[a+1, b-1] \cap \operatorname{dom} S = \varnothing$. On the other hand, it is not necessary that $[a, b] \subseteq \operatorname{dom} S$ whenever $a < b \in \operatorname{dom} S$. The right condition, I presume, is the following.

$$0 \notin \operatorname{dom} S \;\&\; (\forall a \in \operatorname{dom} S)(a-1 \in \operatorname{dom} S \;\&\; a+1 \in \operatorname{dom} S). \qquad (\mathrm{w}_4)$$

Given a (nonstandard) natural number a, the *galaxy* of a is the class $G(a) = \{a+n : n \in \omega\}$. The preceding condition says that if $a \in \operatorname{dom} S$, then $G(a) \subseteq \operatorname{dom} S$.

Definition 8.1 A *world* is any class W of transformations, satisfying the conditions w_1–w_4 above.

A world puts a certain amount of the objects available "into time," i.e., makes them historical entities having past, present and future. A world is a possible world in the sense that it turns certain potential entities into actual ones through some—out of many possible—realizations. Let $A/\doteq$ the class of identities over the universe A of objects and let $I \in A/\doteq$. We say that the identity *I exists at time a* with respect to the world W, if $I \cap W(a) \neq \varnothing$. The world W is said to be *complete*, if every identity I exists at some time with respect to W, i.e., if

$$(\forall I \in A/\doteq)(\exists a \in \operatorname{dom} W)(W(a) \cap I \neq \varnothing).$$

We shall see after a while that there is an abundance of complete worlds.

Example 8.2 Using the fact that the world M has cardinality ω_1 and that all uncountable subclasses of M are equipotent, we can enumerate all atoms by an ω_1-sequence and write $\mathrm{Atom} = \{x_\alpha : \alpha < \omega_1\}$. Let also $\{a_\alpha : \alpha < \omega_1\}$ be an enumeration of the class $N - \omega$ of nonstandard numbers. For all $\alpha, \beta < \omega_1$,consider the one-element function $f_{\alpha\beta} = \{(a_\alpha, x_\beta)\}$. This is a trivial transformation with domain the trivial interval $[a_\alpha, a_\alpha]$. Put

$$W = \{f_{\alpha\beta} : \alpha, \beta < \omega_1\}.$$

It is easy to check that W is a world with $\operatorname{dom} W = N - \omega$ and $\operatorname{rng} W = W(a) = \mathrm{Atom}$, for every $a \in N - \omega$. Clearly, W represents a universe at rest with all artifacts disassembled in their atomic parts. Schematically, this motionless, decomposed world is, as shown in Figure 6, a class of lines running parallel to the time-axis.
Instead of atoms in the preceding example, we might choose any class $X = \{y_\alpha : \alpha < \omega_1\}$ of objects such that $\Pi_0(y_\alpha) \cap \Pi_0(y_\beta) = \varnothing$ for all $\alpha \neq \beta$ and instead of the entire N we could take any convex subclass of N.

A world is a subclass of our universe, in general not a set belonging to it. A family of worlds $\{W_i : i \in J\}$ will be said *small* if $|J| < \omega_1$.

Proposition 8.3 *If $\{W_i : i \in J\}$ is a small class of worlds, linearly ordered by inclusion, then the class $W = \bigcup_i W_i$ is a world.*

Proof: Clearly, W satisfies conditions w_2, w_3, w_4. $W(a) = \bigcup\{W_i(a) : i \in J\}$ and $\Pi_0(W(a)) = \bigcup\{\Pi_0(W_i(a)) : i \in J\}$. Since for all i, a, $\Pi_0(W_i(a+1)) = \Pi_0(W_i(a))$, it follows that w_1 holds too. □

