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Urquhart's C with Intuitionistic Negation: Dummett's LC without the Contraction Axiom

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Abstract This paper offers a particular intuitionistic negation completion of Urquhart's system **C** resulting in a super-intuitionistic contractionless propositional logic equivalent to Dummett's **LC** without contraction.

1 Introduction Ono and Komori [3] is a general study of propositional contractionless logic, i.e., propositional logics without the rule

$$\frac{\Gamma,\ \alpha, \alpha,\ \Delta \to \chi}{\Gamma,\ \alpha,\ \Delta \to \chi}$$

in a Gentzen-type formulation, or without the axiom

$$[A \to (A \to B)] \to (A \to B)$$

in a Hilbert-type one.

In the "concluding remarks" of their paper, Ono and Komori encourage the study of intermediate logics (i.e., logics between the intuitionistic and the classical logic) without the contraction principle. Moreover, in Urquhart [5] a most interesting positive propositional logic **C** is introduced, which can intuitively be described as the positive fragment of Dummett's **LC** (see [1]) minus the contraction axiom. There are essentially two possibilities for extending **C** with a negation connective. The first one, a kind of "semiclassical negation," gives as a result Łukasiewicz's infinite-valued logic **Lw**. The second, a kind of semi-intuitionistic negation, generates a logic **CI**, which is, from a intuitive point of view, Dummett's **LC** without the contraction or reductio (i.e., $(A \to \neg A) \to \neg A$) axioms (a complete semantics for **CI** is offered in Méndez and Salto [2]).

But there is still a third possibility left, namely adding the reductio axiom to **CI**. The resulting system (let us refer to it by **CIr**) is, intuitively, Urquhart's **C** with intuitionistic negation or, alternatively, Dummett's **LC** without the contraction axiom.

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So **CIr** is a prominent component in the class of super-intuitionistic logics Ono and Komori refer to.

In what follows, we shall slightly modify the standard techniques of Routley-Meyer type semantics (see [4]) so as to deal with **CIr**-nonrelevant consistent theories. We shall introduce negation as a primitive connective, but it would be easy to define it by means of a falsity constant (see [2]). In the development of these semantics a point of interest is, we think, to show that the contraction axiom is not derivable from **CIr**. So in §5 the reader can find the simplest **CIr**-model falsifying the contraction axiom. The results of [2] are not presupposed in this paper as far as **CIr** is concerned.

***2 Urquhart's* C *with semi-intuitionistic negation: the system* CI** Urquhart's **C** can be axiomatized as follows.

Axioms:

A1. $(B \to C) \to [(A \to B) \to (A \to C)]$
A2. $[A \to (B \to C)] \to [B \to (A \to C)]$
A3. $(A \wedge B) \to A \qquad (A \wedge B) \to B$
A4. $A \to [B \to (A \wedge B)]$
A5. $A \to (A \vee B) \qquad B \to (A \vee B)$
A6. $[(A \to B) \wedge (A \to C)] \to [(A \vee B) \to C]$
A7. $(A \to B) \vee (B \to A)$

Rule:

modus ponens: If $\vdash A$ and $\vdash A \to B$, then $\vdash B$.

In order to formulate **CI** we add to the sentential language of **C** the unary connective $\neg$ (negation) and the following axioms.

A8. $(A \to \neg B) \to (B \to \neg A)$
A9. $A \to (\neg A \to B)$

***3 Semantics for* CI** A **CI**-model is the structure $\langle K, R, \models \rangle$ where K is a set and R is a ternary relation on K subject to the following definitions and postulates for all $a, b, c, d \in K$ with quantifiers ranging over K.

d1. $a \leq b =_{def} \exists x\, Rxab$
d2. $R^2abcd =_{def} \exists x[Rabx \text{ and } Rxcd]$
P1. $a \leq a$
P2. $a \leq b$ and $Rbcd \Rightarrow Racd$
P3. $R^2abcd \Rightarrow \exists x[Rbcx \text{ and } Raxd]$
P4. $Rabc \Rightarrow Rbac$
P5. $Rabc$ and $Rade \Rightarrow b \leq e$ or $d \leq c$.

