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Consequence and Confirmation

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Abstract Gentzen's account of logical consequence is extended so as to become a matter of degree. We characterize and study two kinds of function G, where $G(X, Y)$ takes values between 0 and 1, which represent the degree to which the set X of statements (understood conjunctively) logically implies the set Y of statements (understood disjunctively). It is then shown that these functions are essentially the same as the absolute and the relative probability functions described by Carnap.

1 Prior to Gentzen's[1] 1934–35 paper [2], *logical consequence* was thought of as a relation from a conjunctively interpreted set X of statements to a *single* statement A. However, in the paper in question Gentzen introduced a new notion of derivability which permits one to derive a disjunctively interpreted sequence of statements from a conjunctively interpreted one. Sets may of course substitute in Gentzen for sequences and often do, which makes for a new notion of logical consequence, one according to which—given sets X and Y of statements—X has Y as a logical consequence if (i) it is impossible for all the statements in X to be true and, simultaneously, all those in Y to be false, or equivalently (ii) at least one statement in Y must be true if all the statements in X are true. It is Gentzen's notion of logical consequence which we shall extend here to one of *degrees of logical consequence* or—equivalently, as we shall prove—to one of *degrees of confirmation* in the sense of Carnap [1].

In Sections 2 and 3 we formulate characterizations of two classes of functions (G_1-functions and G_2-functions) which, in different ways, assign degrees of logical consequence to pairs $\langle X, Y\rangle$ of sets of statements. This is done axiomatically by extending Gentzen's structural rules to apply to functions from finite antecedent and succedent sets of statements to real numbers in the interval $[0, 1]$, and these generalized constraints are then justified in terms of weighted truth-value assignments. In Section 4 the constraints are further extended to cover infinite antecedent and succedent sets of statements. We next turn to Gentzen's introduction rules for connectives and extend them also to G_1 and G_2 functions, and we show in Section 5 that, as long

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as the restrictions of G_1 and G_2 functions to sets of atomic statements meet the respective structural constraints, the full functions obtained by using the generalized introduction rules in a recursive definition do, too, a result that is analogous to, in fact a generalization of, Gentzen's Cut Elimination Theorem. In Section 6 lastly it is shown that the G_1 and G_2 functions can be construed, respectively, as absolute and relative probability functions in Carnap's and Popper's sense.

Suppose $\mathcal{L}$ is a language that has only *finitely many* statements, a restriction which, as indicated earlier, will be lifted from Section 4 on; and understand by a truth-value assignment α for $\mathcal{L}$ any result of assigning either 0 (i.e., False) or 1 (i.e., True) to each statement of $\mathcal{L}$. Then, w will be for us a *weight* function with the truth-value assignments for $\mathcal{L}$ as its arguments and such that:

W1. $0 \leq w(\alpha)$ for every α, and

W2. $\sum_\alpha w(\alpha) = 1$, where $\sum_\alpha w(\alpha)$ is the sum of the weights of the various truth-value assignments for $\mathcal{L}$.

A truth-value assignment for $\mathcal{L}$ will be called *possible* if the statements of $\mathcal{L}$ to which it assigns 1 can all be true and, simultaneously, those to which it assigns 0 can all be false; otherwise it will be called *impossible*. And when—as at this point—$\mathcal{L}$ is a finite language, all and only the impossible truth-value assignments for $\mathcal{L}$ can have weight 0, the possible ones in contrast having positive weights which reflect, if you will, their respective *likelihood*. Given the notion of an impossible truth-value assignment for $\mathcal{L}$ formulation (i) of the account of logical consequence in paragraph one above can be restated as follows: X has Y as a logical consequence, for short,

$$X \models Y,$$

if any truth-value assignment for $\mathcal{L}$ that assigns 1 to all the members of X and 0 to all those of Y is an impossible one, i.e., if

$$\sum\{w(\alpha) : X \subseteq T_\alpha \text{ and } Y \subseteq F_\alpha\} = 0,$$

where $T_\alpha = \{A \in \mathcal{L} : \alpha(A) = 1\}$ and $F_\alpha = \{A \in \mathcal{L} : \alpha(A) = 0\}$.[2]

We shall concurrently be interested in a second, *implicit*, characterization of logical consequence, one which is an adaptation of Gentzen's own rules of derivability to suit sets rather than sequences of statements. The following three are versions of his so-called *structural rules*, with '$\models$,' the standard symbol for logical consequence, in place of his arrow '$\rightarrow$':

R1. If $X \cap Y \neq \varnothing$, then $X \models Y$ (Overlap)

R2. If $X \models Y$, then $\{A\} \cup X \models Y$ and $X \models Y \cup \{A\}$ (Thinning)

R3. If $X \models Y \cup \{A\}$ and $\{A\} \cup X \models Y$, then $X \models Y$ (Cut).

Any relation satisfying these three constraints will constitute for us a consequence relation, and raised with respect to such a relation is the question: "Can it be so generalized as to become a matter of degree, measurable by a number in the interval $[0, 1]$ (end-points included)?"

2 A generalization of $\models$ whereby the logical relationship between sets X and Y becomes "a matter of degree" would make use of a binary function G with values in the above mentioned interval $[0, 1]$ and such that $G(X, Y) = 1$ if and only if $X \models Y$. The first characterization of $\models$ in Section 1 provides the best starting point for the generalization.