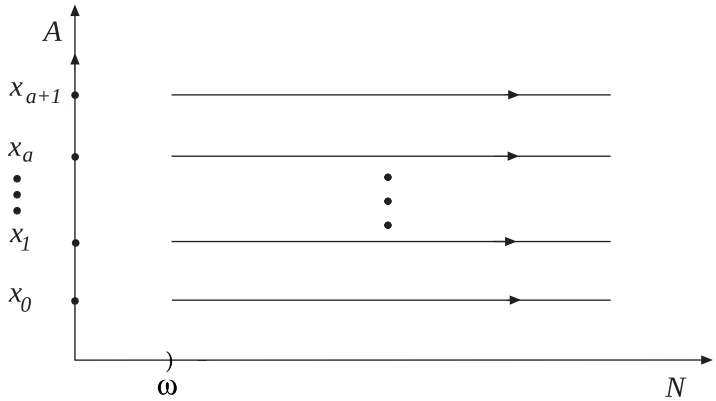


Figure 6: Universe at rest

Proposition 8.4 *If* $\{W_i : i \in J\}$ *is a class of worlds whose lives* dom W *are pairwise disjoint, then* $\bigcup_i W_i$ *is a world.*

Proof: Straightforward. □

Given a world W with bounded dom W, we can *translate* it along the time-axis as far as we wish. To be specific, let $c >$ dom W, $c \in N$, be a bound for dom W. For every $f \in W$, let f' be the function such that

$$(a, x) \in f' \text{ iff } (a - c, x) \in f.$$

Then, f' is the translation of f to the right of length c. Obviously, the class $W' = \{f' : f \in W\}$ is a world such that dom $W \cap$ dom $W' = \varnothing$. W' is an exact copy of W, i.e., a "return" of W in some future time. Moreover the union $W \cup W'$ is also a world according to Proposition 8.3.

Condition w_4 implies that there is no finite world. Hence the smallest size of a world is the countable one.

Proposition 8.5 *For every transformation* f *with* dom $f \in N - \omega$*, there is a world* W *containing it. If* f *is simple, then* W *can be countable.*

Proof (Sketch): Let dom $f = [a, b]$. If $G(a)$ is the galaxy of a, put $X = G(a) \cup [a, b] \cup G(b)$. We can define a world W with dom $W = X$. To this effect, it suffices:

1. To give and take away lives, accordingly, at the right time, to all spare atoms (if any) involved in the transformation f. That is, if at some time a, $f(a+1) = f(a)[y'/y]$, then y' should exist until a and then disappear; on the contrary, y should come into existence exactly at time $a+1$.
2. To fill the domain X with transformations of this kind; for instance, consider the atoms at rest before their death and after their birth.

If f is simple, then only a standard number of atoms are involved in f and since $G(a), G(b)$ are countable classes, we can fill X by using countably many transformations of the kind described previously. □

Proposition 8.6 *Let* W *be a world and let* I *be an identity. Then, there is a world* W' *such that* rng $W \subseteq$ rng W', rng $W' \cap I \neq \varnothing$ *and* dom $W' =$ dom W.

Proof: If $W \cap I \neq \varnothing$ we have nothing to prove. Let $W \cap I = \varnothing$. Choose an object $x \in I$. If $\Pi_0(x) \cap \Pi_0(W(a)) = \varnothing$ for some a, then consider the constant instant transformations $\{(b, x)\}$ for each $b \in G(a)$ and put $W' = W \cup \{\{(b, x)\} : b \in G(a)\}$. Clearly, W' is a world meeting the identity I as required.

Suppose, now, that $\Pi_0(x) \cap \Pi_0(W(a)) \neq \varnothing$ for all $a \in \mathrm{dom}\, W$. Fix some $a \in \mathrm{dom}\, W$ and let $X = \Pi_0(x) \cap \Pi_0(W(a))$ and $Y = \Pi_0(x) - X$. Fix also enumerations $X = \{x_\alpha : \alpha < \omega_1\}$, $Y = \{y_\alpha : \alpha < \omega_1\}$ of X, Y. Then, for each $\alpha < \omega_1$ there is a unique transformation $f_\alpha \in W$ such that $x_\alpha \in \Pi_0(f_\alpha(a))$.

