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Expansions of Ultrahomogeneous Graphs

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Abstract Lachlan and Woodrow have completly classified the countable ultrahomogeneous graphs. We expand the language of graphs to include a new unary predicate. In this expanded language, ultrahomogeneous vertex 2-colorings of ultrahomogeneous graphs are classified.

1 Introduction A countable structure **M** for a first order, purely relational language $\mathcal{L}$ is *ultrahomogeneous* provided any isomorphism between finite substructures of **M** extends to an automorphism of the entire structure. Since a countable ultrahomogeneous structure is determined up to isomorphism (among the countable ultrahomogeneous structures) by its finite substructures, many questions about a countable ultrahomogeneous structure can be reduced to questions about its finite substructures. The particular ultrahomogeneous structures we will be concerned with are countable (undirected) graphs. Lachlan and Woodrow [4] have completely classified the countably infinite ultrahomogeneous graphs.

We address the question of how the vertex set of ultrahomogeneous graphs may be two colored so that the colored graph (in the expanded language formed by adding a unary predicate U) remains ultrahomogeneous. It is not difficult to see what each colored portion of the graph must be isomorphic to. In fact, with one minor exception, the type of each color together with the underlying graph type uniquely determines the entire ultrahomogeneous structure.

There is a one-one correspondence between countably infinite ultrahomogeneous structures and coherent classes of finite models. The following definition follows (cf. Kueker and Laskowski [3]).

Definition 1 A class $\mathcal{K}$ of finite structures is *coherent* provided it is (*i*) closed under substructures and isomorphisms; (*ii*) has countably many isomorphism types; (*iii*) contains arbitrarily large finite structures, and (*iv*) satisfies the amalgamation and joint embedding properties.

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Fact 2 A coherent class $\mathcal{K}$ has a unique countably infinite ultrahomogeneous structure associated with it that embeds every element of $\mathcal{K}$ and may be written as a countable increasing union of elements of $\mathcal{K}$. Such a structure is said to be $\mathcal{K}$ *generic*. Conversely, the class of all finite structures that embed into a given ultrahomogeneous structure is coherent (cf. Fraïssé [1], Kueker and Laskowski [3]).

Definition 3 Let A, B be graphs with vertex sets $\{a_\alpha : \alpha < \delta\}$ and $\{b_\beta : \beta < \gamma\}$. The *wreath product* $A(B)$ is the graph with vertex set $A \times B$ and edge relation E such that $((a_{\alpha_1}, b_{\beta_1}), (a_{\alpha_2}, b_{\beta_2})) \in E$ if and only if $a_{\alpha_1} \neq a_{\alpha_2}$ and $(a_{\alpha_1}, a_{\alpha_2}) \in E^{\mathbf{A}}$, or $a_{\alpha_1} = a_{\alpha_2}$ and $(b_{\beta_1}, b_{\beta_2}) \in E^{\mathbf{B}}$.

If N_α and K_α are null and complete graphs on α vertices respectively, then it is easy to see that $N_\alpha(K_\beta)$ and $K_\alpha(N_\beta)$ where $\alpha, \beta \leq \omega$ are ultrahomogeneous graphs. There are also two additional finite ultrahomogeneous graphs, $K_3 \times K_3$ and the pentagon (cf. Gardner [2], Sheehan [5]).

The countable ultrahomogeneous graphs that remain may best be described through the coherent classes that produce them as generics.

Definition 4 Fix $1 \leq n < \omega$. Then Σ_n is the class of all finite graphs A (for the language $\mathcal{L} = \{E\}$) in which K_{n+1}, the complete graph on $n+1$ vertices, does not embed. Similarly, we have the dual class $\widetilde{\Sigma}_n$ of finite graphs which do not embed N_{n+1}, the null graph on $n+1$ vertices.

It is easy to see that Σ_n ($\widetilde{\Sigma}_n$) is coherent (as will generally be the case in this paper free amalgamation will suffice) and so has the associated ultrahomogeneous generic structure $\mathbf{G}_n(\widetilde{\mathbf{G}}_n)$. When $n = 2$, $\mathbf{G}_2$ is the ultrahomogeneous graph which embeds all triangle free graphs. In the case where $n = 1$, $\mathbf{G}_1$ is an infinite null graph, $\widetilde{\mathbf{G}}_1$ an infinite complete graph. Finally, the class of all finite graphs Σ_ω is easily seen to be coherent, and has ultrahomogeneous generic $\mathbf{G}_\omega$, the random graph. Lachlan and Woodrow [4] have shown that the above list is a complete classification of all countable ultrahomogeneous graphs.

In the following, graphs are structures for the language $\mathcal{L} = \{E\}$; the two colored graphs we consider are structures for $\mathcal{L}' = \mathcal{L} \cup \{U\}$. A nontrivial structure $\mathbf{M}$ for the language $\mathcal{L}'$ is one where the interpretations of U and $\neg U$ are both nonempty. Throughout, in an abuse of notation, $U^{\mathbf{M}}$ and $\neg U^{\mathbf{M}}$ will refer to the graph structure (ie $\langle U^{\mathbf{M}}, E\rangle$) inherited from $\mathbf{M}$ on these subsets of its universe.

