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Partition Principles and Infinite Sums of Cardinal Numbers

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Abstract The Axiom of Choice implies the Partition Principle and the existence, uniqueness, and monotonicity of (possibly infinite) sums of cardinal numbers. We establish several deductive relations among those principles and their variants: the monotonicity follows from the existence plus uniqueness; the uniqueness implies the Partition Principle; the Weak Partition Principle is strictly stronger than the Well-Ordered Choice.

1 Introduction The Partition Principle states that the size of any partition of a set is at most that of the original set. The uniqueness of the sums of cardinal numbers is the principle that the direct sums of equipollent sets are also equipollent. (They are PP and FB, respectively, in the next section.) They are immediate consequences of the Axiom of Choice and the first two of seven applications presented by Zermelo [18] to indicate the indispensability of the Axiom.

The deductive relations have not been settled among those three principles except for the above-mentioned trivial ones. We partly answer by showing the implication FB $\Rightarrow$ PP (Theorem 3.2).

We also establish that the Weak Partition Principle is strictly stronger than the Axiom of Choice restricted to well-orderable families of sets; this solves two of problems in Banaschewski and Moore [1]. For other results, see the end of the next section.

2 Preliminaries We work in the theory ZFU (the Zermelo-Fraenkel set theory with atoms and without the Axiom of Choice) $-$ Regularity, or ZF$^\circ$, unless otherwise stated.

Some of our notation is borrowed from [1] or Rubin and Rubin [13]. We define the relations $\approx$, $\preceq$, $\prec$ and $\preceq\!*$ by

$$x \approx y \quad \Leftrightarrow \quad \text{there exists a bijection } x \to y,$$

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$$
\begin{array}{rcl}
x \preceq y & \Leftrightarrow & \text{there exists an injection } x \to y, \\
x \prec y & \Leftrightarrow & x \preceq y \text{ and not } y \preceq x, \\
x \preceq^* y & \Leftrightarrow & x \text{ is empty or there exists a surjection } y \to x.
\end{array}
$$

We use $+$ and $\sum$ to denote direct sums; if necessary to be specific,

$$
\begin{aligned}
x + y &= (x \times \{0\}) \cup (y \times \{1\}), \\
\sum_{i \in I} x_i &= \bigcup_{i \in I} (x_i \times \{i\}).
\end{aligned}
$$

A set x is said to be *idemmultiple* if $x + x \approx x$.

In ZF$^\circ$, the notion of cardinality is known to be undefinable (cf. Jech [7], Theorem 11.2, see also Remark 2.1 below). So we use a *local cardinal number* instead, by which we mean a nonempty set X such that $(\forall x, y \in X)(x \approx y)$. Let

$$x \widetilde{\in} X \Leftrightarrow \text{ for some } y \in X, x \approx y.$$

Nevertheless for x well-orderable, $|x|$ denotes the least ordinal equipollent to x.

We consider following statements.

AC: The Axiom of Choice.

$\aleph_\alpha$-AC: Well-ordered choice of length ω_α.

DC: The Principle of Dependent Choices.

$\aleph_0$-TC: Every Dedekind-finite set is finite.

PP: If $x \preceq^* y$, then $x \preceq y$. (The Partition Principle.)

WPP: If $x \preceq^* y$, then $y \not\prec x$. (The Weak Partition Principle. An equivalent formulation: if $x \preceq^* y \preceq x$, then $x \approx y$.)

PPIdm: If y is idemmultiple and $x \preceq^* y$, then $x \preceq y$.

FI: For every pair $\langle x_i : i \in I \rangle$, $\langle y_i : i \in I \rangle$ of families of sets with the same index set, $(\forall i \in I)(x_i \preceq y_i)$ implies $\sum_{i \in I} x_i \preceq \sum_{i \in I} y_i$.

FB: For $\langle x_i : i \in I \rangle$, $\langle y_i : i \in I \rangle$ as above, $(\forall i \in I)(x_i \approx y_i)$ implies $\sum_{i \in I} x_i \approx \sum_{i \in I} y_i$.