It should be a simple matter to calculate $G(X, Y)$ in terms of the weights that we presumed in that section to be assigned to the various truth-value assignments for $\mathcal{L}$. There are two different ways, it quickly appears, of doing the calculation. The *first* exploits the formulation (i) of the consequence relation: "It is impossible for all the statements in X to be true and, simultaneously, all those in Y to be false." The formulation suggests that in calculating $G(X, Y)$ we add up the weights of the truth-value assignments for $\mathcal{L}$ in which either not all the statements in X are true or else not all those in Y are false. The *second* exploits formulation (ii) of the consequence relation: "If all the statements in X are true, then at least one statement in Y must be true." This formulation suggests that to calculate $G(X, Y)$ is to answer the question: "To exactly which degree is it possible to find a member of Y that is true in a truth-value assignment for $\mathcal{L}$ in which all the members of X are true?"

We pursue both alternatives, and for clarity's sake we shall write '$G_1(X, Y)$' when attending to the first and '$G_2(X, Y)$' when attending to the second. $G_1(X, Y)$ is calculated by adding up the weights of all the truth-value assignments for $\mathcal{L}$ in which either some member of X is false or else some member of Y is true. This means of course leaving out of the sum the weights of just those truth-value assignments in which all the statements in X are true and all those in Y are false. It proves convenient to give a label of its own, 'F_1', to the sum of the latter weights. So

1. $F_1(X, Y) = \sum\{w(\alpha) : X \subseteq T_\alpha \text{ and } Y \subseteq F_\alpha\}$,

and hence

2. $G_1(X, Y) = 1 - F_1(X, Y)$,

which of course yields

3. $F_1(X, Y) = 1 - G_1(X, Y)$.

For a given finite language $\mathcal{L}$, each and every weight w thus determines a binary function F_1 and a binary one G_1, $G_1(X, Y)$ measuring the extent to which X (understood conjunctively) has Y (understood disjunctively) as a consequence, and $F_1(X, Y)$ measuring the extent to which X fails to do so. Note incidentally that $G_1(X, Y) = 0$ if and only if $\varnothing \models \{A\}$ for every A in X and $\varnothing \models \{\neg B\}$ for every B in Y, whereas $G_2(X, Y) = 0$ if and only if $X \models \{\neg B\}$ for every B in Y.

The various functions G_1 on a language $\mathcal{L}$ can also be specified implicitly by placing on G_1 these three constraints:

$G_1$1. $G_1(X, Y) \leq 1$
$G_1$2. $G_1(\varnothing, \varnothing) = 0$
$G_1$3. $F_1(X, Y \cup \{A\}) + F_1(\{A\} \cup X, Y) = F_1(X, Y)$.

In $G_1$3, a constraint reminiscent of Gentzen's *Cut*, 'F_1' can in view of (3) be thought of as short for '$1 - G_1$'. So appearances to the contrary notwithstanding, $G_1$3 is a constraint on G_1.[3]

Theorems 2.1 and 2.5 below show that the present characterization of degrees of logical consequence is equivalent to the one using (1) – (2) in the preceding paragraph. And Theorem 2.6 shows that the implicit characterization of logical consequence in Section 1 utilizing constraints R1–R3 issues via (3) from constraints $G_1$1–$G_1$3, this when Y being a logical consequence of X is defined as $G_1(X, Y) = 1$.

Theorem 2.1 *Let F_1 be defined as in (1), i.e.,*

$$F_1(X, Y) = \sum\{w(\alpha) : X \subseteq T_\alpha \text{ and } Y \subseteq F_\alpha\},$$

and G_1 as in (2). Then G_1 meets constraints $G_1$1–$G_1$3.

Proof: (i) G_1 meets $G_1$1 by W_1 and the definition of G_1.

(ii) $\sum\{w(\alpha) : \varnothing \subseteq T_\alpha$ and $\varnothing \subseteq F_\alpha\} = 1$ by W_2. So $F_1(\varnothing, \varnothing) = 1$ by the definition of F_1. So G_1 meets $G_1$2.

(iii) For any α, $X \subseteq T_\alpha$ and $Y \subseteq F_\alpha$ if and only if either $X \subseteq T_\alpha$ and $Y \cup \{A\} \subseteq F_\alpha$ or $\{A\} \cup X \subseteq T_\alpha$ and $Y \subseteq F_\alpha$. So G_1 meets $G_1$3. □

In Lemmas 2.2–2.4 we presume G_1 to meet constraints $G_1$1–$G_1$3.

Lemma 2.2 *If $X \cap Y \neq \varnothing$, then $F_1(X, Y) = 0$ and $G_1(X, Y) = 1$.*

Proof: Suppose $X \cap Y \neq \varnothing$, and let A belong to both X and Y. Then $X = \{A\} \cup X$ and $Y = Y \cup \{A\}$. Hence $F_1(X, Y) + F_1(X, Y) = F_1(X, Y)$ by $G_1$3. Hence $F_1(X, Y) = 0$ and $G_1(X, Y) = 1$. □

The next lemma amounts to a generalization of constraint $G_1$3.

Lemma 2.3 *(a) $F_1(X, Y) = \sum\{F_1(Z' \cup X, Y \cup (Z - Z')) : Z' \subseteq Z\}$;*
(b) $F_1(X, Y) = \sum\{F_1(T_\alpha, F_\alpha) : X \subseteq T_\alpha$ and $Y \subseteq F_\alpha\}$.

