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Binary Quantification Systems

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Abstract We investigate the formal theory of binary quantifiers, that is, quantifiers that take seriously the surface structure of natural language quantifier phrases. We show how to develop a natural deduction system for logics of this sort and demonstrate soundness and completeness results.

1 Introduction There is a natural sense in which ordinary language quantification is binary. We say of every man that he is mortal; we even say of every thing that it exists. It is not just universal quantification that works like this. All the standard and nonstandard quantifiers seem to behave similarly—consider 'most,' 'many,' 'few,' 'some' and 'the (one and only).' It is well known that some of these quantifiers—'most,' 'many' and 'few' in particular—cannot be represented in symbolic languages that do not take seriously their binary structure. It is also well known that if definite descriptions are to be treated as a species of quantifier, and not analyzed away in Russellian style, then again the binary structure evident in ordinary English will have to be respected. So it is appropriate to investigate formal languages that reflect the binary character of natural language quantification.[1]

There are two superficially different approaches that take this structure seriously: so-called resticted or sortal quantification and binary quantifiers. Contrary to the claim made by Evans [4], p. 59, these seem to be syntactic variants at best. We propose to speak of "binary quantifers", and choose syntax appropriate to a category of expression that takes a pair of open sentences, each containing one or more occurrences of a single free variable, to form a closed sentence (that is, (S/(S/N,S/N)) rather than (S/(S/N))/(S/N)). But no serious theoretical commitment hangs on that choice.

In this paper, we develop two systems of proof for a first-order language, *FL*, augmented with existential and universal binary quantifiers. The systems turn out to be equivalent in the sense that they are both sound and complete relative to the same semantics. We can think of them as alternative presentations of the same consequence relation. There is a sense in which the systems are redundant. For, given suitable definitions, every formula in which binary quantifiers occur is interderivable with one

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having only the standard unary quantifiers of first order theory. But formulas obtained using just binary quantifiers and negation in *FL* require other truth-functional connectives on translation into a classical first-order language. So interesting subsystems, for quantifiers and negation (with or without identity), are possible, that are sound and complete with respect to their semantics, and to which no similar subsystem of standard first-order logic corresponds. This shows that the systems stand on their own feet.

There are, moreover, reasons for thinking that the binary quantifier approach provides a superior environment for the study of natural language quantification. A binary quantifier takes a pair of open sentences, which we call the 'prefix' and 'matrix' of the formula, to form a single closed sentence. Intuitively, the prefix can be thought of as specifying a sub-domain about which the matrix, i.e., the second open sentence, makes some claim. This structure provides scope for semantic variation in the treatment of quantifiers that can be exploited, where philosophical or linguistic concerns dictate, without the necessity for accompanying variation in the semantics for the sentential connectives. The standard unary quantifiers, by contrast, allow markedly less scope for variation. This is one reason why the study of alternative logics has so often concentrated on varying the semantics for the connectives. A striking case is to be found in Blamey [2].

To see how this further degree of semantic freedom can be exploited, consider the following argument.

> Every man who loves all his enemies is a saint. Thomas is a man who has no enemies. Therefore Thomas is a saint.

There is a strong intuition that this argument is invalid, an intuition that can only be correct if 'Thomas loves all his enemies' is not implied by 'Thomas has no enemies.' This will be so if the natural language quantifier expressions, 'every' and 'all,' carry existential commitment, that is, if 'Every F is G' and 'All Fs are G' both imply that there are Fs. Then, if we try to paraphrase the argument in a standard first-order symbolic language, but in a way that respects the intuition of invalidity, we have to work with formulas of forbidding complexity—the first premise now requires a formula having six quantifiers, namely

$$\exists x \exists y[(Mx \mathbin{\&} Eyx) \mathbin{\&} \forall z(Ezx \to Lxz)] \mathbin{\&}$$
$$\forall x[((Mx \mathbin{\&} \exists y Eyx) \mathbin{\&} \forall z(Ezx \to Lxz)) \to Sx]$$

or some equivalent. On the other hand, if we work with a symbolic language employing binary quantifiers, we can get a simple and natural representation of the first premise of the argument with only two quantifiers (in our notation '$\forall x[Mx \mathbin{\&} \forall y(Eyx : Lxy) : Sx]$) for we can then adopt semantics for the binary universal quantifier stipulating that the sub-domain determined by the prefix must be nonempty. That is the approach we shall follow below. It should be stressed, however, that while recognition of the existential commitment of certain quantifiers leads naturally to a binary treatment, the binary approach as such is not committed to the existential commitment of the natural language universal quantifier. We can easily introduce a universal binary quantifier into our system that is not existentially committing. Indeed, such a quantifier is required to represent English 'any,' used as a universal quantifier that is not existentially committing. We shall indicate some other, more radical, ways of varying the quantifier clauses in the concluding section of this paper.

2 Syntax We take our language *FL* to be couched in terms of a denumerable set of individual constants $(a, b, c, \ldots)$, a denumerable set of individual variables $(x, y, z, x^1, y^1, \ldots)$, and a denumerable set of n-ary predicates $(P_n^1, P_n^2, P_n^3, \ldots)$ for each finite n. For the logical vocabulary we shall have the truth-functional connectives, the colon, the brackets, and the quantifiers $\forall$ and $\exists$. We also include one special two-place predicate letter, $=$, interpreted as identity. We take over the usual definition of well-formedness of formulas, except for the clause for the quantifiers. The syntactic formation rule for for them proceeds as follows.

> If $\Phi(t)$ and $\Psi(t)$ are both wffs with no occurrences of the variable ω, then $\forall\omega(\Phi(\omega) : \Psi(\omega))$ and $\exists\omega(\Phi(\omega) : \Psi(\omega))$ are both wffs, where $\Phi(\omega)$ and $\Psi(\omega)$ are formed by replacing at least one occurrence of t in $\Phi(t)$ and $\Psi(t)$ with ω.

Sentences are wffs with no free variables. Unary quantifiers can be introduced into *FL*, when required, by definition. $\forall\omega(\Phi(\omega))$ can abbreviate $\forall\omega(\omega = \omega : \Phi(\omega))$; and $\exists\omega(\Phi(\omega))$ can abbreviate $\exists\omega(\omega = \omega : \Phi(\omega))$

3 Semantics An interpretation $\mathcal{M}$ for *FL* is an ordered pair $\langle D, V\rangle$ such that:

1. D is a nonempty set;
2. V is a function defined over the individual constants and predicate letters of *FL* such that
   (a) V assigns an element of D to each individual constant of FL, and
   (b) V assigns a set of n-tuples whose elements are drawn from D to each n-ary predicate letter of *FL*.