Finally, $\models$ is a valuation relation from K to the sentences of **C** satisfying the following conditions for all $a \in K$:

1. For each propositional variable p and $a, b \in K$, $a \models p$ and $a \leq b \Rightarrow b \models p$;
2. $a \models A \wedge B$ iff $a \models A$ and $a \models B$;

3. $a \models A \vee B$ iff $a \models A$ or $a \models B$;
4. $a \models A \to B$ iff for all $b, c \in K$, $Rabc$ and $b \models A \Rightarrow c \models B$;
5. $a \models \neg A$ iff for all $b, c \in K$ not-$Rabc$ or $b \not\models A$.

A formula is *valid* iff $a \models A$ for all $a \in K$ in all models. We have shown in [2] that A is a theorem of **CI** if A is valid.

***4 Adding the reductio axiom to* CI*: the system* CIr** To formulate **CIr** we add to **CI** the reductio axiom:

A10. $(A \to \neg A) \to \neg A$.

Now, we note:

1. **CIr** and the Łukasiewicz's n-valued logic are independent systems: **A10** is not a theorem of **Ln**; **CIr** does not count with nonintuitionistic principles such as, e.g., strong De Morgan Laws.
2. **CIr** clearly includes **CI** (which is, of course, included in **Ln**) but for purposes of comparison only (see §5 below), we describe a **CI**-model falsifying the reductio axiom. Consider a **CI**-model $\langle K, R, \models \rangle$ with $K = \{a, b\}$ and let $Rabb$, $Raaa$, but not-$Raba$, not-$Rbba$, not-$Rbbb$; $b \models A$, but $a \not\models A$. It is clear that $a \not\models \neg A$, and it is not difficult to show that $a \models A \to \neg A$. Thus, $a \not\models (A \to \neg A) \to \neg A$, and so **A10** is not valid.
3. As shown in §5 below, the contraction axiom is not a theorem of **CIr**.

5 Semantics for* CIr *together with a model falsifying the contraction axiom

A **CIr**-model is just like a **CI**-model but with the addition of the postulate:

P6. $Rabc \Rightarrow \exists x Rcbx$.

Now, semantic consistency is easy. As an illustration, we show the validity of **A10**, for which we use the equivalence between the propositions "**A10** is valid" and "if $a \models A \to \neg A$, then $a \models \neg A$ for all $a \in K$ in all models" (see [4]). So suppose for reductio a model with some $a \in K$ such that $a \models A \to \neg A$ and $a \not\models \neg A$. By clause (5) there are some $b, c \in K$ such that $Rabc$ and $b \models A$. Thus, $c \models \neg A$ (since $a \models A \to \neg A$, $Rabc$, $b \models A$), which contradicts $Rcbd$ (since $Rabc$, P6) and $b \models A$ (by clause (5)). Therefore, **A10** is valid.

Now, we provide a **CIr**-model falsifying the contraction axiom. Consider a **CIr**-model $\langle K, R, \models \rangle$ where $K = \{a, b, c, d\}$; $a \models A$, $b \models A$, $c \models A$, $d \not\models A$, $a \not\models B$, $b \not\models B$, $c \models B$ and $d \not\models B$; $Raab$, $Raac$, $Rada$, $Rabc$, $Racc$, $Radb$, $Radc$, $Rbac$, $Rbbc$, $Rbdb$, $Rbcc$, $Rbdc$, $Rccc$, $Rcac$, $Rcbc$, $Rcdc$, $Rdaa$, $Rdbb$, $Rddd$, $Rdab$, $Rdac$, $Rdda$, $Rdbc$, $Rdcc$, $Rddb$, and $Rddc$.