Proof: (a) By induction on the cardinality n of Z.
Basis: $n = 0$. Then (a) holds trivially.
Inductive Step: $n > 0$. Let A be a member of Z and Z^* be $Z - \{A\}$. Then $F_1(X, Y) = F_1(X, Y \cup \{A\}) + F_1(\{A\} \cup X, Y)$ by $G_1$3. But $F_1(X, Y \cup \{A\}) = \sum\{F_1(Z' \cup X, Y \cup \{A\} \cup (Z^* - Z')) : Z' \subseteq Z^*\}$ and $F_1(\{A\} \cup X, Y) = \sum\{F_1(Z' \cup \{A\} \cup X, Y \cup (Z^* - Z')) : Z' \subseteq Z^*\}$ by the inductive hypothesis. So (a).

(b) Let Z in (a) be $\mathcal{L}$. Then $F_1(X, Y) = \sum_\alpha F_1(T_\alpha \cup X, Y \cup F_\alpha)$. But $(T_\alpha \cup X) \cap (Y \cup F_\alpha) \neq \varnothing$ if and only if $X \subseteq T_\alpha$ and $Y \subseteq F_\alpha$. Hence (b) by Lemma 2.2. □

Lemma 2.4 *(a) $G_1(X, Y) \leq G_1(\{A\} \cup X, Y)$ and $G_1(X, Y) \leq G_1(X, Y \cup \{A\})$;*
(b) $G_1(X, Y) \leq G_1(Z \cup X, Y)$ and $G_1(X, Y) \leq G_1(X, Y \cup Z)$;
(c) $0 \leq G_1(X, Y)$.

Proof: (a) $F_1(\{A\} \cup X, Y) \leq F_1(X, Y)$ and $F_1(X, Y \cup \{A\}) \leq F_1(X, Y)$ by $G_1$3 and $G_1$1. So (a) by (2).

(b) by repeated use of (a).

(c) $G_1(\varnothing, \varnothing) = 0$ by $G_1$2. Hence $0 \leq G_1(X, Y)$ by (b). □

Theorem 2.5 *If G_1 meets constraints $G_1$1–$G_1$3, then there exists a weight function w such that $F_1(X, Y) = \sum\{w(\alpha) : X \subseteq T_\alpha$ and $Y \subseteq F_\alpha\}$.*

Proof: Suppose G_1 meets constraints G_11–G_13, and let $w_{G_1}(\alpha) = F_1(T_\alpha, F_\alpha)$. Then $w_{G_1}(\alpha) \geq 0$ by G_11. And $\sum_\alpha w_{G_1}(\alpha)$ equals $\sum_\alpha F_1(T_\alpha, F_\alpha)$, which itself equals $\sum\{F_1(T_\alpha, F_\alpha) : \varnothing \subseteq T_\alpha \text{ and } \varnothing \subseteq F_\alpha\}$. So $\sum_\alpha w_{G_1}(\alpha) = F(\varnothing, \varnothing)$ by Lemma 2.3(b), and hence $\sum_\alpha w_{G_1}(\alpha) = 1$ by G_12 and statement (2) above. So w_{G_1} is a weight function. Now let $F_1'(X, Y)$ be $\sum\{w_{G_1}(\alpha) : X \subseteq T_\alpha \text{ and} Y \subseteq F_\alpha\}$. Then $F_1'(X, Y)$ equals $\sum\{F(T_\alpha, F_\alpha) : X \subseteq T_\alpha \text{ and } Y \subseteq F_\alpha\}$, and hence $F_1(X, Y)$ by Lemma 2.3(b) again. So Theorem 2.5. □

Theorems 2.1 and 2.5 thus guarantee that the characterizations (1) and (2) of the degrees of logical consequence in terms of weight functions and that in terms of constraints G_11–G_13 are equivalent.

As for our second task, let $\models_{G_1}$ be this binary relation on subsets of $\mathcal{L}$:

$$X \models_{G_1} Y \text{ if and only if } G_1(X, Y) = 1,$$

or, equivalently,

$$X \models_{G_1} Y \text{ if and only if } F_1(X, Y) = 0.$$

Theorem 2.6 *If G_1 meets constraints G_11–G_13, then the relation $\models_{G_1}$ satisfies R1–R3.*

Proof: (i) Suppose $X \cap Y \neq \varnothing$. Then $G_1(X, Y) = 1$ by Lemma 2.2, i.e., $X \models_{G_1} Y$. Hence $\models_{G_1}$ meets *Overlap*. (ii) Suppose $X \models_{G_1} Y$, i.e., $G_1(X, Y) = 1$. Then both $G_1(\{A\} \cup X, Y) \geq 1$ and $G_1(X, Y \cup \{A\}) \geq 1$ by Lemma 2.4(a). Hence $G_1(\{A\} \cup X, Y) = 1$ and $G_1(X, Y \cup \{A\}) = 1$ by G_11. Hence $\models_{G_1}$ meets *Thinning*. (iii) Suppose $X \models_{G_1} Y \cup \{A\}$ and $\{A\} \cup X \models_{G_1} Y$, i.e., $F_1(X, Y \cup \{A\}) = 0$ and $F_1(\{A\} \cup X, Y) = 0$. Then $F_1(X, Y) = 0$ by G_13. Hence $X \models_{G_1} Y$. Hence $\models_{G_1}$ meets *Cut*. □