LCR: For every family $\langle X_i : i \in I \rangle$ of local cardinal numbers, there exists a family $\langle x_i : i \in I \rangle$ of sets such that $(\forall i \in I)(x_i \widetilde{\in} X_i)$.

Idm: Every infinite set is idemmultiple.

PW: If a set has at least two elements, then it can be partitioned into well-orderable blocks with at least two elements.

WU: The union of a well-orderable family of well-orderable sets is also well-orderable.

Remark 2.1 Assume temporarily the Regularity Axiom and that the class of atoms is a set. In this case every set x is assigned its cardinal number card(x) such that $x \approx y \Leftrightarrow \text{card}(x) = \text{card}(y)$. For cardinal numbers $\mathbf{m}_i, i \in I$ and $\mathbf{m}$, we define $\mathbf{m}$ to be a sum of $\langle \mathbf{m}_i : i \in I \rangle$ if there exists a family of sets $\langle x_i : i \in I \rangle$ such that $(\forall i \in I)(\text{card}(x_i) = \mathbf{m}_i)$ and $\text{card}\left(\sum_{i \in I} x_i\right) = \mathbf{m}$. Then existence, uniqueness and monotonicity of the sum are equivalent to LCR, FB and FI, respectively.

These principles are all consequences of AC and independent of ZF. The following deductive relations are well known or easily seen (cf. Halpern and Howard [3], Häussler [4], Howard [5], Jech [7], Moore [9], Pelc [10], Pincus [12], Rubin and Rubin [13], and Sageev [14]), where arrows denote implications in ZF$^\circ$ and negated arrows mean that the implications are independent of ZF.[1]

$$
\begin{array}{cccccccc}
 & \mathrm{FI} & \rightarrow & \mathrm{FB} & & & & \\
 & \downarrow & & & \searrow & & & \\
 & \mathrm{PP} & & & & \mathrm{WU} & \substack{\rightarrow \\ \not\leftarrow} & (*) \\
\swarrow & \downarrow & \searrow & & \nearrow & & & \uparrow\not\downarrow \\
\mathrm{WPP} & \mathrm{PPIdm} & & (\forall\alpha)\aleph_\alpha\text{-AC} & \substack{\rightarrow \\ \not\leftarrow} & \mathrm{DC} & \substack{\rightarrow \\ \not\leftarrow} & \aleph_0\text{-AC} \\
 & & & & \not\searrow & & & \downarrow\not\uparrow \\
 & & & & & \mathrm{Idm} & \substack{\rightarrow \\ \not\leftarrow} & \aleph_0\text{-TC} \\
 & & & & \swarrow & \not\uparrow & \not\searrow & \not\uparrow\not\downarrow \\
 & & & \mathrm{PW} & & \mathrm{LCR} & \not\rightarrow & (*)
\end{array}
$$

The symbol $(*)$ stands for the Axiom of Choice restricted to countable families of countable sets, which we include here to simplify the diagram.

Our results are indicated below by double arrows.

$$
\begin{array}{ccccccccccccc}
\mathrm{FB}+\mathrm{LCR} & & & & & & & & & & & & \\
\Downarrow & & & & & & & & & & & & \\
\mathrm{FI} & \rightarrow & \mathrm{FB} & \Rightarrow & \mathrm{PP} & \rightarrow & \mathrm{WPP} & \Rightarrow & \mathrm{PPIdm} & \substack{\Rightarrow \\ \not\Leftarrow} & (\forall\alpha)\aleph_\alpha\text{-AC} & . \\
\Uparrow & & & & & & & & & & \not\Downarrow & \\
\mathrm{FB}+\mathrm{PW} & \Rightarrow & \mathrm{Idm} & & & & & & & & \mathrm{PW} &
\end{array}
$$

We thus answer two of the open problems mentioned in [1]:

(13) Does CB* (or even WPP) imply DC?

(14) Does PP follow from the proposition that for all α, $\aleph_\alpha$-PP?

the former affirmatively (Corollary 4.2), the latter negatively (Corollary 5.2).

3 Partition principles and direct sums Pincus (see [10]) proved that

$$\mathrm{PP} \Rightarrow (\forall\alpha)\aleph_\alpha\text{-AC}.$$

Here PP can be weakened to the following principle.