The situation with respect to the structured case is straightforward; we present the following without proof.

Theorem 5 *Fix n, m, $1 \leq n \leq \omega$, $1 \leq m \leq \omega$. Then:*

1. *For any n', $1 \leq n' < n$, there exists a unique (up to isomorphism) ultrahomogeneous $\mathcal{L}'$ structure $\mathbf{M}$ such that $\mathbf{M}|_{\mathcal{L}} \cong \mathbf{K}_n(\mathbf{N}_m)$ and $U^{\mathbf{M}} \cong \mathbf{K}_{n'}(\mathbf{N}_m)$, $\neg U^{\mathbf{M}} \cong \mathbf{K}_{n-n'}(\mathbf{N}_m)$. In the case $n = \omega$ and $m < \omega$, then possibly $n' = \omega$ and $\neg U^{\mathbf{M}} \cong \mathbf{K}_\omega(\mathbf{N}_m)$.*
2. *For any m', $1 \leq m' < m$, there is a unique (up to isomorphism) ultrahomogeneous $\mathcal{L}'$ structure $\mathbf{M}$ such that $\mathbf{M}|_{\mathcal{L}} \cong \mathbf{K}_n(\mathbf{N}_m)$ and $U^{\mathbf{M}} \cong \mathbf{K}_n(\mathbf{N}_{m'})$, $\neg U^{\mathbf{M}} \cong \mathbf{K}_n(\mathbf{N}_{m-m'})$. In the case $m = \omega$ and $n < \omega$, then possibly $m' = \omega$ and $\neg U^{\mathbf{M}} \cong \mathbf{K}_n(\mathbf{N}_\omega)$.*

3. *If $n = m = \omega$, there are ultrahomogeneous $\mathcal{L}'$ structures $\mathbf{M}_1 \ncong \mathbf{M}_2$ such that $\mathbf{M}_i|_{\mathcal{L}} \cong U^{\mathbf{M}_i} \cong \neg U^{\mathbf{M}_i} \cong \mathbf{K}_\omega(\mathbf{N}_\omega)$, $i = 1, 2$, where $\mathbf{M}_1$ is the limiting case of (1) above, $\mathbf{M}_2$ is the limiting case of (2) above. That is, for every $a \in U^{\mathbf{M}_1}$ and every $b \in \neg U^{\mathbf{M}_1}$, $\mathbf{M}_1 \models E(a, b)$ while for every $a \in U^{\mathbf{M}_2}$, there is a $b \in \neg U^{\mathbf{M}_2}$ such that $\mathbf{M}_2 \models \neg E(a, b)$.*
4. *Given any nontrivial ultrahomogeneous $\mathcal{L}'$ structure $\mathbf{N}$ expanding $\mathbf{K}_n(\mathbf{N}_m)$ there is an $\mathbf{M}$ such that (up to interchanging $U^{\mathbf{M}}$ and $\neg U^{\mathbf{M}}$) $\mathbf{N} \cong \mathbf{M}$, where $\mathbf{M}$ is as in (1), (2), or (3) above.*

We have a similar statement for the dual of $\mathbf{K}_n(\mathbf{N}_m)$, $\mathbf{N}_m(\mathbf{K}_n)$.

The main theorem concerns the situation in the unstructured case.

Theorem 6 *Fix n, $2 \leq n < \omega$. Then:*

1. *For any m, $1 \leq m \leq n$, there exists a unique (up to isomorphism) ultrahomogeneous $\mathcal{L}'$ structure $\mathbf{M}$ such that $\mathbf{M}|_{\mathcal{L}} \cong \mathbf{G}_n$, $U^{\mathbf{M}} \cong \mathbf{G}_m$, and $\neg U^{\mathbf{M}} \cong \mathbf{G}_n$.*
2. *For any nontrivial ultrahomogeneous $\mathcal{L}'$ structure $\mathbf{N}$ expanding $\mathbf{G}_n$ there is an m such that (up to interchanging the interpretations of U and $\neg U$) $\mathbf{N} \cong \mathbf{M}$, where $\mathbf{M}$ is as described in (1) above.*

A similar statement applies for the dual, $\widetilde{\mathbf{G}}_n$.

Theorem 6 can be extended to the cases $n = 1$ and $n = \omega$, where the underlying structure is respectively an infinite null or random graph. If $n = 1$ we have $\mathbf{G}_1$, an infinite null graph. Since there really is no graph structure, $U^{\mathbf{M}}$ could be any subset of $\mathbf{M}$. If $n = \omega$, then to the possibilities described in Theorem 6, there are the additional cases: $U^{\mathbf{M}} \cong \mathbf{N}_l(\mathbf{K}_\omega)$, (symmetrically, $\mathbf{K}_l(\mathbf{N}_\omega)$) for $2 \leq l \leq \omega$.