3 Turning now to formulation (ii) of the consequence relation in paragraph one of Section 1, hence to the second way described in Section 2 of calculating $G(X, Y)$ in terms of weights, let the function F_2 be defined thus, w being of course the weight function described in Section 1:

$$1'.\ F_2(X, Y) = \begin{cases} 0 \text{ if } \sum\{w(\alpha) : X \subseteq T_\alpha\} = 0 \\ \sum\{w(\alpha) : X \subseteq T_\alpha \text{ and } Y \subseteq F_\alpha\} / \sum\{w(\alpha) : X \subseteq T_\alpha\} \text{ otherwise,} \end{cases}$$

and G_2 defined thus:

$$2'.\ G_2(X, Y) = 1 - F_2(X, Y),$$

which of course yields:

$$3'.\ F_2(X, Y) = 1 - G_2(X, Y).$$

The resulting functions G_2 on $\mathcal{L}$ can also be specified *implicitly* by placing on them these five constraints:

G_21. $0 \leq G_2(X, Y)$
G_22. $G_2(X, Y) \leq 1$
G_23. $G_2(\varnothing, \varnothing) = 0$
G_24. $F_2(X, Y \cup \{A\}) + F_2(\{A\} \cup X, Y) \times G_2(X, \{A\}) = F_2(X, Y)$
G_25. If $G_2(X, \{A\}) = 0$, then $G_2(\{A\} \cup X, \varnothing) = 1$.

In $G_2 4$, a constraint reminiscent like $G_1 3$ of Gentzen's *Cut*, 'F_2' can in view of (3′) be thought of as short for '$1 - G_2$'. So $G_2 4$ is a constraint on G_2, the way $G_1 3$ was a constraint on G_1.[4]

Proceeding as in Section 2, we go on to show that the characterization of degrees of logical consequence using constraints $G_2 1$–$G_2 5$ is equivalent to the one using (1′)–(2′) in the preceding paragraph (= Theorems 3.2 and 3.12 below). We then show that the implicit characterization of logical consequence in Section 1 that utilizes R1–R3 issues via (3′) from constraints $G_2 1$–$G_2 5$, this when Y being a logical consequence of X is defined as $G_2(X, Y) = 1$ (= Theorem 3.13 below). Accomplishing the second task will be a lengthy affair, one that will call in particular for an auxiliary function, the function H introduced after Lemma 3.7.

Lemma 3.1 *Let F_2 and G_2 be defined as in (1′) and (2′), and let $\sum\{w(\alpha) : X \subseteq T_\alpha\} \neq 0$. Then $G_2(X, \{A\}) = \sum\{w(\alpha) : \{A\} \cup X \subseteq T_\alpha\} / \sum\{w(\alpha) : X \subseteq T_\alpha\}$.*

Proof: Since $\sum\{w(\alpha) : X \subseteq T_\alpha\} \neq 0$, $F_2(X, \{A\}) = \sum\{w(\alpha) : X \subseteq T_\alpha \text{ and } A \in F_\alpha\} / \sum\{w(\alpha) : X \subseteq T_\alpha\}$. So $G_2(X, \{A\}) = [\sum\{w(\alpha) : X \subseteq T_\alpha\} - \sum\{w(\alpha) : X \subseteq T_\alpha \text{ and } A \in F_\alpha\}] / \sum\{w(\alpha) : X \subseteq T_\alpha\}$. So $G_2(X, \{A\})$ equals $\sum\{w(\alpha) : X \subseteq T_\alpha \text{ and } A \notin F_\alpha\} / \sum\{w(\alpha) : X \subseteq T_\alpha\}$, which equals $\sum\{w(\alpha) : \{A\} \cup X \subseteq T_\alpha\} / \sum\{w(\alpha) : X \subseteq T_\alpha\}$. □

Theorem 3.2 *Let F_2 and G_2 be defined as in (1′) and (2′). Then G_2 meets constraints $G_2 1$–$G_2 5$.*

Proof: (i) That G_2 meets $G_2 1$, $G_2 2$, and $G_2 3$ is obvious from (1′) and (2′).

(ii) For any a, $X \subseteq T_\alpha$ and $Y \subseteq F_\alpha$ if and only if either $\{A\} \cup X \subseteq T_\alpha$ and $Y \subseteq F_\alpha$ or $X \subseteq T_\alpha$ and $Y \cup \{A\} \subseteq F_\alpha$. So assume first that $\sum\{w(\alpha) : X \subseteq T_\alpha\} = 0$. Then $\sum\{w(\alpha) : \{A\} \cup X \subseteq T_\alpha\} = 0$. Hence all of $F_2(X, Y \cup \{A\})$, $F_2(\{A\} \cup X, Y)$, and $F_2(X, Y)$ equal 0, and G_2 meets $G_2 4$. Assume next that $\sum\{w(\alpha) : X \subseteq T_\alpha\} \neq 0$. Then, owing to Lemma 3.1 in the first step,