PP$^-$: If y is idemmultiple and there exists a surjection onto an ordinal $f : y \to \lambda$ such that for each $\xi < \lambda$, $f^{-1}[\{\xi\}]$ is Dedekind-infinite, then $\lambda \preceq y$.

The condition "y is idemmultiple" above, which makes PP$^-$ a consequence of PPIdm, is not relevant to the arguments in this section but is necessary for the proof of Corollary 4.2.

Lemma 3.1 PP$^-$ *is equivalent to* $(\forall\alpha)\aleph_\alpha$-AC.

Proof (Sketch): The necessity is straightforward. For the other direction, a slight modification of Pincus's proof will do. Let $\langle x_\gamma : \gamma < \omega_\alpha \rangle$ be a family of nonempty sets. Assuming PP$^-$ and $(\forall\beta < \alpha)\aleph_\beta$-AC, we prove $\prod_{\gamma<\omega_\alpha} x_\gamma \neq \varnothing$.

Let C_γ for $\gamma < \omega_\alpha$ denote the set $\prod_{\delta<\gamma} x_\delta$, which is nonempty by $(\forall\beta < \alpha)\aleph_\beta$-AC. We define families $\langle D_\gamma : \gamma < \omega_\alpha\rangle$ of sets and $\langle \lambda_\gamma : \gamma < \omega_\alpha\rangle$ of cardinals inductively as follows.

$$\begin{aligned} \lambda_\gamma &= \max\left\{\aleph\left(\bigcup_{\delta<\gamma} D_\delta\right), \sup_{\delta<\gamma} \lambda_\delta^+\right\}, \\ D_\gamma &= \omega \times C_\gamma \times \lambda_\gamma, \end{aligned}$$

where $\aleph(\cdot)$ denotes the Hartogs function:

$$\aleph(x) = \min\{\alpha \in \mathit{Ord} : \alpha \not\preceq x\}.$$

Let $D = \bigcup_{\gamma<\omega_\alpha} D_\gamma$, $\lambda = \sup_{\gamma<\omega_\alpha} \lambda_\gamma$.

Since the projection $f : D \to \lambda$ such that $f(*, *, \mu) = \mu$ satisfies the premise of PP$^-$, we get an injection $g : \lambda \to D$. Using g, we can define a choice function in $\prod_{\gamma<\omega_\alpha} x_\gamma$. □

Theorem 3.2 FB *implies* PP.

Proof: Consider the following auxiliary statement.

PP$'$: If there exists a surjection $f : y \to x$ such that for each $u \in x$, $f^{-1}[\{u\}]$ is finite or Dedekind-infinite, then $x \preceq y$.

PP$'$ implies PP$^-$ and so $\aleph_0$-TC by Lemma 3.1. Thus PP$'$ is in fact equivalent to PP.

Let x, y and f be as in the premise of PP$'$. Assuming FB, we show $x \preceq y$.

We define families $\langle y_u : u \in x\rangle$ and $\langle z_u : u \in x\rangle$ as follows.

$$\begin{aligned} y_u &= f^{-1}[\{u\}], \\ z_u &= \begin{cases} |y_u|, & y_u \text{ is finite}, \\ y_u \cup \{0\}, & y_u \text{ is Dedekind-infinite}. \end{cases} \end{aligned}$$

Then we have $(\forall u \in x)(y_u \approx z_u)$ and, by using FB,

$$\sum_{u\in x} y_u \approx \sum_{u\in x} z_u.$$

On the other hand, surjectivity of f implies $(\forall u \in x)(0 \in z_u)$. Hence we get

$$y = \bigsqcup_{u\in x} y_u \approx \sum_{u\in x} y_u \approx \sum_{u\in x} z_u \supseteq \{0\} \times x \approx x,$$

accordingly $x \preceq y$. □

Here we refer to two cancellation laws.