Theorem 7

1. *For any m, $1 \leq m \leq \omega$, there exists a unique ultrahomogeneous $\mathcal{L}'$ structure $\mathbf{M}$ such that $\mathbf{M}|_{\mathcal{L}} \cong \mathbf{G}_\omega$, $\neg U^{\mathbf{M}} \cong \mathbf{G}_\omega$ and exactly one of the following four cases holds: (i) $U^{\mathbf{M}} \cong \mathbf{G}_m$, or (ii) $U^{\mathbf{M}} \cong \widetilde{\mathbf{G}}_m$, or (iii) $U^{\mathbf{M}} \cong \mathbf{N}_m(\mathbf{K}_\omega)$, or (iv) $U^{\mathbf{M}} \cong \mathbf{K}_m(\mathbf{N}_\omega)$.*
2. *For any nontrivial ultrahomogeneous $\mathcal{L}'$ structure $\mathbf{N}$ expanding $\mathbf{G}_n$ there is an m such that (up to interchanging the interpretations of U and $\neg U$) $\mathbf{N} \cong \mathbf{M}$, where $\mathbf{M}$ is as described in (1) above.*

We devote the remainder of the paper to a proof of Theorem 6, followed by a brief indication of how Theorem 7 may be proven using similar techniques.

The proof of Theorem 6 is comprised of three parts.

(A) The ultrahomogeneous $\mathcal{L}'$ structures as described in Theorem 6 exist.

(B) No other interpretations of $U^{\mathbf{M}}$ are possible.

(C) The uniqueness of such expansions.

Begin with $\mathbf{G}_n$ and divide it into two pieces, one of which looks like $\mathbf{G}_n$ again and the other of which looks like $\mathbf{G}_m$. *A priori*, it would seem possible to do this in several ways, differing in the edge structure between the two distinguished subgraphs. In fact it is not; every graph that "should" occur across the boundary does occur. In the case where $m < n$, this is relatively easy to show. The difficulty lies in the $m = n$ case.

For part (A) it is easy to see $\mathbf{M}$ as described exists. Fix n and m, and let $\mathcal{K}_{n,m}$ be the class of all finite $\mathcal{L}'$ structures $\mathbf{C}$ such that $\mathbf{C}|_{\{E\}} \in \Sigma_n$ and $U^{\mathbf{C}} \in \Sigma_m$. Recall that Σ_n and Σ_m are the coherent classes generating $\mathbf{G}_n$ and $\mathbf{G}_m$ respectively. Then $\mathcal{K}_{n,m}$ is coherent, noting in particular that free amalgamation holds.

2 The possibilities for U^M In this section we address part (B) of the proof of Theorem 6. The possibilities for the interpretation of $U^{\mathbf{M}}$ that are consistent with $\mathbf{G}_n$ and must be eliminated are (1) $U^{\mathbf{M}}$ finite, (2) $U^{\mathbf{M}} \cong \mathbf{N}_\omega(\mathbf{K}_l)$, $l \leq n$, and (3) $U^{\mathbf{M}} \cong \mathbf{K}_l(\mathbf{N}_\omega)$, $l \leq n$.

Case 1: $U^{\mathbf{M}}$ is finite. Then it is possible to find $a \notin U^{\mathbf{M}}$ such that $\neg E(a,u)$ all $u \in U^{\mathbf{M}}$, and to find $a' \notin U^{\mathbf{M}}$ such that $E(a', u_0)$ some $u_0 \in U^{\mathbf{M}}$. But $a \cong a'$ (as finite $\mathcal{L}'$ structures) yet this isomorphism does not extend to an automorphism of $\mathbf{M}$.

Case 2: $U^{\mathbf{M}} \cong \mathbf{N}_\omega(\mathbf{K}_l)$. As in the first case, find $a \in \neg U^{\mathbf{M}}$ such that there is $K_l \in U^{\mathbf{M}}$ with no edges to a, and another copy of $K_l \in U^{\mathbf{M}}$ with exactly one edge to a. As elements of any coherent class that would have such an $\mathbf{M}$ (with $U^{\mathbf{M}} \cong \mathbf{N}_\omega(\mathbf{K}_l)$) as generic, these two versions of aK_l should satisfy the amalgamation property. But amalgamation fails of the two over aK_{l-1}, leaving on one hand a vertex of K_l with no edge to a outside of the base, and on the other hand the vertex with the edge to a outside (see Figure 1).

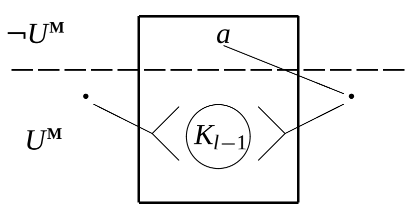

Figure 1: $\mathbf{N}_\omega(\mathbf{K}_l)$

Case 3: $U^{\mathbf{M}} \cong \mathbf{K}_l(\mathbf{N}_\omega)$. Let $a_0a_1b_1K_{n-1}$ be the graph with edges $E(a_0,a_1)$, $E(a_0,b_1)$, and $E(b_1,c)$ all $c \in K_{n-1}$. We can embed this into $\mathbf{M}|_{\mathcal{L}}$ with a_0, a_1 and $b_1 \in U^{\mathbf{M}}$ (hence a_1 and b_1 are in the same copy of N_ω) forcing necessarily $K_{n-1} \in \neg U^{\mathbf{M}}$. Similarly, find $a_0b_0a_1K_{n-1}$ where $E(a_1,a_0)$, $E(a_1,b_0)$, and $E(b_0,c)$ all $c \in K_{n-1}$. Embed again so that a_0, a_1 and $b_0 \in U^{\mathbf{M}}$ and $K_{n-1} \in \neg U^{\mathbf{M}}$. Thus we should be able to amalgamate these two structures over $a_0a_1K_{n-1}$. We obtain a contradiction as there must be an edge between b_0 and b_1, yielding a copy of K_{n+1} (see Figure 2).