$$
\begin{aligned}
&F_2(X, Y \cup \{A\}) + F_2(\{A\} \cup X, Y) \times G_2(X, \{A\}) = \\
&\quad = \sum\{w(\alpha) : X \subseteq T_\alpha \text{ and } Y \cup \{A\} \subseteq F_\alpha\} / \sum\{w(\alpha) : X \subseteq T_\alpha\} + \\
&\qquad + \Big[\sum\{w(\alpha) : \{A\} \cup X \subseteq T_\alpha \text{ and } Y \subseteq F_\alpha\} / \sum\{w(\alpha) : \{A\} \cup X \subseteq T_\alpha\}\Big] \times \\
&\qquad \times \Big[\sum\{w(\alpha) : \{A\} \cup X \subseteq T_\alpha\} / \sum\{w(\alpha) : X \subseteq T_\alpha\}\Big] = \\
&\quad = \Big[\sum\{w(\alpha) : X \subseteq T_\alpha \text{ and } Y \cup \{A\} \subseteq F_\alpha\} + \\
&\qquad + \sum\{w(\alpha) : \{A\} \cup X \subseteq T_\alpha \text{ and } Y \subseteq F_\alpha\}\Big] / \sum\{w(\alpha) : X \subseteq T_\alpha\} = \\
&\quad = \sum\{w(\alpha) : X \subseteq T_\alpha \text{ and } Y \subseteq F_\alpha\} / \sum\{w(\alpha) : X \subseteq T_\alpha\} = \\
&\quad = F_2(X, Y).
\end{aligned}
$$

Hence G_2 meets $G_2 4$.

(iii) Suppose $G_2(X, \{A\}) = 0$. Then $\sum\{w(\alpha) : X \subseteq T_\alpha\} \neq 0$. So $\sum\{w(\alpha) : \{A\} \cup X \subseteq T_\alpha\} = 0$ by Lemma 3.1. So $F_2(\{A\} \cup X, \varnothing) = 0$, and consequently $G_2(\{A\} \cup X, \varnothing) = 1$ by (2′). So G_2 meets $G_2 5$. □

In Lemmas 3.3–3.11, which will yield Theorem 3.12, we presume G_2 to meet constraints $G_2 1$–$G_2 5$.

Lemma 3.3 *(a)* $F_2(X, Y \cup \{A\}) \leq F_2(X, Y)$ *and* $G_2(X, Y) \leq G_2(X, Y \cup \{A\})$*;*
(b) $F_2(X, Y \cup Z) \leq F_2(X, Y)$ *and* $G_2(X, Y) \leq G_2(X, Y \cup Z)$*;*
(c) If $G_2(X, Y) = 1$*, then* $G_2(X, Y \cup Z) = 1$ *(i.e., if* $F_2(X, Y) = 0$*, then* $F_2(X, Y \cup Z) = 0$*).*

Proof: (a) is by $G_2 4$, $G_2 1$, and $G_2 2$; (b) is by repeated use of (a); and (c) is by (b) and $G_2 2$. □

Lemma 3.4 *If* $X \cap Y \neq \varnothing$*, then* $G_2(X, Y) = 1$.

Proof: With A a member of both X and Y, $F_2(X, Y) + F_2(X, Y) \times G_2(X, \{A\}) = F(X, Y)$ by $G_2 4$. So either $F_2(X, Y) = 0$ or $G_2(X, \{A\}) = 0$. But in the latter case $G_2(X, \varnothing) = 1$ by $G_2 5$, and hence $G_2(X, Y) = 1$ by Lemma 3.3(c). □

Lemma 3.5 *(a) If* $F_2(X, Y) = 0$*, then* $F_2(\{A\} \cup X, Y) = 0$*;*
(b) If $G_2(X, Y) = 1$*, then* $G_2(Z \cup X, Y) = 1$ *(i.e., if* $F_2(X, Y) = 0$*, then* $F_2(Z \cup X, Y) = 0$*).*

Proof: (a) Suppose $F_2(X, Y) = 0$. Then either $F_2(\{A\} \cup X, Y) = 0$ or $G_2(X, \{A\}) = 0$ by $G_2 4$ and $G_2 2$. But in the latter case $G_2(\{A\} \cup X, \varnothing) = 1$ by $G_2 5$, hence $F_2(\{A\} \cup X, \varnothing) = 0$, and hence $F_2(\{A\} \cup X, Y) = 0$ by Lemma 3.3(c). So $F_2(\{A\} \cup X, Y) = 0$ in either case. (b) is by repeated use of (a). □

Lemma 3.6 *(a)* $G_2(X, \varnothing)$ *equals either 0 or 1.*
(b) If $G_2(X, \{A\}) \neq 1$*, then* $G_2(X, \varnothing) = 0$*. (If* $F_2(X, \{A\}) \neq 0$*, then* $F_2(X, \varnothing) = 1$*.)*

Proof: (a) is by induction on the cardinality of X. If $X = \varnothing$, then (a) by $G_2 3$. Assume then that $G_2(X, \varnothing) = 0$ or $G_2(X, \varnothing) = 1$, and let A be a statement not in X. $G_2(X, \varnothing) = G_2(\{A\} \cup X, \varnothing) \times G_2(X, \{A\})$ by $G_2 4$. So if $G_2(X, \varnothing) = 1$, then $G_2(\{A\} \cup X, \varnothing) = 1$, whereas if $G_2(X, \varnothing) = 0$, then $G_2(\{A\} \cup X, \varnothing) = 0$ or $G_2(X, \{A\}) = 0$, and in the latter case $G_2(\{A\} \cup X, \varnothing) = 1$ by $G_2 5$.