Theorem 3.3 (Tarski [16], Corollary 5) *If $x + n \times z \approx y + n \times z$ for some natural number n, then $x + z \approx y + z$.*

Theorem 3.4 (Fillmore [2]) *Assume $\aleph_0$-AC. If $(\forall n < \omega)(n \times x \preceq (n + 1) \times y)$, then $x \preceq y$.*[2]

Lemma 3.5 *Assume* LCR. *If two families of sets* $\langle x_i : i \in I\rangle$ *and* $\langle y_i : i \in I\rangle$ *and a natural number* n *satisfy*

$$(\forall i \in I)((n+1) \times x_i \preceq y_i),$$

then for some $\langle z_i : i \in I\rangle$,

$$(\forall i \in I)(n \times x_i + z_i \approx y_i).$$

Proof: Let $\langle x_i : i \in I\rangle$, $\langle y_i : i \in I\rangle$ and n be as in the hypothesis and $\langle Z_i : i \in I\rangle$ determined by

$$Z_i = \{z \subseteq y_i : z \approx x_i + w \text{ for some } w \text{ such that } (n+1) \times x_i + w \approx y_i\}.$$

Then for each $i \in I$, $Z_i \neq \varnothing$.

Suppose $z, z' \in Z_i$. There exist w and w' satisfying

$$\begin{aligned} (n+1) \times x_i + w &\approx (n+1) \times x_i + w' \ (\approx y_i), \\ z &\approx x_i + w, \\ z' &\approx x_i + w'. \end{aligned}$$

By Theorem 3.3, we have $x_i + w \approx x_i + w'$, and so $z \approx z'$. Hence each Z_i for $i \in I$ is a local cardinal number.

By using LCR, we obtain a family $\langle z_i : i \in I\rangle$ such that $(\forall i \in I)(z_i \tilde{\in} Z_i)$, for which $(\forall i \in I)(n \times x_i + z_i \approx y_i)$ holds. □

Theorem 3.6 FB *plus* LCR *implies* FI.

Proof: Suppose two families of sets $\langle x_i : i \in I\rangle$ and $\langle y_i : i \in I\rangle$ satisfy $(\forall i \in I)(x_i \preceq y_i)$. Then, for each $n < \omega$,

$$(\forall i \in I)((n+1) \times x_i \preceq (n+1) \times y_i).$$

By Lemma 3.5, we get a family $\langle z_i : i \in I\rangle$ such that

$$(\forall i \in I)(n \times x_i + z_i \approx (n+1) \times y_i).$$

Therefore, by FB,

$$n \times \sum_{i \in I} x_i + \sum_{i \in I} z_i \approx (n+1) \times \sum_{i \in I} y_i,$$

and so

$$n \times \sum_{i \in I} x_i \preceq (n+1) \times \sum_{i \in I} y_i.$$

Applying Theorem 3.4, we conclude that

$$\sum_{i \in I} x_i \preceq \sum_{i \in I} y_i.$$

□

4 Idemmultiplicity As far as idemmultiple sets are concerned, some aspects of cardinalities are quite simple.

Proposition 4.1 WPP *implies* PPIdm.[3]

Proof: Let $x \preceq^* y \approx y + y$. We have $y \preceq x + y \preceq^* y + y \approx y$. By WPP, We get $x + y \approx y$ and so $x \preceq y$. □

Corollary 4.2 WPP *implies* $(\forall \alpha)\aleph_\alpha$-AC.

Proof: Combine the proposition with Lemma 3.1. □

Corollary 4.3 *Assume* Idm. *Then* PP *and* WPP *are equivalent.*

Lemma 4.4 FB *plus* Idm *implies* FI.

Proof: Let $\langle x_i : i \in I \rangle$, $\langle y_i : i \in I \rangle$ be such that $(\forall i \in I)(x_i \preceq y_i)$. We define the family $\langle z_i : i \in I \rangle$ by

$$z_i = \begin{cases} |y_i| - |x_i|, & y_i \text{ is finite}, \\ y_i, & y_i \text{ is infinite}. \end{cases}$$

Using Idm, we get $(\forall i \in I)(x_i + z_i \approx y_i)$; and hence, due to FB, $\sum_{i \in I} x_i \preceq \sum_{i \in I} y_i$. □

In the lemma above, Idm can be replaced by (apparently weaker) PW. We shall show this through a generalization of the theorem in König [8].[4]

For partitions y, z of the same set, we denote by $z \sqsubset y$ that

$$(\forall v \in y)(\exists w \in z)(v \subsetneq w).$$

(I.e., z is everywhere strictly coarser than y.)