We extend V to a function v defined over appropriate syntactic objects in the usual way for the truth-functional connectives and display here only the clauses for the quantifiers. $\langle D, v\rangle$ is called a model. It is clear that each interpretation uniquely determines a model. When $\Phi(\omega)$ is a wff with only ω free, $v(\Phi(\omega)) = \{o \in D \mid v_t^o(\Phi(t)) = \mathsf{T}\}$, for t not in $\Phi(\omega)$ and where v_t^o is the function that differs at most from v by assigning the object $o \in D$ to the individual constant t, and where $\Phi(t)$ is a substitution instance of $\Phi(\omega)$, i.e., t is substituted for every free occurrence of ω in $\Phi(\omega)$.

$$\begin{aligned} v(\exists\omega(\Phi(\omega) : \Psi(\omega))) &= \mathsf{T} \text{ iff } v(\Phi(\omega)) \cap v(\Psi(\omega)) \neq \varnothing \\ &= \mathsf{F} \text{ otherwise.} \end{aligned}$$

$$\begin{aligned} v(\forall\omega(\Phi(\omega) : \Psi(\omega))) &= \mathsf{T} \text{ iff } v(\Phi(\omega)) \neq \varnothing \text{ and } v(\Phi(\omega)) \subseteq v(\Psi(\omega)) \\ &= \mathsf{F} \text{ otherwise.} \end{aligned}$$

It should be noticed that the clause for the universal quantifier implements the decision, announced above, to let universally quantified formulas carry "existential import."

4 Proof theory: tableaux The basic notion here is that of a tree or tableaux, Δ, generated by a set of sentences, Γ, according to certain rules, in a manner made familiar in Jeffrey [5] and Boolos and Jeffrey [3]. We adopt standard rules for the truth functional connectives and identity, and add quantifier rules and the rule EM as follows.

[IS]
$$\frac{\begin{array}{c}\forall\omega(\Phi(\omega) : \Psi(\omega))\\ \Phi(t)\end{array}}{\Psi(t)}$$

[HI]
$$\frac{Q\,\omega(\Phi(\omega) : \Psi(\omega))}{\begin{array}{c}\Phi(t)\\ \Psi(t)\end{array}}$$
if t does not occur in the branch for $Q = \forall, \exists$

[QN]
$$\frac{\begin{array}{c}\neg Q^1\omega(\Phi(\omega) : \Psi(\omega))\\ \Phi(t)\end{array}}{Q^2\omega(\Phi(\omega) : \neg\Psi(\omega))}$$
for $Q^1 = \exists,\ Q^2 = \forall$ and $Q^1 = \forall,\ Q^2 = \exists$

[EM]
$$\frac{\vdots}{A \qquad\qquad \neg A}$$

We could weaken EM by requiring that in a use of EM the sentence A be of no greater complexity than some sentence already occurring in the branch and be entirely composed of nonlogical syntactic elements that already occur on the branch. We shall not add these restrictions since the rule given is clearly sound. We shall, however, use the restricted rule in proving completeness, establishing the claim just made.

4.1 Soundness of the tableaux system We define a branch to be *closed* if for some sentence type A, an occurrence of A and of its negation $\neg A$ both occur on the branch. A proof tree is closed iff every branch of the proof tree is closed. A proof tree is a *finished proof tree* iff every branch of the tree is either closed or such that every rule that can be applied has been. We say that there is a *proof* of A from the set Γ (which we signify by $\Gamma \vdash_\tau A$) iff the set $\Gamma \cup \{\neg \mathrm{A}\}$ yields a closed proof tree. We define a relation of semantic consequence $\models$, such that $\models \subseteq \mathcal{P}(FL) \times FL$ and $\Gamma \models A$ iff any model that makes every sentence in Γ true makes A true. We say that a given ordered pair $\langle \Gamma, A\rangle$ is valid iff $\Gamma \models A$, and that it is τ-provable iff $\Gamma \vdash_\tau A$.

Theorem 4.1 (Soundness) *If a set Γ has a model, then any proof tree generated from Γ in accordance with the tableaux rules has at least one branch with a model. (I.e., if $\Gamma \vdash_\tau A$ then $\Gamma \models A$.)*

Proof: The proof proceeds by induction on the number of rule applications in a branch. The base case is trivial: if a branch $\mathcal{B}$ has no rule applications then it must be at most a subset of Γ that *ex hypothesi* has a model. So $\mathcal{B}$ has a model. The induction hypothesis is that we have a branch $\mathcal{B}$ generated from Γ by using n rule applications and that has a model $\langle D, v\rangle$. We need to show that extending $\mathcal{B}$ with any of the rules leaves us with at least one branch $\mathcal{B}'$ that has a model. (We note at this point that some of the rules preserve a stronger property which we will exploit in these cases, namely the property that if $\langle D, v\rangle$ is a model of $\mathcal{B}$ then $\langle D, v\rangle$ is a model of a branch extending $\mathcal{B}$ by that rule.) The cases for the truth-functional connectives are straightforward and omitted here.

Case 1 ([IS]): Suppose that $\mathcal{B}$ contains $\forall\omega(\Phi(\omega) : \Psi(\omega))$ and $\Phi(t)$ and that $\langle D, v\rangle$ is a model for $\mathcal{B}$. Then, $v(\forall\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T}$ and $v(\Phi(t)) = \mathsf{T}$ and so $v(t) \in v(\Phi(\omega))$. By semantic clause for the universal quantifier, $v(\Phi(\omega)) \subseteq v(\Psi(\omega))$. Hence $v(t) \in v(\Psi(\omega))$ and so $v(\Psi(\mathrm{t})) = \mathsf{T}$, whence it follows that $\langle D, v\rangle$ is a model of $\mathcal{B} \cup \{\Psi(t)\}$.

Case 2 ([QN_1]): Suppose that $\mathcal{B}$ contains $\neg\forall\omega(\Phi(\omega) : \Psi(\omega))$ and $\Phi(t)$ and that $\langle D, v\rangle$ is a model of $\mathcal{B}$. Then since $v(t) \in v(\Phi(\omega))$ it must be that $v(\Phi(\omega)) \not\subset v(\Psi(\omega))$. So for some $o \in v(\Phi(\omega))$, $o \notin v(\Psi(\omega))$ hence $o \in v(\neg\Psi(\omega))$. Thus $v(\exists\omega(\Phi(\omega) : \neg\Psi(\omega))) = \mathsf{T}$, hence $\langle D, v\rangle$ is a model for $\mathcal{B} \cup \{\exists\omega(\Phi(\omega) : \neg\Psi(\omega))\}$.

Case 3 ([QN_2]): Suppose that $\mathcal{B}$ contains $\neg\exists\omega(\Phi(\omega) : \Psi(\omega))$ and $\Phi(t)$ and that $\langle D, v\rangle$ is a model of $\mathcal{B}$. Since $v(\exists\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{F}$, $v(\Phi(\omega)) \cap v(\Psi(\omega)) \neq \varnothing$, and since $v(t) \in v(\Phi(\omega)), v(\Phi(\omega)) \neq \varnothing$ whence it follows that $v(\Phi(\omega)) \subset v(\neg\Psi(\omega))$, so $v(\forall\omega(\Phi(\omega) : \neg\Psi(\omega))) = \mathsf{T}$. So $\langle D, v\rangle$ is a model of $\mathcal{B} \cup \{\forall\omega(\Phi(\omega) : \neg\Psi(\omega))\}$.