(b) Suppose $G_2(X, \varnothing) \neq 0$. Then $G_2(X, \varnothing) = 1$ by (a), and hence $G(X, \{A\}) = 1$ by Lemma 3.3(c). □

Lemma 3.7 *(a)* $G_2(X, \{A\}) \times G_2(\{A\} \cup X, \{B\}) = G_2(X, \{B\}) \times G_2(\{B\} \cup X, \{A\})$*;*
(b) Let $A_1, \ldots, A_n$ *be the members of* Y *in some order or other, let* Y_0 *be* $\varnothing$*, and let* Y_i *be* $Y_{i-1} \cup \{A_i\}$ *for* $i = 1, \ldots, n$*. Then* $\prod_{i=1}^{n} G_2(X \cup Y_{i-1}, \{Ai\})$ *is independent of the order of the members of* Y*.*

Proof: (a) By $G_2 4$, $1 - F_2(X, \{A\}) - F_2(X, \{B\}) + F_2(X, \{B\} \cup \{A\})$ equals $[1 - F_2(X, \{A\})] \times [1 - F_2(\{A\} \cup X, \{B\})]$ as well as $[1 - F_2(X, \{B\})] \times [1 - F_2(\{B\} \cup X, \{A\})]$. So (a). (b) is by (a). □

Needed at this point to abbreviate matters is an auxiliary function $H(X, Y)$, whose definition is sanctioned by Lemma 3.7(b).

$$H(X, Y) = \begin{cases} \prod_{i=1}^{n} G_2(X \cup Y_{i-1}, \{A_i\}) \text{ if } Y \neq \varnothing \\ 1 \text{ if } Y = \varnothing.^5 \end{cases}$$

The following lemma amounts to a generalization of constraint $G_2 4$.

Lemma 3.8 $F_2(X, Y) = \sum\{F_2(Z' \cup X, Y \cup (Z - Z')) \times H(X, Z') : Z' \subseteq Z\}$.

Proof: By induction on the cardinality n of Z.
Basis: $n = 0$. $F_2(X, Y) = F_2(X, Y) \times H(X, \varnothing)$, since $H(X, \varnothing) = 1$ by the definition of H.
Inductive Step: $n > 0$. Let A be a member of Z and let $Z^* = Z - \{A\}$. Then

$$F_2(X, Y) = F_2(X, Y \cup \{A\}) + F_2(\{A\} \cup X, Y) \times G_2(X, \{A\})$$

by $G_2 4$. But

$$F_2(X, Y \cup \{A\}) = \sum\{F_2(X \cup Z', Y \cup \{A\} \cup (Z^* - Z')) \times H(X, Z') : Z' \subseteq Z^*\}$$

and

$$\begin{aligned} F_2(\{A\} \cup X, Y) \times G_2(X, \{A\}) \quad = \quad & \sum\{F_2(\{A\} \cup X \cup Z', Y \cup (Z^* - Z')) \times \\ & \times H(\{A\} \cup X, Z') \times G_2(X, \{A\}) : Z' \subseteq Z^*\} \end{aligned}$$

by the inductive hypothesis. But $H(\{A\} \cup X, Z') \times G_2(X, \{A\}) = H(X, Z')$. Hence Lemma 3.8. □

Lemma 3.9 *If $H(\varnothing, X) = 0$, then $G(X, \varnothing) = 1$.*

Proof: Let $A_1, \ldots, A_n$ be the members of X in any order one pleases, let X_0 be $\varnothing$, and let X_i be $X_{i-1} \cup \{A_i\}$ for $i = 1, \ldots, n$. Suppose $H(\varnothing, X) = 0$. Then $G_2(X_{i-1}, \{A_i\}) = 0$ for some i $(1 \leq i \leq n)$. So $G_2(X_i, \varnothing) = 1$ by $G_2 5$, and hence $G_2(X, \varnothing) = 1$ by Lemma 3.5(b). □

Lemma 3.10 *Let $F^*(X, Y)$ be $F_2(X, Y) \times H(\varnothing, X)$, and let $G^*(X, Y)$ be $1 - F^*(X, Y)$. Then G^* meets constraints $G_1 1$–$G_1 3$.*

Proof: (i) $G_1 1$ by $G_2 1$ and $G_2 2$. (ii) $G_1 2$ by $G_2 3$. (iii) Since $H(\varnothing, X \cup \{A\}) = H(\varnothing, X) \times G(X, \{A\})$, $F^*(X, Y \cup \{A\}) + F^*(\{A\} \cup X, Y)$ equals $[F_2(X, Y \cup \{A\}) + F_2(\{A\} \cup X, Y) \times G_2(X, \{A\})] \times H(\varnothing, X)$, and hence $F_2(X, Y) \times H(\varnothing, X)$ by $G_2 4$. Hence $G_1 3$. □

Lemma 3.11 *$F_2(X, Y) = F^*(X, Y)/F^*(X, \varnothing)$, if $F^*(X, \varnothing) \neq 0$, otherwise $F_2(X, Y) = 0$.*

Proof: Assume $F^*(X, \varnothing) = 0$. Then either $F_2(X, \varnothing) = 0$ or $H(\varnothing, X) = 0$. But in the latter case $F_2(X, \varnothing) = 0$ as well by Lemma 3.9. Hence $F_2(X, Y) = 0$ by Lemma 3.3(c). Now assume $F^*(X, \varnothing) \neq 0$. Then $F_2(X, \varnothing) \neq 0$, and hence $F_2(X, \varnothing) = 1$ by Lemma 3.6(a). Hence $F^*(X, \varnothing) = H(\varnothing, X)$, and thus $F_2(X, Y) = F^*(X, Y)/F^*(X, \varnothing)$. □