This exhausts the possibilities for $U^{\mathbf{M}}$ other than as allowed by Theorem 6.

3 Uniqueness In this section we address part (C) of the proof of Theorem 6, the uniqueness of these expansions.

Fix n, $2 \leq n < \omega$. Assume $\mathbf{M}$ is an ultrahomogeneous $\mathcal{L}'$ structure such that $\mathbf{M}|_{\mathcal{L}}$ is isomorphic to $\mathbf{G}_n$, $U^{\mathbf{M}} \cong \mathbf{G}_m$, and $\neg U^{\mathbf{M}} \cong \mathbf{G}_n$. In the following, AB, A, and B are

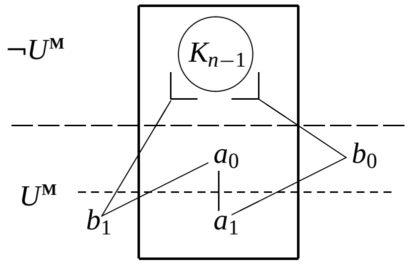

Figure 2: $\mathbf{K}_l(\mathbf{N}_\omega)$

finite graphs. Here AB is a graph comprised of the subgraphs A and B, possibly with edges between the two parts.

First, suppose $m < n$. Let $\mathbf{M}$ be as described above. It suffices to show for any finite graph AB, where $B \hookrightarrow \mathbf{G}_m$ and $AB \hookrightarrow \mathbf{G}_n$, that $A \hookrightarrow \neg U^{\mathbf{M}}$, $B \hookrightarrow U^{\mathbf{M}}$, as this uniquely describes the coherent class generating $\mathbf{M}$. To each vertex of A, attach a disjoint copy of K_m (so that with the vertex of A we have a copy of K_{m+1}), making sure any two of the $|A|$ copies of K_m are completely unconnected. Then the entire new graph is still compatible with $\mathbf{G}_n$, so embeds into $\mathbf{M}|_{\mathcal{L}}$, and we may assume B and all new copies of K_m embed into $U^{\mathbf{M}}$. This forces all of A to be in $\neg U^{\mathbf{M}}$ as desired; we have Theorem 6 for $m < n$.

Hence for the remainder of the proof we may assume $m = n$. Then we have $\mathbf{M}|_{\mathcal{L}} \cong U^{\mathbf{M}} \cong \neg U^{\mathbf{M}} \cong \mathbf{G}_n$. Again we must show for an arbitrary graph AB where $AB \hookrightarrow \mathbf{G}_n$, that $AB \hookrightarrow \mathbf{M}$, with $A \hookrightarrow \neg U^{\mathbf{M}}$ and $B \hookrightarrow U^{\mathbf{M}}$. Whenever some graph CD embeds into $\mathbf{M}$ in this manner (with $C \hookrightarrow \neg U^{\mathbf{M}}$ and $D \hookrightarrow U^{\mathbf{M}}$) we say CD embeds *properly*. The proof is a double induction on the cardinalities of A and B.

Fix a graph AB such that $AB \hookrightarrow \mathbf{G}_n$. Assume:

(I) For *any* finite graph CB^* with $|B^*| < |B|$, such that the graph $CB^* \hookrightarrow \mathbf{G}_n$, we have CB^* embeds properly.

(II) For any graph A^*B, with $|A^*| < |A|$ such that $A^*B \hookrightarrow \mathbf{G}_n$, A^*B embeds properly.

We wish to show that AB embeds properly.

If either A or B is empty we are done. Suppose then that AB is simply ab, and that $\neg E(a, b)$. Choose any $a \in \neg U^{\mathbf{M}}$, $b \in U^{\mathbf{M}}$ and we may suppose $E(a, b)$. Now $tp(x/ab)$ determined by $\neg E(x, a) \wedge \neg E(x, b)$ is consistent with $\mathbf{G}_n$, hence is realized in $\mathbf{M}|_{\mathcal{L}}$ by some c. If $c \in \neg U^{\mathbf{M}}$, then $cb \cong ab$ and is properly embedded. If $c \in U^{\mathbf{M}}$, then $ac \cong ab$ and is properly embedded. In the alternative where $E(a, b)$ holds, a similar argument will suffice.