Case 4 ([$\mathrm{HI}_\exists$]): Consider $\mathcal{B}$ such that $\exists\omega(\Phi(\omega) : \Psi(\omega)) \in \mathcal{B}$, T occurs in no sentence in $\mathcal{B}$ and $\langle D, v\rangle$ is a model for $\mathcal{B}$. Now consider $\mathcal{B} \cup \{\Phi(t), \Psi(t)\}$. We shall show that $\mathcal{B} \cup \{\Phi(t), \Psi(t)\}$ has a model if $\mathcal{B}$ does. Since $v(\exists\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T}$, it must be that $v(\Phi(\omega)) \cap v(\Psi(\omega)) \neq \varnothing$, so choose $o \in v(\Phi(\omega)) \cap v(\Psi(\omega))$ and then consider $\langle D, v_t^o\rangle$. Clearly $\langle D, v_t^o\rangle$ is a model of $\mathcal{B}$ since $\langle D, v_t^o\rangle$ differs not at all from $\langle D, v\rangle$ on the vocabulary used in $\mathcal{B}$ and by definition $v_t^o(t) \in v_t^o(\Phi(\omega))$ so $v_t^o(\Phi(t)) = \mathsf{T}$ and $v_t^o(t) \in v_t^o(\Psi(\omega))$, so $v_t^o(\Psi(t)) = \mathsf{T}$ hence $\langle D, v_t^o\rangle$ is a model of $\mathcal{B} \cup \{\Phi(t), \Psi(t)\}$.

Case 5 ([$\mathrm{HI}_\forall$]): The argument is essentially the same as that for $\mathrm{HI}_\exists$ except that we utilize the fact that if $v(\forall\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T}$, then $v(\Phi(\omega)) \cap v(\Psi(\omega)) \neq \varnothing$ and proceed as before.

This completes the soundness proof for the proof tableaux. □

4.2 Completeness of the tableaux system We prove completeness for our proof tableaux system restricted in the manner indicated above, namely EM will be restricted to form a rule $\mathrm{EM_R}$ that is just like EM except that the sentence A on which the branching is done is (i) no more complex than some sentence already in $\mathcal{B}$, and (ii) contains no nonlogical vocabulary that does not already occur in $\mathcal{B}$ apart from sentences introduced by = I. The effect of this restriction is to allow open branches that are well short of being maximal consistent sets of sentences.

Theorem 4.2 (Completeness Theorem) *Any set of sentences Γ that generates a finished proof tree (fpt) with at least one open branch has a model. If $\Gamma \models A$ then $\Gamma \vdash_\tau A$.*

We prove this theorem by finding a model $\langle D, v\rangle$ of an open branch $\mathcal{B}$ of the fpt for Γ. Since Γ is included in $\mathcal{B}$, $\langle D, v\rangle$ will be model of Γ as well. Now we construct our model $\langle D, v\rangle$ as follows: We define a set $\mathcal{B}^+$ to be $\mathcal{B} \cup \{A \mid \neg A \in \mathcal{B}\}$. We define a relation $\simeq$ on names such that $t \simeq t'$ iff $t = t' \in \mathcal{B}$. Now let $|t| = \{t' \mid t \simeq t'\}$. Then $D = \{|t| \mid t \text{ occurs in } \mathcal{B}\}$. Let V be the smallest function such that (i) for each term t, $V(t) = |t|$, (ii) for each n-ary predicate P, $\langle V(t_1), \ldots, V(t_n)\rangle \in V(\mathrm{P})$ iff $\mathrm{P}t_1, \ldots, t_n \in \mathcal{B}$. Then v is obtained by extending V in accordance with the semantics specified in Section 3.

Establishing completeness comes down to establishing the following result: for every $C \in \mathcal{B}^+$, $v(C) = \mathsf{T}$ iff $C \in \mathcal{B}$. That result will be proved by induction on the complexity of the sentence C, which we define thus: let $\kappa(A)$ be the complexity of A. Then

(a) if A is atomic then $\kappa(A) = 0$;
(b) if A is of the form $\neg B$ then $\kappa(\neg B) = \kappa(B) + 1$;
(c) if A is of the form $B \,\&\, C$ then $\kappa(B \,\&\, C) = \max[\kappa(B), \kappa(C)] + 2$;
(d) if A is of the form $Q\,\omega(\Phi(\omega) : \Psi(\omega))$ then $\kappa[Q\,\omega(\Phi(\omega) : \Psi(\omega))] = \max[\kappa(\Phi(t)), \kappa(\Psi(t))] + 2$ for $Q = \forall, \exists$.

The proof now follows.

Proof:
Base Case: If $C \in \mathcal{B}^+$ and $\kappa(C) = 0$ then $v(C) = \mathsf{T}$ iff $C \in \mathcal{B}$. (a) If $C \in \mathcal{B}$ then by definition of $\langle D, v\rangle$, $v(C) = \mathsf{T}$. (b) *If* $v(C) = \mathsf{T}$ and $C \in \mathcal{B}^+$ *then either* $C \in \mathcal{B}$ *or* $\neg C \in \mathcal{B}$. So suppose $v(C) = \mathsf{T}$ and $\neg C \in \mathcal{B}$. If $v(C) = \mathsf{T}$ and C is atomic then $C = P\,t_1 \ldots t_n$; so $\langle v(t_1), \ldots v(t_n)\rangle \in v(P)$. So, by definition of model, for some $t'_1 \ldots t'_n, t_1 = t'_1, \ldots, t_n = t'_n \in \mathcal{B}$ and $P\,t'_1 \ldots t'_n \in \mathcal{B}$. Whence, by at most n uses of [=E], we have it that $P\,t_1 \ldots t_n \in \mathcal{B}$, which contradicts hypothesis that $\mathcal{B}$ is an open branch. So $C \in \mathcal{B}$.

Inductive hypothesis: *If* $A \in \mathcal{B}^+$ is of complexity less than n, then A is true on $\langle D, v\rangle$ iff $A \in \mathcal{B}$.

Inductive step: Suppose C is of complexity n.

1. Proofs for the case of $C = \neg D$ and $C = D \,\&\, E$ are straightforward and are omitted here.
2. Case of $C = \exists\omega(\Phi(\omega) : \Psi(\omega))$
   (a) Suppose that $\exists\omega(\Phi(\omega) : \Psi(\omega)) \in \mathcal{B}$. We want to show

   $$v(\exists\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T};$$

   that is, we want to show that $v(\Phi(\omega)) \cap v(\Psi(\omega)) \neq \varnothing$.

   By [HI] we have for some t, not in $\exists\omega(\Phi(\omega) : \Psi(\omega))$, that $\Phi(t)$, $\Psi(t) \in \mathcal{B}$. The complexity of $\Phi(t)$ and$\Psi(t)$ is less than that of $\exists\omega(\Phi(\omega) : \Psi(\omega))$, so by IH $v(\Phi(t)) = v(\Psi(t)) = \mathsf{T}$. Thus $|t| \in v(\Phi(\omega))$ and $|t| \in v(\Psi(\omega))$ which suffices to show that $v(\Phi(\omega)) \cap v(\Psi(\omega)) \neq \varnothing$.