Theorem 3.12 *If G_2 meets constraints $G_2 1$–$G_2 5$, then there exists a function w such that*

$$F_2(X, Y) = \begin{cases} 0 \textit{ if } \sum\{w(\alpha) : X \subseteq T_\alpha\} = 0 \\ \sum\{w(\alpha) : X \subseteq T_\alpha \textit{ and } Y \subseteq F_\alpha\} / \sum\{w(\alpha) : X \subseteq T_\alpha\} \textit{ otherwise.} \end{cases}$$

Proof: Suppose G_2 meets G_21–G_25, and let $w_{G_2}(\alpha)$ be $F^*(T_\alpha, F_\alpha)$. Then w_{G_2} is a weight function by Theorem 2.5. Now let $F_2'(X, Y)$ be 0 if $\sum\{w_{G_2}(\alpha) : X \subseteq T_\alpha\} = 0$ and $\sum\{w_{G_2}(a) : X \subseteq T_\alpha \text{ and } Y \subseteq F_\alpha\}/\sum\{w_{G_2}(a) : X \subseteq T_\alpha\}$ otherwise. Then $F_2'(X, Y)$ equals $\sum\{F^*(T_\alpha, F_\alpha) : X \subseteq T_\alpha \text{ and } Y \subseteq F_\alpha\}/\sum\{F^*(T_\alpha, F_\alpha) : X \subseteq T_\alpha\}$ by the definition of w_{G_2}. So $F_2'(X, Y) = F^*(X, Y)/F^*(X, \varnothing)$ by the same reasoning as in Theorem 2.5. Hence $F_2'(X, Y) = F_2(X, Y)$ by Lemma 3.11. □

So Theorems 3.2 and 3.12 guarantee that the characterization (1′) and (2′) of the degrees of logical consequence in terms of weight functions and that in terms of constraints G_21–G_25 are equivalent. And since (i) by Theorem 3.12 the functions G_2 and the weight functions w match one-to-one, and (ii) by Theorem 2.5 the functions G_1 and the functions w also do, the G_1 functions and the G_2 functions stand in one-to-one correspondence.

As for our second task, let $\models_{G_2}$ be this relation on pairs of statements of $\mathcal{L}$: $X \models_{G_2} Y$ if and only if $G_2(X, Y) = 1$, or equivalently, $X \models_{G_2} Y$ if and only if $F_2(X, Y) = 0$.

Theorem 3.13 *If G_2 meets G_21–G_25, then the relation $\models_{G_2}$ satisfies R1–R3.*

Proof: (i) $\models_{G_2}$ meets *Overlap* by Lemma 3.4. (ii) $\models_{G_2}$ meets *Thinning* by Lemma 3.3(a) and Lemma 3.5(a). (iii) Suppose $X \models_{G_2} Y \cup \{A\}$ and $\{A\} \cup X \models_{G_2} Y$, i.e., $F_2(X, Y \cup \{A\}) = 0$ and $F_2(\{A\} \cup X, Y) = 0$. Then $F_2(X, Y) = 0$ by G_24. Hence $X \models_{G_2} Y$. So $\models_{G_2}$ meets *Cut*. □

4 As we turn to languages with infinitely many statements, we extend the counterparts R1–R3 of Gentzen's structural rules to R1′–R3′, where the earlier sets X and Y may now be infinite as well as finite and so may the new sets Z and Z':

R1′. If $X \cap Y \neq \varnothing$, then $X \models Y$

R2′. If $X \models Y$, then $Z \cup X \models Y$ and $X \models Y \cup Z$

R3′. If $Z' \cup X \models Y \cup (Z - Z')$ for every $Z' \subseteq Z$, then $X \models Y$.[6]

In Sections 1–3 we defined $\models_G$ (with the subscript 'G' here short for either of 'G_1' and 'G_2') in the following manner:

$$X \models_G Y \text{ if and only if } G(X, Y) = 1.$$

The definition clearly does not suit infinite sets, for, as indicated by Lemmas 2.4(b) and 3.3(b), $G(X, Y)$ will frequently equal 1 even though $X \not\models Y$. One alternative way of defining logical consequence is suggested by the class of compact consequence relations, relations that are fully determined by their restrictions to finite sets of statements. So we will deal with infinite languages but restrict ourselves to compact consequence relations and define $\models_G$ thus:

$$X \models_G Y \text{ if and only if } G(X', Y') = 1$$
$$\text{for some finite subset } X' \text{ of } X \text{ and some finite subset } Y' \text{ of } Y.$$

In the same spirit it is our next task to design an extra constraint to be placed on the G_1 consequence functions and one to be placed on the G_2 functions, constraints that will fix the values of these functions for infinite sets in terms of their values for finite ones. As regards the first functions, we have from Lemma 2.4(b)

$$G_1(X, Y) \leq G_1(X', Y') \text{ when } X \subseteq X' \text{ and } Y \subseteq Y',$$

which suggests this constraint:

$G_1$4. $G_1(X, Y) = \sup\{G_1(X', Y') : X' \subseteq_f X \text{ and } Y' \subseteq_f Y\}$,

where '$\subseteq_f$' is to be read 'is a finite subset of'. This is of course equivalent to

$$F_1(X, Y) = \inf\{F_1(X', Y') : X' \subseteq_f X \text{ and } Y' \subseteq_f Y\}.$$