Now assume $|A| > 1$ and that B has but one vertex b. Choose a vertex $a \in A$ arbitrarily. Let $A_a c A_b$ be a graph such that: $A_a \cong A \cong A_b$; for all vertices $d \in A_a$ and $e \in A_b$, $\neg E(d, e)$ holds; $E(d, c)$ holds if and only if $E(d', a)$ holds in A (d' the image of d in A); $E(e, c)$ holds if and only if $E(e', b)$ holds in AB (e' the image of e in A). Then $A_a c A_b \hookrightarrow \mathbf{G}_n$ and by (I), $A_a c A_b \hookrightarrow \neg U^{\mathbf{M}}$. Let $A_a^* b \cong A \backslash \{a\} B$. By (II) then $A_a^* b \hookrightarrow \mathbf{M}$, with $A_a^* \hookrightarrow \neg U^{\mathbf{M}}$ and $b \hookrightarrow U^{\mathbf{M}}$. Furthermore we may assume A_a^* embeds into the copy of A_a in the obvious way. Note that we now have $A_a^* b c A_b$ embedded in $\mathbf{M}$, but we do not know the edge relation between b and cA_b. Now $tp(x/A_a^*bcA_b)$ where x "looks like" b to A_b, like a to A_a^*b, and has no edge to c is consistent with $\mathbf{G}_n$, hence realized in $\mathbf{M}|_{\mathcal{L}}$ by f, say. If $f \in \neg U^{\mathbf{M}}$ then $A_a^* fb \cong AB$, properly embedded, and if $f \in U^{\mathbf{M}}$ then $A_a f \cong AB$, properly embedded again (f playing the role of a in the first case, b in the second). Hence AB embeds into $\mathbf{M}$ properly.

We may now assume that $|A| > 1$, $|B| > 1$, and (I) and (II) hold for the graph AB. The strategy to be followed is similar to that used above in the last case: choose a vertex $a \in A$ and remove it, calling the remainder A^*. Then by (II) A^*B embeds properly into $\mathbf{M}$, to a copy of A^*B we will call $A_a^*B_a$. Choose a $b \in B$ and remove it, calling the remainder B^*. Then by (I), AB^* embeds properly into $\mathbf{M}$, without loss of generality to a copy $A_bB_b^*$ disjoint from $A_a^*B_a$ chosen earlier. So now there are "nearly" two copies of AB embedded properly: $(A_a^*B_a)(A_bB_b^*)$. Suppose the $\mathcal{L}$-type $p(x) = tp(x/A_a^*B_aA_bB_b^*)$ is consistent with $\mathbf{G}_n$, where $p|_{A_a^*B_a} = tp(a/A^*B)$ and $p|_{A_bB_b^*} = tp(b/AB^*)$. Then as the underlying structure is $\mathbf{G}_n$, $p(x)$ is realized in $\mathbf{M}|_{\mathcal{L}}$. If the realization c is in $\neg U^{\mathbf{M}}$, then $(A_a^*c)B_a$ is a copy of AB properly embedded in $\mathbf{M}$. If the realization c is in $U^{\mathbf{M}}$, then $A_b(B_b^*c)$ is a copy of AB properly embedded.

All that remains, then, is to find an embedding of $(A_a^*B_a)(A_bB_b^*)$ so that the type $p(x)$ is consistent (with $\mathbf{G}_n$). The problem of course is that there is no reason that $p(x)$ necessarily should be consistent. That is, there could be a copy of K_n inside $A_a^*B_b^*$, say, and $p(x)$ might demand that x be attached to it, forming a K_{n+1}. The trick is to embed $(A_a^*B_a)(A_bB_b^*)$ very carefully, maintaining control over the edges between $(A_a^*A_b)$ and $(A_bB_b^*)$ in such a manner that the consistency of $p(x)$ with $\mathbf{G}_n$ is assured.

Claim 8 *It is possible to embed $(A_a^*B_a)(A_bB_b^*)$ into $\mathbf{M}$ so that the type $p(x)$ is consistent.*

Proof of Claim: The proof splits into three cases.

Case 1: There are $a \in A$, $b_1 \in B$ such that $\neg E(a, b_1)$ and $B \ncong K_l$, any $l \leq n$.

Fix $a \in A$, $b, b_1, b_2 \in B$ such that $\neg E(a, b_1)$, $\neg E(b, b_2)$. In the following, as described above, there will be two copies of AB: an 'a' copy and a 'b' copy. The individual parts of each copy will be distinguished by A_a, B_a, and A_b, B_b. When the vertex 'a' is referred to it will be implicitly understood to refer to the copy of this vertex in A_a. Similarly the vertex 'b' will refer to the copy of b in B_b (though there are copies of a and b in A_b and B_a respectively, it will never be necessary to make use of them). In the same vein, 'b_1' will be considered to be in B_a and 'b_2' will be considered to be in B_b. Let

$$A_a^+ = \{c \in A_a : E(a, c)\} \qquad A_a^- = \{c \in A_a : \neg E(a, c)\},$$

and then $A_a^* = A_a \setminus \{a\} = A_a^+ A_a^-$. Similarly,

$$B_a^+ = \{c \in B_a : E(a, c)\} \qquad B_a^- = \{c \in B_a : \neg E(a, c)\}.$$

Since $b_1 \in B_a$, let $B_a^{-*} = B_a^- \setminus \{b_1\}$.