   (b) Suppose that $v(\exists\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T}$ and $\exists\omega(\Phi(\omega) : \Psi(\omega)) \in \mathcal{B}^+$. We want to show that $\exists\omega(\Phi(\omega) : \Psi(\omega)) \in \mathcal{B}$.

   Since $v(\exists\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T}$ it follows that $v(\Phi(t)) = v(\Psi(t)) = \mathsf{T}$ for some t occurring in $\mathcal{B}$, by definition of $\langle D, v\rangle$. $\Phi(t)$ and $\Psi(t)$ are of lesser complexity than $\exists\omega(\Phi(\omega) : \Psi(\omega))$ and so meet restriction on

$[\mathrm{EM_R}]$. So by this rule and possibly [DNE], $\Phi(t), \Psi(t) \in \mathcal{B}^+$; whence by IH, $\Phi(t), \Psi(t) \in \mathcal{B}$. Now suppose *per absurdum* that

$$\exists\omega(\Phi(\omega) : \Psi(\omega)) \notin \mathcal{B},$$

then by definition of $\mathcal{B}^+$, $\neg\exists\omega(\Phi(\omega) : \Psi(\omega)) \in \mathcal{B}$. But since $\Phi(t) \in \mathcal{B}$ by [QN] we have $\forall\omega(\Phi(\omega) : \neg\Psi(\omega)) \in \mathcal{B}$. So by [IS], $\neg\Psi(t) \in \mathcal{B}$, which contradicts assumption that $\mathcal{B}$ is open. So $\exists\omega(\Phi(\omega) : \Psi(\omega)) \in \mathcal{B}$.

3. Case of $C = \forall\omega(\Phi(\omega) : \Psi(\omega))$

   (a) Suppose that $\forall\omega(\Phi(\omega) : \Psi(\omega)) \in \mathcal{B}$. We want to show

   $$v(\forall\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T}.$$

   By [HI] we have it that for some t not in $\Phi(\omega)$, $\Psi(\omega)$ that $\Phi(t), \Psi(t) \in \mathcal{B}$. By IH, $v(\Phi(t)) = v(\Psi(t)) = \mathsf{T}$. Thus, since $|t| \in v(\Phi(\omega))$, $v(\Phi(\omega)) \neq \varnothing$. Suppose *per absurdum* that $v(\Phi(\omega)) \not\subset v(\Psi(\omega))$. By definition of $\langle D, v\rangle$ for some t', $|t'| \in v(\Phi(\omega))$ and $|t'| \notin v(\Psi(\omega))$; so by definition of $\langle D, v\rangle$ (and possibly use of [=E] as in Case (1)), $v(\Phi(t')) = \mathsf{T}$ and $v(\Psi(t')) = \mathsf{F}$. Now since $\Phi(t')$ is composed of syntactical elements occurring in $\mathcal{B}$ and is of lesser complexity than $\forall\omega(\Phi(\omega) : \Psi(\omega))$ we have it by $[\mathrm{EM_R}]$ that either $\Phi(t') \in \mathcal{B}$ or $\neg\Phi(t') \in \mathcal{B}$. But by IH the latter case contradicts $v(\Phi(t')) = \mathsf{T}$; so $\Phi(t') \in \mathcal{B}$. Then by [IS] $\Psi(t') \in \mathcal{B}$ and by IH $v((\Psi(t')) = \mathsf{T}$, which contradicts our assumption. Hence $v(\forall\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T}$.

   (b) Suppose that $v(\forall\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T}$ and $\forall\omega(\Phi(\omega) : \Psi(\omega)) \in \mathcal{B}^+$. We want to show that $\forall\omega(\Phi(\omega) : \Psi(\omega)) \in \mathcal{B}$.

   Since $v(\forall\omega(\Phi(\omega) : \Psi(\omega)))=\mathsf{T}$, $v(\Phi(\omega))\neq\varnothing$ and $v(\Phi(\omega)) \subseteq v(\Psi(\omega))$. So for some t, $|t| \in v(\Phi(\omega))$, and since $t \in |t|$, $v(\Phi(t)) = \mathsf{T}$ and by definition of $\langle D, v\rangle$ t occurs in $\mathcal{B}$. $\Phi(t)$ is of lesser complexity than $\forall\omega(\Phi(\omega) : \Psi(\omega))$ and $\neg\forall\omega(\Phi(\omega) : \Psi(\omega))$ one of which is in $\mathcal{B}$ since $\forall\omega(\Phi(\omega) : \Psi(\omega)) \in \mathcal{B}^+$. So either $\Phi(t) \in \mathcal{B}$ or $\neg\Phi(t) \in \mathcal{B}$ by $[\mathrm{EM_R}]$. But if $\neg\Phi(t) \in \mathcal{B}$ then by IH $v(\Phi(t)) = \mathsf{F}$, which contradicts our assumption, so $\Phi(t) \in \mathcal{B}$. Now suppose $\neg\forall\omega(\Phi(\omega) : \Psi(\omega)) \in \mathcal{B}$, then by [QN] since $\Phi(t) \in \mathcal{B}$, $\exists\omega(\Phi(\omega) : \neg\Psi(\omega)) \in \mathcal{B}$ and by [HI], for some new t', $\Phi(t'), \neg\Psi(t') \in \mathcal{B}$. But then by IH $|t'| \in v(\Phi(\omega))$ and $|t'| \notin v(\Psi(\omega))$ which contradicts the assumption that $v(\forall\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T}$. So $\forall\omega(\Phi(\omega) : \Psi(\omega)) \in \mathcal{B}$

This ends the completeness proof. □

5 Proof theory: sequent system In this section, we present a system of first-order logic with binary quantifiers in the framework of sequents developed by Tarski (as consequence relations) and Gentzen (as object language sequents). The particular systems we present are given in a form influenced by Scott. We differ from his presentation by treating the sequents as not themselves metalinguistic. This is more in keeping with Gentzen's original motivation. We differ from Gentzen by not treating the sequent sign $\vdash$ as definable in terms already in the language *FL*. It ought to be clear that, since we do not restrict the size of the premise sets to finite sets, none of the connectives in *FL* and no combination of them alone will afford the generality we seek here. So a sequent is a linguistic entity that has the form '$\Gamma \vdash A$' where Γ is a set of sentences, $\vdash$ is the sequent sign, and A is a sentence. We shall abbreviate '$\{A\} \cup \Gamma \vdash B$ by '$A, \Gamma \vdash B$. We shall distinguish two consequence relations, $\vdash$, which is to hold between finite sets of sentences, and an infinitary relation, $\vdash_\alpha$, to be be defined later. We now define $\vdash$ to be the smallest relation on (finite subsets of *FL* $\times$ *FL*) closed under the following rules.