Lemma 4.5 *Assume* PW. *Suppose* y *is a partition (of its union) with at least two blocks. Then there exists a coarser one* z *such that*

$$z \sqsubset y,$$
$$(\forall w \in z)(\{v \in y : v \subseteq w\} \textit{ is well-orderable}).$$

Proof: Due to PW, there exists a partition z' of y such that each $w' \in z'$ is well-orderable and consists of at least two blocks of y. Then

$$z = \left\{ \bigcup w' : w' \in z' \right\}$$

suffices. □

Theorem 4.6 PW *plus* FB *implies* Idm.

Proof: Suppose x is an infinite set. Assuming PW and FB, we shall show that x is idemmultiple. If x is well-orderable, then we are done. So assume otherwise.

We denote by P the set

$$\{y : y \text{ is a partition of } x \text{ into well-orderable blocks}\},$$

Then P is nonempty, and each $y \in P$ has infinitely many blocks. For $y \in P$, let z be a coarser partition as in Lemma 4.5. Thus for each $w \in z$,

$$\{v \in y : v \subseteq w\} \text{ is well-orderable,}$$
$$w = \bigcup\{v \in y : v \subseteq w\};$$

by WU, w is well-orderable. Therefore $z \in P$. Accordingly we have shown that

$$(\forall y \in P)(\exists z \in P)(z \sqsubset w).$$

By using DC, we get a sequence $\langle y_n : n < \omega \rangle$ in P such that

$$(\forall n < \omega)(y_{n+1} \sqsubset y_n).$$

We define the partition y_ω by

$$y_\omega = \left\{ \bigcup_{n<\omega} v_n : \langle v_n : n < \omega \rangle \in \prod_{n<\omega} y_n \text{ and } (\forall n < \omega)(v_n \subseteq v_{n+1}) \right\}.$$

Then each block v of y_ω is the union of a strictly increasing sequence of well-orderable sets. Again by WU, v is well-orderable and infinite, and thus idemmultiple.

By virtue of FB, we have

$$\begin{aligned} \sum_{v \in y_\omega} v &\approx \sum_{v \in y_\omega} (v + v) \\ &\approx \sum_{v \in y_\omega} v + \sum_{v \in y_\omega} v. \end{aligned}$$

On the other hand,

$$x = \bigsqcup_{v \in y_\omega} v \approx \sum_{v \in y_\omega} v.$$

Therefore x is idemmultiple. □

Corollary 4.7 FB *plus* PW *implies* FI.

5 *Levy's model* Recall the model described in [7], Theorem 8.9. We begin with the universe V of ZFU + AC + "the set A of atoms is of size $\aleph_{\alpha+1}$." The permutation model $\mathcal{V}$ is determined by the group $\mathcal{G}$ of all permutations of A and the normal ideal $I = \{X \subseteq A : |A| \leq \aleph_\alpha\}$: $\mathcal{V}$ is the class satisfying

$$A \subset \mathcal{V}$$

and

$$(\forall x)(x \in \mathcal{V} \leftrightarrow x \subset \mathcal{V} \text{ and } (\exists E \in I)(\text{fix}(E) \subseteq \text{sym}(x))),$$

where

$$\begin{aligned} \text{sym}(x) &= \{\pi \in \mathcal{G} : \pi x = x\}, \\ \text{fix}(x) &= \{\pi \in \mathcal{G} : (\forall y \in x)(\pi y = y)\}. \end{aligned}$$

$\mathcal{V}$ is known to be a model of ZFU + $(\forall\alpha)\aleph_\alpha$-AC + $\neg$AC (and more). The transfer into ZF is obtained by Pincus (see also Pincus [11]).

Theorem 5.1 *In the model $\mathcal{V}$,*

1. PW *does not hold;*[5]
2. PPIdm *does not hold.*

Proof: We work in the universe V.