It is an easy but certainly tedious task to prove that P1–P6 are verified. It is no more difficult either to show that $a \models A \to (A \to B)$ (note that $b \models A \to B$ and $c \models A \to B$) and that $a \not\models A \to B$ ($Raab$, $a \models A$, $b \not\models B$). So $[A \to (A \to B)] \to (A \to B)$ is not true in this model, which is the smallest falsifying the contraction axiom, since there is no model with $K = \{a, b, c\}$ falsifying the principle under consideration.

We finish this section by noting that the postulate P6′,

P6′. $\exists x\, Raax$ (for each $a \in K$), is equivalent to P6 in presence of the intuitionistic postulate.

6 *Completeness of* CIr We begin with some definitions and then we prove some previous lemmas.

A theory is a set of formulas of **CIr** closed under adjunction and provable entailment (that is, a is a *theory* if whenever $A, B \in a$, then $A \wedge B \in a$; if whenever $A \to B$ is a theorem and $A \in a$, then $B \in a$); a theory a is *null* iff no wff belongs to a; *prime* iff whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; *regular* if all theorems of **CIr** belongs to a; finally, a is *consistent* if a does not contain the negation of a theorem of **CIr**.

We now define the **CIr**-canonical structure as the pair $\langle K^c, R^c \rangle$ where K^c is the set of all nonnull prime consistent theories, and R^c is defined on K^c as follows: for all formulas A, B and $a, b, c \in K^c$, $Rabc$ iff if $A \to B \in a$ and $A \in a$, then $B \in c$.

Lemma 6.1 *If a is a nonnull theory, then a is regular.*

Proof: Suppose A is a theorem, and let $B \in a$. By the theorem $A \to (B \to A)$, $B \to A$ is a theorem. Then, $B \in a$. □

Lemma 6.2 *For any wff A and theory a, a is inconsistent iff $A \wedge \neg A \in a$.*

Proof: ($\Rightarrow$) Suppose a is inconsistent. Then, $\neg B \in a$ for some theorem B. By **A9**, $\neg B \to (A \wedge \neg A)$ is a theorem. Thus, $A \wedge \neg A \in a$. ($\Leftarrow$) Suppose $A \wedge \neg A \in a$. Given **CIr**, $\neg(A \wedge \neg A)$ and $(A \to \neg A) \to \neg A$ are interchangeable. So $\neg(A \wedge \neg A)$ and $(A \wedge \neg A) \to \neg B$ (with B a theorem) are theorems. Thus, $\neg B \in a$. □

Lemma 6.3 *If A is not provable in* **CIr**, *then there is a nonnull prime consistent theory T which does not contain A.*

Proof: **CIr** is a nonnull consistent theory; by Zorn's lemma, there is a maximal nonnull consistent theory T without A. If T is not prime, then $B \vee C \in T$, $B \notin T$, and $C \notin T$. Define $[T, B] = \{E | \exists D[D \in T \text{ and } (B \wedge D) \to E \in \mathbf{CIr}]\}$, $[T, C] = \{E | \exists D[D \in T \text{ and } (C \wedge D) \to E \in \mathbf{CIr}]\}$. It is easy to show that $[T, B]$ and $[T, C]$ are nonnull theories that strictly include T. By the maximality of T, there are three possible cases.

Case 1: $[T, B]$ and $[T, C]$ are inconsistent.
By definition and Lemma 6.2, $(B \wedge D) \to (E \wedge \neg E')$, $(C \wedge D') \to (E \wedge \neg E') \in$ **CIr** for some wffs E, E' and $D, D' \in T$. By elementary properties of $\wedge$, $\vee$, and $\neg$, $[(B \vee C) \wedge (D \wedge D')] \to (E \wedge \neg E) \in \mathbf{CIr}$. Then, $\neg(E \wedge \neg E) \to \neg[(B \vee C) \wedge (D \wedge D')] \in \mathbf{CIr}$ by contrapositition. But then $\neg[(B \vee C) \wedge (D \wedge D')] \in \mathbf{CIr}$. Now, since $\neg\neg[(B \vee C) \wedge (D \wedge D')] \in T$ (by $(B \vee C) \wedge (D \wedge D') \in \mathbf{CIr}$ and double negation), we conclude that T is inconsistent, which is impossible.