Structural Rules:

$\mathcal{R}$ $\quad \Gamma \vdash A \quad$ if $\quad A \in \Gamma$

$\mathcal{M}$ $\quad \dfrac{\Gamma \vdash A}{\Gamma, \Gamma' \vdash A}$

$\mathcal{T}$ $\quad \dfrac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$

System Rules:

[&E] $\quad \dfrac{\Gamma \vdash A \,\&\, B}{\Gamma \vdash A} \qquad \dfrac{\Gamma \vdash A \,\&\, B}{\Gamma \vdash B}$

[&I] $\quad \dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \,\&\, B}$

[EM] $\quad \dfrac{\Gamma, A \vdash C \qquad \Gamma, \neg A \vdash C}{\Gamma \vdash C}$

[PN] $\quad \dfrac{\Gamma, A \vdash B}{\Gamma, \neg B \vdash \neg A}$

[DN] $\quad \dfrac{\Gamma \vdash \neg\neg B}{\Gamma \vdash B} \qquad \dfrac{\Gamma \vdash B}{\Gamma \vdash \neg\neg B}$

[$\exists$I] $\quad \dfrac{\Gamma \vdash \Phi(t) \qquad \Delta \vdash \Psi(t)}{\Gamma, \Delta \vdash \exists\omega(\Phi(\omega) : \Psi(\omega))}$

[$\exists$HI] $\quad \dfrac{\Gamma, \Phi(t), \Psi(t) \vdash B}{\Gamma, \exists\omega(\Phi(\omega) : \Psi(\omega)) \vdash B}$ $\quad$ if t does not occur in Γ, B, or $\exists\omega(\Phi(\omega) : \Psi(\omega))$.

[$\forall$I] $\quad \dfrac{\Gamma, \Phi(t) \vdash \Psi(t)}{\Gamma, \Phi(r) \vdash \forall\omega(\Phi(\omega) : \Psi(\omega))}$ $\quad$ if t does not occur in Γ or $\forall\omega(\Phi\omega) : \Psi(\omega))$, any r .

[$\forall$HI] $\quad \dfrac{\Gamma, \Phi(t), \Psi(t) \vdash B}{\Gamma, \forall\omega(\Phi(\omega) : \Psi(\omega)) \vdash B}$ $\quad$ if t does not occur in Γ B, or $\forall\omega(\Phi(\omega) : \Psi(\omega))$.

[$\forall$E] $\quad \dfrac{\Gamma \vdash \forall\omega(\Phi(\omega) : \Psi(\omega)) \qquad \Delta \vdash \Phi(t)}{\Gamma, \Delta \vdash \Psi(t)}$

[$=$ I] $\quad \vdash t = t \quad$ for any t

[$=$ E] $\quad \dfrac{\Gamma \vdash \Phi(t) \qquad \Delta \vdash t = r}{\Gamma, \Delta \vdash \Phi(r)}$

We define the relation $\vdash_\alpha$, between a possibly infinite set of sentences Γ and a sentence C, to be the smallest such that $\Gamma \vdash_\alpha C$ iff for some finite (possibly empty) subset Γ' of Γ, $\Gamma' \vdash C$. This definition ensures that $\vdash_\alpha$ is a compact relation.

Note that if two models $\langle D^1, v^1\rangle$ and $\langle D^2, v^2\rangle$ are such that $D^1 = D^2$, and v^1 and v^2 differ at most by their assignments to a set S of primitive nonlogical vocabulary, then $\langle D^1, v^1\rangle$ and $\langle D^2, v^2\rangle$ agree on the truth values of any sentences of *FL* that contain none of the vocabulary in S. This is the property Boolos and Jeffrey [3], p. 125 call continuity.

5.1 Soundness for sequent formulation

Theorem 5.1 (Soundness Theorem) *If* $\Gamma \vdash A$ *then* $\Gamma \models A$.

Proof: We show that each of the rules is validity preserving, omitting as well known the cases for the structural, truth functional, and identity rules.

Case 1 ($\forall$ E): Suppose that [$\forall$E] is not validity preserving. That means that for some $\langle D, v\rangle$, $v(\Gamma) = v(\Delta) = \mathsf{T}$ and $v(\Psi(t)) = \mathsf{F}$ whereas $\Gamma \models \forall\omega(\Phi(\omega) : \Psi(\omega))$ and $\Delta \models \Phi(t)$. So we have $v(\forall\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T}$ and $v(\Phi(t)) = \mathsf{T}$. By the semantics, since $v(t) \in v(\Phi(\omega))$ and we have $v(\Phi(\omega)) \subseteq v(\Psi(\omega))$, we therefore have $v(t) \in v(\Psi(\omega))$ and so $v(\Psi(t)) = \mathsf{T}$, contradicting the hypothesis.

Case 2 ($\forall$ I): Suppose that $\Gamma, \Phi(t) \models \Psi(t)$ and $\Gamma, \Phi(r) \not\models \forall\omega(\Phi(\omega) : \Psi(\omega))$ and the restrictions are met. Then for some $\langle D, v\rangle$, $v(\Gamma) = v(\Phi(t)) = \mathsf{T}$ and $v(\forall\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{F}$. Given that $v(r) \in v(\Phi(\omega))$, it must be that $v(\Phi(\omega)) \not\subseteq v(\Psi(\omega))$. So for some $o \in D$, $o \in v(\Phi(\omega))$ and $o \notin v(\Psi(\omega))$. Now consider v^o_t. $v^o_t(\Gamma) = \mathsf{T}$ by continuity and $v^o_t(\Phi(t)) = \mathsf{T}$ by selection of o. So since $\Gamma, \Phi(t) \models \Psi(t)$ we have it that $v^o_t(\Psi(t)) = \mathsf{T}$ but then $|t| \in v^o_t(\Psi(\omega))$ which contradicts our assumption.

Case 3 ($\forall$ HI): Suppose that $\Gamma, \Phi(t), \Psi(t) \models B$ and $\Gamma, \forall\omega(\Phi(\omega) : \Psi(\omega)) \not\models B$. So for some $\langle D, v\rangle$, $v(\Gamma) = \mathsf{T}$ and $v(\mathrm{B}) = \mathsf{F}$. Thus $v(\Phi(\omega)) \neq \varnothing$ and $v(\Phi(\omega)) \subset v(\Psi(\omega))$. Now choose an $o \in D$ such that $o \in v(\Phi(\omega))$ and consider v^o_t. By continuity, $v^o_t(B) = \mathsf{F}$ and $v^o_t(\Gamma) = \mathsf{T}$. Since $o \in v(\Phi(\omega))$ it follows that $o \in v^o_t(\Phi(\omega))$ and since $o \in v(\Psi(\omega))$ it follows that $o \in v^o_t(\Psi(\omega))$ whence it is immediate that $v^o_t(\Phi(t)) = \mathsf{T}$ and $v^o_t(\Psi(t)) = \mathsf{T}$, so $\Gamma, \Phi(t), \Psi(t) \not\models B$, contradicting the initial assumption.