(1) Suppose x is a partition of A into $(\text{well-orderable})^{\mathcal{V}}$ nontrivial blocks. Let E be in I. Since for every $y \subseteq A$,

$$(y \text{ is well-orderable})^{\mathcal{V}} \Leftrightarrow |y| \leq \aleph_\alpha,$$

there exist two atoms $a, b \in A \setminus E$ which do not belong to the same block of x. We denote by π the transposition of a and b. Then $\pi \in \text{fix}(E)$ and $\pi x \neq x$. Therefore $x \notin \mathcal{V}$.

(2) Let, for each $X \subseteq A$, $[X] = \{Y \subseteq A : X \triangle Y \text{ is finite}\}$, and let $\mathcal{P}(A)/\text{fin} = \{[X] : X \subseteq A\}$. Note that

$$\mathcal{V} \models (\mathcal{P}(A)/\text{fin})^{\mathcal{V}} \preceq_* \mathcal{P}(A)^{\mathcal{V}} \approx \mathcal{P}(A)^{\mathcal{V}} + \mathcal{P}(A)^{\mathcal{V}}.$$

We want to show that

$$\mathcal{V} \models (\mathcal{P}(A)/\text{fin})^{\mathcal{V}} \not\preceq \mathcal{P}(A)^{\mathcal{V}}.$$

Suppose $f : (\mathcal{P}(A)/\text{fin})^{\mathcal{V}} \to \mathcal{P}(A)^{\mathcal{V}}$ is an injection. ($\mathcal{P}(A)^{\mathcal{V}} = \{X \subseteq A : X \in I \text{ or } A \setminus X \in I\}$ and for each $X \in \mathcal{P}(A)^{\mathcal{V}}$, $[X]^{\mathcal{V}} = [X]$.) We show that $f \notin \mathcal{V}$, i.e., for each $E \in I$, we find $\pi \in \text{fix}(E)$ such that $\pi f \neq f$. We define the function $\widetilde{\ }$: $\mathcal{P}(A)^{\mathcal{V}} \to I$ by

$$\widetilde{X} = \begin{cases} X, & X \in I, \\ A \setminus X, & \text{otherwise}. \end{cases}$$

Case 1: For some $X \in \mathcal{P}(A)^{\mathcal{V}}$, $\widetilde{f[X]} \not\subseteq E$. Let π be the transposition of an element of $\widetilde{f[X]} \setminus E$ and one in $A \setminus (\widetilde{f[X]} \cup E)$. Then $\pi \in \text{fix}(E)$. On the other hand, $[X] = [\pi X]$ and $\pi(f[X]) \neq f[X]$, so $(\pi f)[X] = (\pi f)[\pi X] = \pi(f[X]) \neq f[X]$, hence $\pi f \neq f$.

Case 2: For all $X \in \mathcal{P}(A)^{\mathcal{V}}$, $\widetilde{f[X]} \subseteq E$. Let $X \in \mathcal{P}(A)^{\mathcal{V}}$ and $\pi \in \text{fix}(E)$ be such that $[X] \neq [\pi X]$. Then $f[\pi X] \neq f[X] = \pi(f[X]) = (\pi f)[\pi X]$. We also get $\pi f \neq f$. □

Corollary 5.2 *Assume that* ZF *is consistent. Then in* ZF, $(\forall\alpha)\aleph_\alpha$-AC *does not imply* PW *nor* PPIdm, *a fortiori* WPP.

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NOTES

1. By virtue of Howard [6], Theorem 2.3, the implication WU $\to \aleph_0$-AC is independent of ZFU.
2. Fillmore's result is formulated in the language of cardinal algebras (cf. Tarski [17]).
3. Sierpiński [15] shows that $\omega_1 \preceq \mathbf{R}$ and $\mathbf{R}/\mathbf{Q} \preceq \mathbf{R}$, instances of PPIdm, follow from WPP. Our proof is essentially the same as the arguments therein.

4. König deduced Idm from the principle "every infinite set has a partition whose blocks are all at most countable and not singletons" by implicitly using FB and DC.

5. The author thanks Tatsuya Shimura for pointing out that this model witnesses ZF $\nvdash$ PW.

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