The idea is to cut vertically the world W along the line $t = a$ and translate the right-hand side one step to the right (or possibly more steps if we wish). The axis $t = a+1$ moves, then, to $t = a+2$ and in the open strip along $t = a+1$ we insert properly the object x (see Figure 7). We merely have to arrange things so that not to disturb the existing transformations. Thus, let us define for every $f \in W$ a transformation f' as follows:

1. $f' = f$ if $\mathrm{dom}\, f < a$.
2. If $a < \mathrm{dom}\, f$, put $f' = \{(b+1, z) : (b, z) \in f\}$.
3. Let $a \in \mathrm{dom}\, f$ and $f \neq f_a$ for all $\alpha < \omega_1$. We insert a moment of rest in the life of f changing it into

$$f'(b) = \begin{cases} f(b) & \text{for } b \leq a, \\ f(a) & \text{for } b = a+1, \\ f(b-1) & \text{for } b > a+1. \end{cases}$$

4. If $f = f_\alpha$, interrupt f at $a+1$, i.e., put $f' = f_\alpha | a+1$.

Next, put for each $\alpha < \omega_1$ $f''_\alpha = f_\alpha \cap [a+2, c]$ if $\mathrm{dom}\, f$ ends at c. Let also $g = \{(a+1, x)\}$, $g_{b\alpha} = \{(b, y_\alpha)\}$ for each $b \in G(a)$, $b \neq a$. $g_{b\alpha}$ are the orbits of the atoms $y_\alpha, \alpha < \omega_1$ which at the time a cease to exist, since they are assembled together with the atoms $x_\alpha, \alpha < \omega_1$ to form x. x, on the other hand, lives for just one unit of time. Finally put

$$W' = \{f' : f \in W\} \cup \{f''_\alpha : \alpha < \omega_1\} \cup \{g\} \cup \{g_{b\alpha} : b \in G(a) \ \&\ b \neq a \ \&\ \alpha < \omega_1\}.$$

The transformation of W to W' is shown in Figure 7. It is not hard to verify that W' is a world having the required properties. □

Let us call a world W bounded if $\mathrm{dom}\, W$ is bounded in N.

Corollary 8.7 *Let W be bounded. Then for any identity I, there is a bounded world $W' \supseteq W$ such that $I \cap \mathrm{rng}\, W' \neq \varnothing$.*

Proof: Since W is bounded we can translate it along the time-axis and get a copy W_1 of W such that $\mathrm{dom}\, W \cap \mathrm{dom}\, W_1 = \varnothing$. Then transform W_1 by the process described in the preceding proposition into a world W_1' such that $\mathrm{dom}\, W_1' = \mathrm{dom}\, W_1$ and $\mathrm{rng}\, W_1' \cap I \neq \varnothing$. Put $W = W \cup W_1'$. According to Proposition 8.4, this is a world with the demanded properties. □