Case 4 ($\exists$ I): Suppose that $\Gamma \models \Phi(t)$, $\Delta \models \Psi(t)$ and $\Gamma, \Delta \not\models \exists\omega(\Phi(\omega) : \Psi(\omega))$ and that the restrictions are met. Then, for some $\langle D, v\rangle$ $v(\Gamma) = v(\exists\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T}$ and $v(\exists\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{F}$. Since $\Gamma \models \Phi(t)$ we have it that $v(\Phi(t)) = \mathsf{T}$. And since $\Delta \models \Psi(t)$ we have it that $v(\Psi(t)) = \mathsf{T}$. So $v(t) \in v(\Phi(\omega))$ and $v(t) \in v(\Psi(\omega))$. Whence $v(\Phi(\omega)) \cap v(\Psi(\omega)) \neq \varnothing$. Thus, $v(\exists\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T}$, contradicting the hypothesis.

Case 5 ($\exists$ HI): Suppose $\Gamma, \Phi(t), \Psi(t) \models B$ and $\Gamma, \exists\omega(\Phi(\omega) : \Psi(\omega)) \not\models B$, and that the restrictions are met. Then, for some $\langle D, v\rangle$, $v(\Gamma) = v(\exists\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T}$ and $v(\mathrm{B}) = \mathsf{F}$, since $v(\exists\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T}$ and $v(\Phi(\omega)) \cap v(\Psi(\omega)) \neq \varnothing$. Choose $o \in D$ such that $o \in v(\Phi(\omega)) \cap v(\Psi(\omega))$ then consider v^o_t: $v^o_t(\Gamma) = \mathsf{T}$ and $v^o_t(B) = \mathsf{F}$ by continuity, and $v^o_t(\Phi(t)) = v^o_t(\Psi(t)) = \mathsf{T}$. So $\Gamma, \Phi(t), \Psi(t) \not\models B$, contrary to the hypothesis.

This completes the soundness theorem for the sequent system. $\square$

5.2 Completeness theorem for the sequent system Before we go on to the Completeness Theorem for the sequent system, we state and prove a lemma and note that two sequents corresponding to the [QN] rules of the tableaux system are provable. The sequents are $Fa, \neg\exists\omega(\Phi(\omega) : \Psi(\omega)) \vdash \forall\omega(\Phi(\omega) : \neg\Psi(\omega))$ and $Fa, \neg\forall\omega(\Phi(\omega) : \Psi(\omega)) \vdash \exists\omega(\Phi(\omega) : \neg\Psi(\omega))$. The proofs of these are straightforward.

Call a set, Γ, *happy* iff if $Q\,\omega(\Phi(\omega) : \Psi(\omega)) \in \Gamma$ (for $Q = \exists, \forall$), then for some t, $\Phi(t), \Psi(t) \in \Gamma$.

Lemma 5.2 (Happy Set Lemma) *Consider an unhappy finite set Δ, such that $\Delta \nvdash C$. There is a happy $\Delta^* \supset \Delta$ such that $\Delta^* \nvdash C$.*

Proof: Suppose that there is a sentence of the form $Q\,\omega(\Phi(\omega) : \Psi(\omega)) \in \Delta$ such that for no term t do we have $\Phi(t), \Psi(t) \in \Delta$. Choose a new term r that does not occur in $\Delta \cup \{C\}$, then set Δ' to be $\Delta \cup \{\Phi(r), \Psi(r)\}$. Now suppose *per absurdum* that $\Delta' \vdash C$. Then $\Delta, \Phi(r), \Psi(r) \vdash C$. So $\Delta, Q\,\omega(\Phi(\omega) : \Psi(\omega)) \vdash C$ by [HI] for $Q = \exists, \forall$. But $\Delta \cup \{Q\,\omega(\Phi(\omega) : \Psi(\omega))\} = \Delta$. So $\Delta \vdash C$, contradicting our hypothesis. If Δ' is happy, we have found our Δ^*. If Δ' is unhappy we repeat the procedure until we have a happy $\Delta^{\prime\prime\cdots\prime}$ which is our Δ^*. The procedure must end after a finite number of steps since the sentences we add to our set are less complicated than the sentence that occasioned their entry. Given that the initial Δ was finite and each sentence has at most a finite number of quantifiers, we somewhere reach the end with a happy non-C-entailing set. $\square$

Recall that $\vdash_\alpha$ was defined as the smallest relation on $\mathcal{P}(FL) \times FL$ such that $\Gamma \vdash_\alpha C$ iff for some Γ', Γ' is finite and $\Gamma' \subseteq \Gamma$ and $\Gamma' \vdash C$. A set is $\neg$-consistent iff for no sentence A are A and $\neg A$ in the set.

Lemma 5.3 (Extension Lemma) *Given Γ and C such that Γ is finite and $\Gamma \nvdash C$, there is a maximal $\neg$-consistent and happy Γ^* such that $\Gamma \subseteq \Gamma^*$ and $\Gamma \nvdash_\alpha C$.*

Proof: The following procedure guarantees the result:

Step 1. Set $\Gamma = \Gamma_1^1$

Step 2. If Γ_k^n is not happy then extend Γ_k^n to a happy Γ_{k+1}^n as per the Happy Set Lemma.

Step 3. If Γ_k^n is happy then extend Γ_k^n by considering the nth sentence, S, on a standard enumeration E of all the sentences of FL and if $\Gamma_k^n, S \nvdash_\alpha C$ then $\Gamma_1^{n+1} = \Gamma_k^n \cup \{S\}$ else $\Gamma_1^{n+1} = \Gamma_k^n \cup \{\neg S\}$. Set $\Gamma^* = \bigcup_i \bigcup_j \Gamma_j^i$ for $i, j = 1\ldots$

To show that this construction does the job we suppose that we are given a finite Γ and a C such that $\Gamma \nvdash C$. Clearly, since Γ is finite and $\Gamma \nvdash C$, we have it that $\Gamma \nvdash_\alpha C$. So could the output of Step 2 fail to preserve the unprovability of C? No, the Happy set Lemma shows that it cannot. What of Step 3? Could $\Gamma_k^n \nvdash_\alpha C$ but $\Gamma_1^{n+1} \vdash_\alpha C$? If that did happen then $\Gamma_k^n, S \vdash_\alpha C$ and $\Gamma_k^n, \neg S \vdash_\alpha C$. (In the following we do not assume Γ_k^n to be finite.) So for some finite Δ, Θ such that $\Delta \subset \Gamma_k^n$ and $\Theta \subset \Gamma_k^n$ and$\Delta, S \vdash C$ and $\Theta, \neg S \vdash C$. By $\mathcal{M}\,\Delta, \Theta, S \vdash C$ and $\Delta, \Theta, \neg S \vdash C$. By [EM] $\Delta, \Theta \vdash C$. But $\Delta \cup \Theta$ is finite and a subset of Γ_k^n, so $\Gamma_k^n \vdash_\alpha C$, contradicting the hypothesis. So Step

3 is such that if $\Gamma_k^n \nvdash_\alpha C$ then $\Gamma_1^{n+1} \nvdash_\alpha C$. We know that Γ^* is maximal since E is an enumeration of all sentences of *FL* and we consider each sentence. By Step 3 at least one of each sentence and its negation gets put into some $\Gamma_{\mathrm{k}}^{\mathrm{n}}$. And finally, if some sentence and its negation both occurred in Γ^*, then there would be a finite point in the construction at which they first occurred together and at that point the set would provably entail C—something we have shown that we could always avoid. It is obvious that if $\Gamma^* \nvdash_\alpha C$ then $\neg \mathrm{C} \in \Gamma^*$. □

We show that Γ^* is consistent by proving our *Completeness Theorem*.