Continuing in the same fashion with respect to A_b and B_b, define

$$A_b^+ = \{c \in A_b : E(c, b)\} \qquad A_b^- = \{c \in A_b : \neg E(c, b)\}$$

$$B_b^+ = \{c \in B_b : E(c, b)\} \qquad B_b^- = \{c \in B_b : \neg E(c, b)\}$$

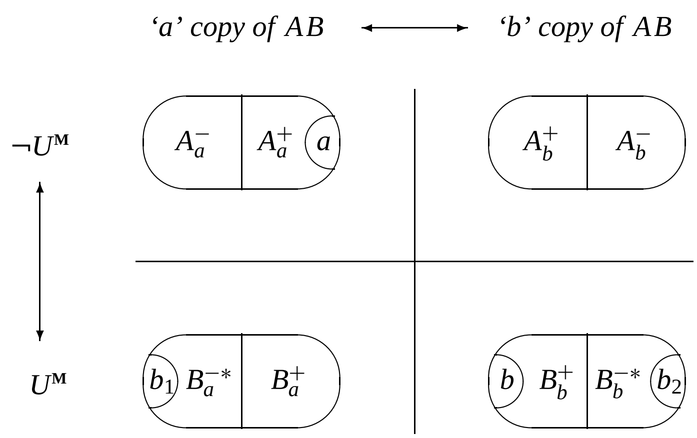

Figure 3: Basic Setup

and then $B_b^* = B_b \setminus \{b\} = B_b^+ B_b^-$. Finally, remembering that $b_2 \in B_b^-$, define $B_b^{**} = B_b^* \setminus \{b_2\} = B_b \setminus \{b,b_2\}$ and $B_b^{-*} = B_b^- \setminus \{b_2\}$ (see Figure 3).

The embedding is made in 5 steps.

Step 1: In $U^{\mathbf{M}}$, find a copy of B_a^+ and a copy of B_b^{**} with no edges between them. Find a new vertex c with edges to every vertex of B_a^+ that has no edges to B_b^{**} (see Figure 4).

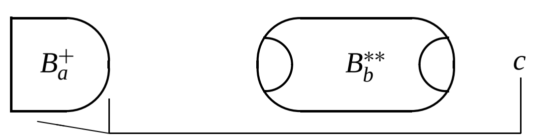

Figure 4: Step 1

Step 2: Then $|B_b^{**}c| < |B|$, hence by inductive hypothesis (I) it is possible to find the following in $\neg U^{\mathbf{M}}$: a copy of A_b, chosen so that A_b is connected to B_b^{**} as appropriate, (i.e., so $A_b B_b^{**} \cong AB \backslash \{b, b_2\}$) and such that c has edges to all vertices of A_b^+ yet has no edges to any vertex of A_b^-. Resolving B_b^{**} into the two components B_b^{-*} and B_b^+, find in $\neg U^{\mathbf{M}}$ a vertex d which is not connected to A_b, to c, or to B_b^{-*} but is connected to all of B_b^+. This latter is consistent (with respect to $\mathbf{G}_n$), as in the worst case d is connected to all of B_a^+, but then $tp(d/B_a^+) = tp(a/B_a^+)$ (see Figure 5). The edges between d and B_a^+ and between A_b and B_a^+ are unknown.

Step 3: It is now possible to find B_a^{-*} in $U^{\mathbf{M}}$ so that $B_a^{-*} B_a^+ \cong B_a \setminus \{b_1\} = B_a^*$. In $\neg U^{\mathbf{M}}$ find A_a so that $A_a B_a^* \cong AB \setminus \{b_1\}$, with A_a not connected to A_b in any way. Recall that A_a is composed of three parts, A_a^-, A_a^+, and a itself. In addition, require the choice of A_a to be made so that there are no edges from A_a^- and a to d, while A_a^+ is connected in its entirety to d. Finally, it is possible to insist that B_a^{-*} not be connected to d or to A_b (see Figure 6). Again, the edges between A_a and $B_b^{**}c$ are unknown.

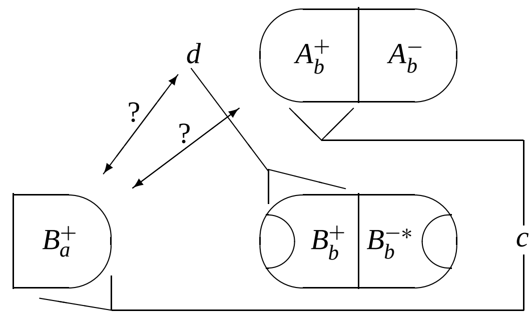

Figure 5: Step 2

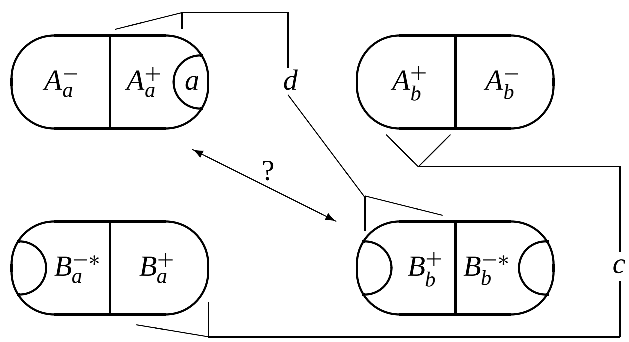

Figure 6: Step 3

Step 4: In $U^{\mathbf{M}}$ find a copy of b_1 to complete $B_a^{-*}B_a^+$ to form B_a, connected to A_a^* as appropriate. Presumably $E(b_1, a)$, as otherwise $A_a B_a \cong AB$ showing AB embeds properly and the proof is complete. In what manner b_1 is connected to d, A_b, B_b^{**}, or c is irrelevant.