Case 2: $A \in [T, B]$ and $A \in [T, C]$.
By definition, $(B \wedge D) \to A$, $(C \wedge D') \to A \in \mathbf{CIr}$ for some $D, D' \in T$. Then, $[(B \vee C) \wedge (D \wedge D')] \to A \in \mathbf{CIr}$, hence $A \in T$, which is impossible.

Case 3: $[T, B]$ is inconsistent and $A \in [T, C]$, or $[T, C]$ is inconsistent and $A \in [T, B]$.
We consider the first alternative, the second being similar. By definition, $(B \wedge D) \to (E \wedge E')$, $(C \wedge D') \to A \in \mathbf{CIr}$ for some wffs E and $D, D' \in T$. Now, it is clear that $(B \wedge D) \to A \in \mathbf{CIr}$. So $A \in T$, which is impossible (as in Case (2) above).

Each of Cases (1), (2), and (3) is untenable. Therefore, T is prime, which ends the proof of Lemma 6.3. □

Lemma 6.4 *Let $\langle K^c, R^c\rangle$ be the canonical structure. For all $a, b \in K^c$, $a \leq b$ iff $a \subseteq b$.*

Proof: Suppose $a \leq b$. By definition, $Rxab$ for some $x \in K^c$. Since $A \to A \in x$, whenever $A \in a$ we have $A \in b$, i.e., $a \subseteq b$. Suppose now $a \subseteq b$. It is clear that $R\mathbf{CI}ab$ (because $R\mathbf{CI}aa$ and $a \subseteq b$). So $a \leq b$ by definition.

Next we prove that x can be extended to a prime nonnull consistent theory x' such that $Rx'ab$. Thus, consider the set of all nonnull consistent theories y such that $x \subseteq y$ and $Ryab$. By Zorn's Lemma, there is a maximal element x' in this set such that $x \subseteq x'$ and $Rx'ab$. If x' is not prime, then $A \vee B \in x'$, $A \notin x'$, $B \notin x'$ for some wffs A, B. Then, define the nonnull theories $[x', A], [x', B]$ that strictly include x' (cf. Lemma 6.3).

By the maximality of x', there are three possible cases.

Case 1: $[x', A]$ and $[x', B]$ are inconsistent.
Then x' is inconsistent (cf. Lemma 6.2).

Case 2: not-$R[x', A]ab$ and not-$R[x', B]ab$.
By definition, $(A \wedge E) \to (C \to D)$, $(B \wedge E') \to (C' \to D') \in \mathbf{CIr}$, $C, C' \in a$, $E, E' \in x$, $D \notin b$, $D' \notin b$ for some wffs C, C', E, E', D, D'. Hence, $[(A \vee B) \wedge (E \wedge E')] \to [(C \to D) \vee (C' \to D')] \in \mathbf{CIr}$ by elimination of disjunction and distribution. Then, $[(C \to D) \vee (C' \to D')] \in x'$ (since $(A \vee B) \wedge (E \wedge E') \in x'$), and so $(C \wedge C') \to (D \vee D') \in x'$. Thus, $D \vee D' \in b$ (since $Rx'ab, C \wedge C' \in a$). But b is prime. Therefore, $D \in b$ or $D' \in b$, contradicting our hypothesis.