Theorem 5.4 (Completeness Theorem) *If* $\Gamma \nvdash_\alpha C$ *then* $\Gamma \not\models C$.

Proof: Assume we have $\Gamma^* \nvdash_\alpha C$. Then we want to show that $\Gamma^* \not\models C$. We do that by showing there to be a model of Γ^* that makes C false. We define the model $\langle D^*, v^* \rangle$ as follows. We first define a relation on names $\simeq$ such that $t \simeq t'$ iff $t = t' \in \Gamma^*$ and $|t| = \{t' \mid t \simeq t'\}$. Let $D^* = \{|t| \mid t$ occurs in $\Gamma^*\}$. Let V^* be the smallest function such that for each name t, $v^*(t) = |t|$, and for each $n-$ary predicate P, $\langle V^*(t_1), \ldots, V^*(t_n) \rangle \in V^*(P)$ iff $Pt_1 \ldots t_n \in \Gamma^*$. $\langle D^*, V^* \rangle$ can be extended in the usual way to $\langle D^*, v^* \rangle$. We prove by induction on the complexity of sentences, defined as before, that $\langle D^*, v^* \rangle$ makes all and only the sentences of Γ^* true.

Base Case: complexity $= 0$, $A \in \Gamma^*$ and A is atomic.

1. A is of the form $Pt_1 \ldots t_n$, so by definition of v^*, $\langle v^*(t_1), \ldots, v^*(t_n) \rangle \in v^*(P)$, so $v^*(\mathrm{A}) = \mathsf{T}$.
2. Suppose $v^*(\mathrm{A}) = \mathsf{T}$, show $\mathrm{A} \in \Gamma^*$. $v^*(Pt_1 \ldots Pt_n) = \mathsf{T}$. So for some terms $r_1, \ldots, r_n$, $Pr_1 \ldots r_n \in \Gamma^*, t_1 = r_n \in \Gamma^*, \ldots, t_n = r_n \in \Gamma^*$. But then, since $Pr_1 \ldots r_n, t_1 = r_1, \ \ldots \ , t_n = r_n, \neg Pt_1 \ldots t_n \vdash C$, it follows that $Pt_1 \ldots t_n \in \Gamma^*$.

Inductive Hypothesis: if $C \in \Gamma^*$ and $\kappa(C)$ is less than n then C is true on $\langle D^*, v^* \rangle$.

Inductive Step: Suppose $A \in \Gamma^*$ and $\kappa(A) = n$.

1. The case for negation and conjunction is trivial. We move on to the quantifiers.
2. A is of the form $\exists\omega(\Phi(\omega) : \Psi(\omega))$
   (a) $\exists\omega(\Phi(\omega) : \Psi(\omega)) \in \Gamma^*$. So, since Γ^* is happy, for some t, $\Phi(t) \in \Gamma^*$ and $\Psi(t) \in \Gamma^*$. Whence it follows by IH that $v^*(\Phi(t)) = v^*(\Psi(t)) = \mathsf{T}$ and so $|t| \in v^*(\Phi(\omega)), |t| \in v^*(\Psi(\omega))$. So $v^*(\exists\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T}$.
   (b) $v^*(\exists\omega(\Phi(\omega) : \Psi(\omega))) = \mathsf{T}$, show that $\exists\omega(\Phi(\omega) : \Psi(\omega)) \in \Gamma^*$. For some t, $|t| \in v^*(\Phi(\omega)) \cup v^*(\Psi(\omega))$, so by IH $\Phi(t)$, $\Psi(t) \in \Gamma^*$. But $\Phi(t), \Psi(t), \neg\exists\omega(\Phi(\omega) : \Psi(\omega)) \vdash C$, so $\neg\exists\omega(\Phi(\omega) : \Psi(\omega)) \notin \Gamma^*$, so $\exists\omega(\Phi(\omega) : \Psi(\omega)) \in \Gamma^*$.
3. A is of the form $\forall\omega(\Phi(\omega) : \Psi(\omega))$
   (a) $\forall\omega(\Phi(\omega) : \Psi(\omega)) \in \Gamma^*$, show $v^*[\forall\omega(\Phi(\omega) : \Psi(\omega))] = \mathsf{T}$. Since Γ^* is happy, for some t, $\Phi(t)$, $\Psi(t) \in \Gamma^*$. So $|t| \in v^*(\Phi(\omega))$. Suppose *per absurdum* that $v^*[\forall\omega(\Phi(\omega) : \Psi(\omega))] = \mathsf{F}$. Then for some $o \in D^*$, $o \in v^*(\Phi(\omega))$ and $o \notin v^*(\Psi(\omega))$, so for some term r, $|r| = o$ so then $v^*(\Phi(r)) = \mathsf{T}$. $v^*(\neg\Psi(r)) = \mathsf{T}$, so by IH $\Phi(r), \neg\Psi(r) \in \Gamma^*$. But $\forall\omega(\Phi(\omega) : \Psi(\omega)), \Phi(r), \neg\Psi(r) \vdash C$; so $v^*[\forall\omega(\Phi(\omega) : \Psi(\omega))] = \mathsf{T}$

(b) $v^*[\forall\omega(\Phi(\omega) : \Psi(\omega))] = \top$, show that $\forall\omega(\Phi(\omega) : \Psi(\omega)) \in \Gamma^*$. So $v^*(\Phi(\omega)) \neq \varnothing$ and $v^*(\Phi(\omega)) \subseteq v^*(\Psi(\omega))$. For some t, $|t| \in v^*(\Phi(\omega))$, whence $|t| \in v^*(\Psi(\omega))$. So by IH $\Phi(t) \in \Gamma^*$ and$\Psi(t) \in \Gamma^*$. Suppose $\neg\forall\omega(\Phi(\omega) : \Psi(\omega)) \in \Gamma^*$, then since $\Phi(t) \in \Gamma^*$ and $\Phi(t), \neg\forall\omega(\Phi(\omega) : \Psi(\omega)), \neg\exists(\Phi(\omega) : \neg\Psi(\omega)) \vdash C$ it follows that $\exists\omega(\Phi(\omega) : \neg\Psi(\omega)) \in \Gamma^*$. Whence for some r, since Γ^* is happy, $\Phi(r), \neg\Psi(r) \in \Gamma^*$. By IH $v^*(\Phi(r)) = v^*(\Psi(r)) = \top$, so $|r| \notin v^*(\Psi(\omega))$. But then $v^*(\Phi(\omega)) \not\subseteq v^*(\Psi(\omega))$ contradicting the hypothesis. So $\neg\forall\omega(\Phi(\omega) : \Psi(\omega)) \notin \Gamma^*$ and so $\forall\omega(\Phi(\omega) : \Psi(\omega)) \in \Gamma^*$.