Step 5: In $U^{\mathbf{M}}$ find a copy of b_2 to complete B_b^{**} to B_b^*, connected to A_b as appropriate and not connected to A_a or to d. Here the edge relations between b_2 and c, B_a are unknown (see Figure 7).

The basic embedding is now complete.

Claim 9 *$p(x) = tp(x/A_a^* B_a A_b B_b^* dc)$ is consistent with $\mathbf{G}_n$, where*

$$tp(x/A_a^* B_a) = tp(a/AB\backslash\{a\}) \qquad tp(x/A_b B_b^*) = tp(b/AB\backslash\{b\})$$

and $\neg E(x,a)$, $\neg E(x,d)$, $\neg E(x,c)$.

It suffices to show $K_n \not\hookrightarrow A_a^+ B_a^+ A_b^+ B_b^+$. Notice that $K_n \not\hookrightarrow A_a^+ B_a^+$ and $K_n \not\hookrightarrow A_b^+ B_b^+$, as otherwise $AB \not\hookrightarrow \mathbf{G}_n$. Also by choice of B_a^+ and B_b^+, $K_n \not\hookrightarrow B_a^+ B_b^+$. By construction, $K_n \not\hookrightarrow A_a^+ A_b^+$. If $K_n \hookrightarrow A_a^+ B_b^+$, then d together with this K_n must already form a K_{n+1}, a contradiction. If $K_n \hookrightarrow B_a^+ A_b^+$, then c together with this K_n must form a K_{n+1}, again a contradiction. Since A_a^+ and A_b^+ have no edges between them, and B_a^+ and B_b^+ also have no edges between them, no K_n can embed into any three (or all four) pieces of $A_a^+ B_a^+ A_b^+ B_b^+$. So Claim 9 is shown.

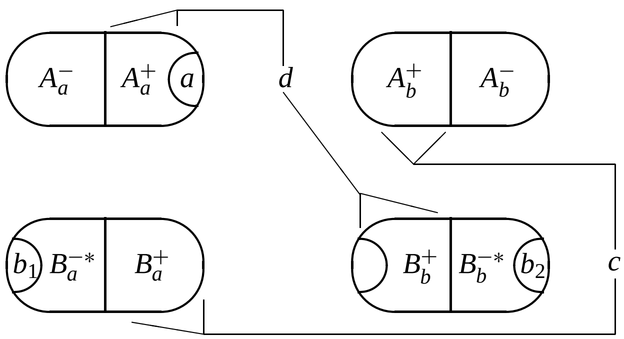

Figure 7: Step 5

The type $p(x)$ is consistent with $\mathbf{G}_n$, hence realized by e say. If e is in $U^{\mathbf{M}}$, then $A_b(B_b^* e) \cong AB$, and if e is in $\neg U^{\mathbf{M}}$ then $(A_a^* e)B_a \cong AB$. In either case, AB embeds properly into $\mathbf{M}$, completing Case 1.

Case 2: There is $a \in A$, $b_1 \in B$ such that $\neg E(a, b_1)$, and $B \cong K_l$, $l \leq n$.

Subcase 1: $B_a^+ \cong K_j$, some $j \leq n-2$. Then the preceding construction will continue to go through, with the exception of requiring the b_2 chosen in Step 5 to be connected to d. All of B_a^+ may be connected to b_2, but this will form only a K_{n-1} at worst (when $j = n-2$).

Subcase 2: $B_a^+ \cong K_{n-1}$. Then $AB\backslash\{b_1\}$ can be embedded properly, following which $AB\backslash\{a\}$ can be embedded over it. The only possible problem would be if $E(b_1, a)$. But then $B_a^+ b_1 a \cong K_{n+1}$, a contradiction. Hence $\neg E(b_1, a)$ and AB embeds properly (see Figure 8).

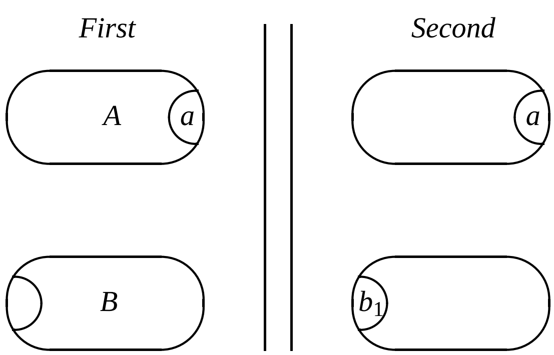

Figure 8: Case 2.2

Case 3: For every $a \in A$, and every $b \in B$, $E(a, b)$.