Case 3: not-$R[x', A]ab$ and $[x', B]$ is inconsistent, or not-$R[x', B]ab$ and $[x', A]$ is inconsistent.
Suppose not-$R[x', A]ab$ and $[x', B]$ is inconsistent. By definition, $(A \wedge E) \to (C \to D)$, $(B \wedge E') \to (H \wedge \neg H) \in \mathbf{CI}$, $E, E' \in x'$, $C \in a$, $D \notin b$ for some wffs E, E', C, D, H. Now, it is clear that $(B \wedge E') \to (C \to D) \in \mathbf{CIr}$. So $C \to D \in x'$ as in Case (2) above, and thus $D \in b$ by $Rx'ab$, contradicting the hypothesis. The proof that not-$R[x', B]ab$ and $[x', A]$ is inconsistent leads also to contradiction is similar.

Each of Cases (1), (2), and (3) is untenable, therefore x' is prime, which ends the proof of Lemma 6.4. □

Lemma 6.5 *The canonical structure is indeed a model structure.*

Proof: We have to prove that the postulates P1–P6 hold in the canonical structure. Now, P1 and P2 are trivial by Lemma 6.4; P4 is easy using the theorem $A \to [(A \to B) \to B]$, and P5 is inmediate by **A7** and Lemma 6.4. Thus, it remains to be proved that P3 and P6 hold.

P3. $R^2abcd \Rightarrow \exists x[Rbcx$ and $Raxd]$.
Given $Raby$ and $Rycd$, we have to show that there is a prime nonnull consistent theory x' such that $Rbcx'$ and $Rax'd$. Thus, define the nonnull theory $x = \{B | \exists A[A \in c$ and $A \to B \in b]\}$. Now, $Rbcx$ is trivial and $Raxd$ easily follows from the hypothesis and **A1**. Next, we prove that x is consistent. Suppose it is not. Then, $B \wedge \neg B \in x$.

But, as $(B \wedge \neg B) \rightarrow \neg A$ (A is a theorem) is a theorem, $(B \wedge \neg B) \rightarrow \neg A \in a$, whence $\neg A \in d$ by $Raxd$, contradicting the consistency of d.

Consider now the set of all nonnull consistent theories y such that $x \subseteq y$ and $Rayd$. By Zorn's Lemma, there is a maximal element x' in this set such that $Rax'd$ and $Rbcx'$ ($Rbcx$ and $x \subseteq x'$). If x' is not prime, define the nonnull theories $[x', A]$, $[x', B]$ strictly including x'. By the maximality of x', there are three possibilities:

1. $[x', A]$ and $[x', B]$ are inconsistent;
2. not-$Ra[x', A]d$ and not-$Ra[x', B]d$;
3. not-$Ra[x', A]d$ and $[x', B]$ is inconsistent, or not-$Ra[x', B]d$ and $[x', A]$ is inconsistent.

As in the proof of Lemma 6.4, it can be shown that each one of these possibilities is impossible. Therefore, x' is a prime nonnull consistent theory, which ends the proof that P3 holds in the canonical model.

P6. $Rabc \Rightarrow \exists x Rcbx$.

Suppose $Rabc$. Define the nonnull theory $x = \{B | \exists A[A \in b$ and $A \rightarrow C \in c]\}$. It is clear that $Rcbx$. Thus, it remains to be proved how to extend x to a prime consistent theory. We begin by proving that x is consistent. Suppose it is not. Then, by definition, $B \rightarrow (A \wedge \neg A) \in C$, $B \in b$. Contraposing, $\neg(A \wedge \neg A) \rightarrow \neg B \in c$, and so $\neg B \in c$ (since $Rcxc$ by P1 and P4, and $\neg(A \wedge \neg A) \in x$ by $x \in K^c$; cf. Lemmas 6.1 and 6.2). Now, $B \rightarrow \neg(B \rightarrow B) \in a$ by $Rabc$ and $B \in b$. Contraposing, $\neg\neg(B \rightarrow B) \rightarrow \neg\neg B \in c$, and thus $\neg\neg B \in c$. Therefore, $\neg B \wedge \neg\neg B \in c$, contradicting the consistency of c.