This ends the proof of the Completeness Theorem. □

By examining the proof of the Completeness Theorem, we note that $\vdash_\alpha$ is compact. As in the tableaux system, compactness falls out of the finite size of any closed proof tree. It also follows from the proof that the logic restricted to the rules for the quantifiers and negation is sound and complete with respect to the semantics. This fragment of the logic is not equivalent to any natural fragment of classical logic with unary quantifiers. This can be verified by noting that any fragment of standard first-order logic that is rich enough to express sentences equivalent to those of the form $\forall\omega(\Phi(\omega) : \Psi(\omega))$ will need to express something equivalent to $\forall\omega(\Phi(\omega) \to \Psi(\omega))$ & $\exists\omega(\Phi(\omega))$. Thus we need at least material implication. But if we add material implication without any other restrictions, we obtain the resources of the complete classical language. And there is no natural restriction of classical language such that, for every sentence of restricted *FL*, there exists a sentence in the restricted classical language that translates it. We cannot, for example, find an equivalent fragment of classical language by restricting the classical language to some number of occurrences of connectives, since there is no limit to the number of quantifiers which can be embedded in a sentence of binary structure.

6 Extensions and revisions We have established that formal systems for binary quantifiers, comparable with those familiar from standard first-order logic, can be constructed that are reasonably elegant and eminently teachable. But the primary interest of binary quantifiers really lies in the possibility of extending the system to cope with new quantifiers, particularly those that require a binary treatment—for example, a Russellian definite description operator, which should be thought of as a quantifier rather than a term-forming operator, and the plurative quantifiers ‘most,’ ‘many,’ and ‘few.’ Binary quantifiers are also of interest because they provide a convenient environment for exploring deviant proposals for the semantics of natural language quantifiers, for example, proposals that involve non-bivalent presuppositional languages. The work we have done in the present paper is intended to establish a base camp from which to start wider explorations. We shall conclude by commenting briefly on where these explorations might lead.

A binary quantifier has to operate on two open sentences, the prefix and matrix, to form a closed formula. We may think of the semantics as determining a pair of conditions, one each for the prefix and matrix, whose joint satisfaction is sufficient for truth. Call the condition associated with the prefix the ‘quorum’ and that associated with the matrix the ‘quota.’ The quorum fixes the sub-domain, the quota determines what condition has to be met by the sub-domain for the formula to be true. The quantifiers we have considered all have a standard quorum, namely that the set of satisfiers for the prefix be nonempty. This accounts for the fact that we have, in HI, a common rule of inference for the quantifiers. The difference between the quantifiers comes by varying the quota.

One way of extending the system is to look at other quantifiers that differ from our basic two only in the quota condition—the plurative quantifiers might be handled in this way. It is best to think of 'Most *F*s are *G*' and 'Many *F*s are not *G*' as near contradictories, i.e., provided the quorum condition is satisfied, each implies and is implied by the negation of the other. (To see that this is plausible, notice that the schema 'Few *F*s are *G*' can be paraphrased equally well either as 'Most *F*s are not *G*' or as 'Not many *F*s are *G*.') The task is then to set an appropriate quota condition. The usual suggestion is that 'most' requires a simple majority of favourable cases. But the link with 'many' suggests that this is too precise; 'most' and 'many' are vague. That a bare majority of *F*s should be *G* is simply the lower bound of the range of possible precisifications of natural language 'most.' Pursuing the logic of plurative quantifiers should, then, lead us into the complexities of vague and, presumably, non-bivalent languages.

Evidently there is room for variation on the quorum as well as the quota. One obvious possibility is to make the quorum condition null, i.e., to drop the requirement that the satisfier set for the prefix be nonempty. Many will think this the natural approach, and there are undoubtedly natural language quantifiers for which this is right—English 'any' treated as a universal quantifier is plausibly such a case. Use $\forall^0$ for this quantifier, which we regard as a universal quantifier that is not existentially committing. The formation rule is to be as for the standard universal quantifier, but its semantics are given thus

$$\begin{aligned} v(\forall^0\omega(\Phi(\omega):\Psi(\omega))) &= \mathsf{T} \text{ iff } v(\Phi(\omega)) \subseteq v(\Psi(\omega)) \\ &= \mathsf{F} \text{ otherwise.} \end{aligned}$$

It should be obvious how to modify the rules of proof for the universal quantifier so as to accommodate this addition to the system; there will, for example, be no HI rule.

In this paper, however, we have been interested mainly in systems in which a positive quorum condition is effective, though we have chosen the most conservative approach, looking at something that is very much a re-presentation of classical first-order logic. A more adventurous approach might vary the quota condition in either of two ways.

First, we can consider quantifiers with a different quorum condition. We might, for example, have a quantifier for which the quorum specifies that there be just one object in the domain satisfying the prefix. Consider language for the Russellian descriptions, i.e., a language for which the following equivalence holds, using '$\mathbf{I}\omega$' for the definite description operator: $\mathbf{I}\omega(\Phi\omega : \Psi\omega)$ iff $\exists\omega[\forall\chi(\Phi\chi : \omega = \chi) : \Psi\omega]$. (A like move would let us handle 'both' and 'neither.') Arguably the plurative quantifiers are better handled in such a setting, this time with the quorum condition set to require (vaguely) a numerous set of satisfiers for the prefix. Finally, quantificational sentences that use the structure 'all the' followed by a plural noun call for similar treatment. Varying the quorum condition in this way will have an impact on the rules of inference appropriate to the quantifier, one that will show up in the form taken by the HI rule.

The other way in which we might vary the semantic condition associated with the prefix is to distinguish the consequence of failure of the quorum condition from failure of the quota. In the classical systems developed in this paper, failure on either condition results in falsehood. But we could let failure of the quorum result in indeterminacy of truth value, leading to a non-bivalent, presuppositional logic. The special problems for this approach now arise from the need to redefine the notion of logical consequence for non-bivalent languages. The matter is best approached by thinking about logical equivalence. It is natural to suppose that if two sentences are logically equivalent, then they should necessarily be alike in truth value. It is also appealing to think that logically equivalent sentences stand in the relation of logical consequence to each other. If both requirements are to be met, then a strong notion of logical consequence is required, one which requires both that truth be passed from premise to conclusion and that falsehood be passed from conclusion to premise. We have a particular interest in languages of this sort and hope to discuss them in a sequel to this paper.

NOTE

1. The suggestion that binary quantifiers should be used to handle 'most,' 'many' and 'few' is to be found in Altham and Tennant [1] and in Wiggins [8]; the use of binary quantifier notation for definite descriptions is suggested by Quine [7] who attributes it to Sharvey, but the idea seems to have originated in Prior [6].

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