Subcase 1: $A \ncong K_l$ any $l \leq n$. Let a, a' in A be such that $\neg E(a, a')$. Embed $A_a A_b B_b^*$ properly (that is, B_b^* in $U^{\mathbf{M}}$ and $A_a A_b$ in $\neg U^{\mathbf{M}}$) so that $A_b B_b^*$ has edges as appropriate, with a' having edges to all of A_b^+ and B_b^+. Otherwise, enforce no edges.

Then embed $A_a^* B_a$ properly, overlapping A_a^* with the copy of A_a chosen earlier. The concern is among $B_a^+ B_b^+ A_b^+$ now, but if a K_n embeds then a' with the K_n forms a K_{n+1}. Hence K_n does not embed in $B_a^+ B_b^+ A_b^+$ (see Figure 9).

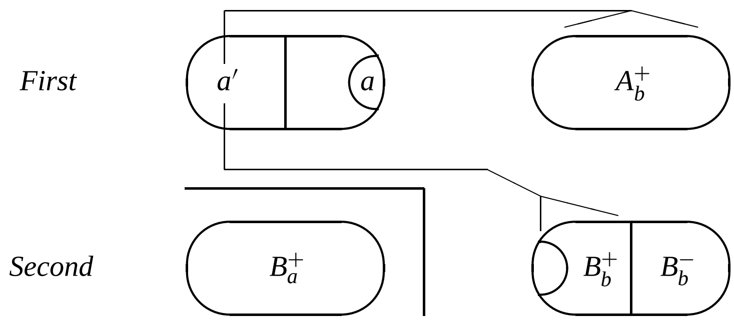

Figure 9: Case 3.1

Subcase 2: $A \cong K_l$, $l \leq n-1$ (so $A_a^* = A_a^+$).

Subsubcase 1: $B \ncong K_l$, any $l \leq n$. Fix $b, b_2 \in B$ such that $\neg E(b, b_2)$. Find B_a and B_b^* in $U^{\mathbf{M}}$ with no edges between them, except that b_2 has edges to all of B_a^+ ($= B_a$ now). Attach A_a^+ to B_a, then A_b to B_b^*, enforcing no edges between A_a^+ and A_b. Then no K_n embeds in $A_a^+ B_b^*$ (as these together are not large enough), and if K_n embeds in $B_a^+ A_b^+$, then as b_2 is connected to all of A_b, $K_n b_2$ forms K_{n+1} (see Figure 10).

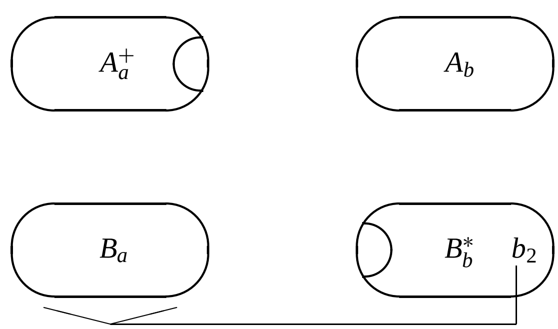

Figure 10: Case 3.2.1

Subsubcase 2: $B \cong K_j$. Thus $AB \cong K_l$, where $l \leq n$. Fix $a \in A$, $b \in B$. Embed AB^* properly, then overlap A^*B. If $E(a, b)$ then the proof is complete. If $\neg E(a, b)$, then the $tp(x/A^* a B^* b)$, where x has edges to all of $A^* a B^* b$, is consistent hence realized by c, say. If c is in $U^{\mathbf{M}}$, then $A^* a B^* c \cong AB$ and if c is in $\neg U^{\mathbf{M}}$, then $A^* c B^* b \cong AB$ (see Figure 11).

This concludes the proof of Claim 8, and thus the proof of Theorem 6.

□

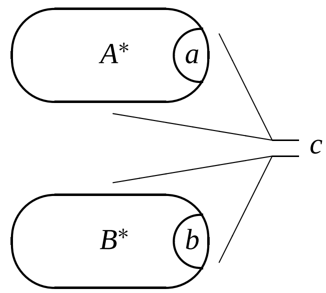

Figure 11: Case 3.2.2

Theorem 7, the extension of Theorem 6, is easy to see. Existence follows for these new cases by appropriately redefining the coherent class $\mathcal{K}_{n,m}$ given in Section 2 above, and redefining amalgamation on the $\mathbf{N}_l(\mathbf{K}_\omega)$ portion of the coherent classes in the obvious manner. Since the basic structure is the random graph, this is never a problem.

The question of the consistency of $tp(x/A_a^* B_a A_b B_b^*)$ is moot; the underlying structure being the random graph and so embedding all finite graphs. Uniqueness in the new cases is easily seen. Let AB be an arbitrary finite graph where $AB \hookrightarrow \mathbf{G}_\omega$, and without loss of generality suppose $B \hookrightarrow \mathbf{N}_m(\mathbf{K}_\omega)$. Then B consists of at most l complete components. Fix one component of cardinality n say, and add two more vertices and edges to it so as to form K_{n+2}. Add edges from one of these new vertices to all of A, none from the other to A. Then the entire graph embeds into $\mathbf{G}_\omega$, and without loss of generality we may assume the enlarged B embeds inside $\mathbf{N}_m(\mathbf{K}_\omega)$, forcing A to embed in $\neg U^{\mathbf{M}}$.

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