Proposition 8.8 *Any bounded world can be extended to a complete world.*

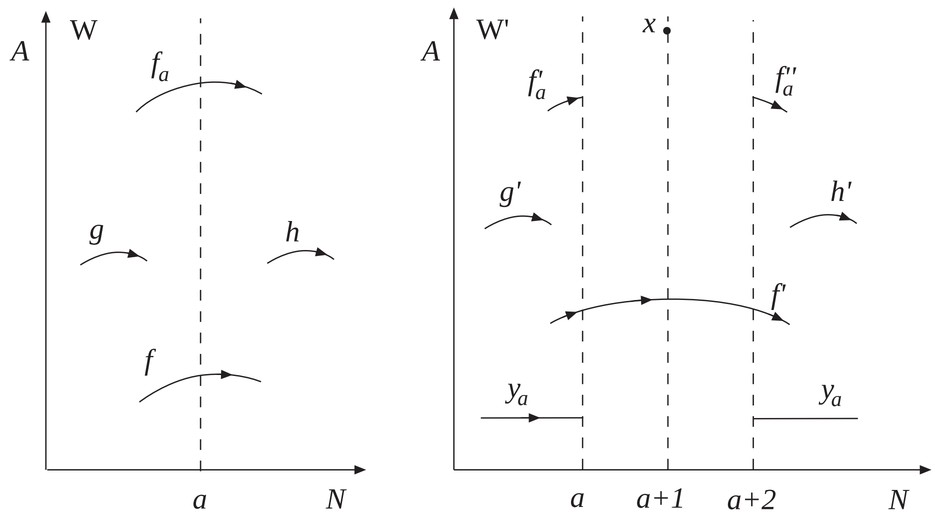


Figure 7: Transformation of $W \to W'$

Proof: The collection $A/\doteq$ of identities can be, clearly, enumerated in the form $(I_\alpha), \alpha < \omega_1$. We shall define inductively an ascending chain of worlds $(W_\alpha), \alpha < \omega_1$, with the following properties:

1. $W_0 = W$ and $W_\alpha \subseteq W_{\alpha+1}$ for every $\alpha < \omega_1$;
2. Each W_α is bounded;
3. $\operatorname{rng} W_\alpha \cap I_\alpha \neq \varnothing$ for every $\alpha < \omega_1$.

If we assume that the sequence is constructed, then the class $W = \bigcup_\alpha W_\alpha$ is a world according to Proposition 8.3 and is complete since $\operatorname{rng} W \cap I_\alpha \neq \varnothing$ for all $\alpha < \omega_1$.

Suppose W_α has been defined and $\operatorname{dom} W_\alpha$ is bounded. By Corollary 8.7 there is a bounded world $W_{\alpha+1} \supseteq W_\alpha$ such that $\operatorname{rng} W_{\alpha+1} \cap I_{\alpha+1} \neq \varnothing$.

Suppose now that α is a limit ordinal and all $W_\beta, \beta < \alpha$, have been defined and are bounded. Then the class $W = \bigcup\{W_\beta : \beta < \alpha\}$ is a world again. Moreover $\operatorname{dom} W = \bigcup\{\operatorname{dom} W_\beta : \beta < \alpha\}$ is bounded. Indeed the classes $\operatorname{dom} W_\beta, \beta < \alpha$, are countably many and bounded, hence, due to the saturation of $\mathfrak{M}$, their union cannot be cofinal to N. Extend W to W' again as in Proposition 8.6 such that $W' \cap I_\alpha \neq \varnothing$ and set $W_\alpha = W'$. This completes the construction of the sequence $(W_\alpha), \alpha < \omega_1$, and the proof. $\square$

A world W is said to be *maximal* if it is maximal with respect to $\subseteq$, i.e., if for every $f \notin W$ there is no world containing $W \cup \{f\}$.

Proposition 8.9 *W is maximal if and only if* $\operatorname{dom} W = N - \omega$ *and for every $a \in N$*, $\Pi_0(W(a)) = \text{Atom}$.

Proof: Let W be maximal. Suppose $\operatorname{dom} W \overset{\subset}{\neq} N - \omega$ and let $\alpha > \omega$ and $a \notin \operatorname{dom} W$. Then, clearly, $G(a) \cap \operatorname{dom} W = \varnothing$. If W_1 is any world with $\operatorname{dom} W_1 = G(a)$, then $W \cup W_1$ is a world properly extending W, a contradiction. Assume next that for some a there is an atom $x \notin \Pi_0(W(a))$. Then $x \notin \Pi_0(W(a+n))$ for every $n \in \omega$. Therefore we can add to W the transformations $\{(b, x) : b \in G(a)\}$, extending properly W again.

Conversely, suppose W is not maximal and dom $W = N - \omega$. It suffices to show that for some a, $\Pi_0(W(a)) \neq$ Atom. Indeed, by hypothesis there is an $f \notin W$ and a world $W' \supseteq W \cup \{f\}$. Let $a \in$ dom f. Then $\Pi_0(f(a)) \cap \Pi_0(y) = \varnothing$ for every $y \in W(a)$. Hence $\Pi_0(f(a)) \cap \Pi_0(W(a)) = \varnothing$. Therefore $\Pi_0(W(a)) \neq$ Atom and the claim is proved. □

Thus, the world at rest of Example 8.2 of this section is a maximal world. We do not know whether this is the only maximal world.

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