Consider now the set of all nonnull consistent theories y such that $x \subseteq y$ and $Rcby$. By Zorn's lemma there is a maximal element x' such that $Rcbx'$. If x' is not prime, define, as in previous lemmas, the nonnull theories $[x', A]$, $[x', B]$ that strictly include x'. Now, we note that $Rcb[x', A]$ and $Rcb[x', B]$ trivially hold, since $Rcbx$ and $x' \subseteq [x', A], [x', B]$. So $[x', A]$ and $[x', B]$ are inconsistent by the maximality of x'. But if $[x', A]$ and $[x', B]$ are inconsistent, then x' is inconsistent (cf. Lemma 6.3), which is impossible. Therefore, x' is a prime nonnull consistent theory; this ends the proof that P6 holds in the canonical structure, and Lemma 6.5 is proved. □

Lemma 6.6 *Let $\langle K^c, R^c, \models^c \rangle$ be the canonical model where $\langle K^c, R^c \rangle$ is the canonical structure and $\models^c$ is a relation from K^c to the sentences of* **CIr** *such that for each wff A and $a \in K^c$, $a \models^c A$ iff $A \in a$. Then, the canonical model is indeed a model.*

Proof: We have to prove that the canonical $\models^c$ satisfies the conditions (1)–(5) of the valuation relation. Now, clauses (1)–(3) are trivial. It remains to prove clauses (4) and (5).

Clause (4): $a \models A \rightarrow B$ iff for all $b, c \in K^c$, if $Rabc$ and $b \models A$, then $c \models B$. Proof from left to right is simple. So suppose $a \not\models A \rightarrow B$. We show that there are $b', c' \in K^c$ such that $Rab'c'$, $b' \models A$ and $c' \models B$. Then, define $b = \{C | A \rightarrow C \in$ **CIr**$\}$, $c = \{C | \exists D[D \in b$ and $D \rightarrow C \in a]\}$. It is easy to prove that b and c are nonnull theories such that $Rabc$. We now prove that b and c are consistent. Suppose that b is inconsistent. Then, $A \rightarrow (C \wedge \neg C) \in$ **CIr** whence, contraposing, $\neg(C \wedge \neg C) \rightarrow \neg A \in$ **CIr**, and so $\neg A \in$ **CIr**. Thus, $A \rightarrow B \in a$, and then $a \models A \rightarrow B$, which contradicts the hypothesis. The proof that c is consistent is similar.

Now, consider the set Z of all nonnull consistent theories x such that $c \subseteq x$ and $B \notin x$. An argument similar to that in the proof of Lemma 6.3 shows that there is a prime nonnull consistent theory c' such that $c \subseteq c'$ and $B \notin c'$. By definition of R, $Rabc'$. Let now Y be the set of all consistent theories y such that $b \subseteq y$ and $Rayc'$. Reasoning as in the proof of Lemma 6.4, it is easy to show that there is a prime consistent theory b' such that $b \subseteq b'$ and $Rab'c$. But since clearly $A \in b$, we have $A \in b'$. Hence, there are prime consistent theories b', c' such that $Rab'c'$, $A \in b'$, and $B \notin c'$. By definition of $\models$, $b \models A$ and $c \not\models B$, which ends the proof of clause (4).

Clause (5): $a \models \neg A$ iff there are $b, c \in K$ such that not-$Rabc$ or $b \not\models A$. Proof from left to right is easy. So suppose $a \not\models \neg A$. We show that there are $b', c' \in K$ such that $Rab'c'$ and $b' \models A$. Define $b = \{B | A \to B \in \mathbf{CIr}\}$, $c = \{C | \exists B [B \in b$ and $B \to C \in a]\}$. The proof is similar to that of clause (4). □

Finally we prove, the following.

Theorem 6.7 (Completeness) *If A is valid, then A is a theorem of* **CIr**.

Proof: Suppose that A is not a theorem. Then, $A \notin T$ by Lemma 6.3. So A is not valid by Lemmas 6.4 and 6.5. □

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