As regards the second functions, we have from Lemma 3.11:

$$F_2(X, Y) = \begin{cases} F^*(X, Y)/F^*(X, \varnothing) \text{ if } F^*(X, \varnothing) \neq 0 \\ 0 \text{ otherwise,} \end{cases}$$

which together with Lemma 2.4 suggests

$G_2$6. $$F_2(X, Y) = \begin{cases} \inf\{F^*(X', Y') : X' \subseteq_f X \text{ and } Y' \subseteq_f Y\}/\inf\{F^*(X', \varnothing) : \\ \quad X' \subseteq_f X\} \text{ if } \inf\{F^*(X', \varnothing) : X' \subseteq_f X\} \neq 0 \\ 0 \text{ otherwise.}^7 \end{cases}$$

Incidentally, we could have avoided placing constraints $G_1$4 and $G_2$6 on the present G_1 and G_2 functions by extending the finitary functions from Sections 1 and 2 to infinitary ones with the help of definitions corresponding to $G_1$4 and $G_2$6. Theorem 4.1 (which we prove) and Theorem 4.2 (proof of which we leave to the reader) establish this.

Theorem 4.1 *Let G_1 be a function on the finite subsets of $\mathcal{L}$ meeting constraints $G_1$1–$G_1$3, and let $G'_1(X, Y) = \sup\{G_1(X', Y') : X' \subseteq_f X \text{ and } Y' \subseteq_f Y\}$. Then G'_1 meets $G_1$1–$G_1$3.*

Proof: That G'_1 meets $G_1$1 and $G_1$2 is trivial. As for $G_1$3, $F'_1(X, Y \cup \{A\}) + F'_1(\{A\} \cup X, Y) = \inf\{F_1(X', Y') : X' \subseteq_f X \text{ and } Y' \subseteq_f Y \cup \{A\}\} + \inf\{F_1(X', Y') : X' \subseteq_f \{A\} \cup X \text{ and } Y' \subseteq_f Y\}$, which by Lemma 2.4(a) equals $\inf\{F_1(X', Y' \cup \{A\}) : X' \subseteq_f X \text{ and } Y' \subseteq_f Y\} + \inf\{F_1(X' \cup \{A\}, Y') : X' \subseteq_f X \text{ and } Y' \subseteq_f Y\}$, which equals $\inf\{F_1(X', Y' \cup \{A\}) + F_1(X' \cup \{A\}, Y') : X' \subseteq_f X \text{ and } Y' \subseteq_f Y\}$, which by $G_1$3 equals $\inf\{F_1(X', Y') : X' \subseteq_f X \text{ and } Y' \subseteq_f Y\}$, which in turn equals $F'_1(X, Y)$. □

Theorem 4.2 *Let G_2 be a function on the finite subsets of $\mathcal{L}$ meeting constraints $G_2$1–$G_2$5, and let $G'_2(X, Y) = 1 - F'_2(X, Y)$, where*

$$F'_2(X, Y) = \begin{cases} \inf\{F^*(X', Y') : X' \subseteq_f X \textit{ and } Y' \subseteq_f Y\}/\inf\{F^*(X, \varnothing) : X' \subseteq_f X\} \\ \quad \textit{if } \inf\{F^*(X', \varnothing) : X' \subseteq_f X\} \neq 0 \\ 0 \textit{ otherwise}. \end{cases}$$

Then G'_2 meets $G_2$1–$G_2$5.

So the present G_1 functions and G_2 functions are *compact* functions in the broader sense that the value of G for any two sets X and Y depends exclusively upon the values of G for finite subsets of X and Y, and hence upon the values of G for finite sets. And thanks to constraints $G_1 4$ and $G_2 6$, and to the definitions of $\models_{G_1}$ and $\models_{G_2}$, the latter relations conform to R1′, R2′, and R3′. For the restrictions to finite sets of $\models_{G_1}$ and of $\models_{G_2}$ meet R1, R2, and R3, as we have established in Theorems 2.6 and 3.13. And proof that a compact relation of this kind satisfies R1′, R2′, and R3′ is to be found in Shoesmith and Smiley [10], p. 37. Hence these two theorems.

Theorem 4.3 *If G_1 meets $G_1 1$–$G_1 4$, then the relation $\models_{G_1}$ satisfies R1′–R3′.*

Theorem 4.4 *If G_2 meets $G_2 1$–$G_2 6$, then the relation $\models_{G_2}$ satisfies R1′–R3′.*

5 The previous section dealt with compact $\models$ and G in general. The present one will concern a particular language, the language $\mathcal{L}_0$ of propositional logic, with denumerably many atomic statements, '$\neg$' and '$\wedge$' as its primitive connectives, and '$\vee$', '$\supset$', and '$\equiv$' defined in terms of them in the customary manner. In Theorems 5.1–5.4 all sets of statements are presumed to be finite; but in view of Theorems 4.1 and 4.2 like results hold when the sets are infinite. As usual, the *complexity* $\mathrm{c}(A)$ of an atomic statement A is 0, that of a negation $\neg A$ is $\mathrm{c}(A)+1$, and that of a conjunction $A \wedge B$ is $\mathrm{c}(A)+\mathrm{c}(B)+1$; and the complexity $\mathrm{c}(\{A_1, A_2, \ldots, A_n\})$ of a finite set of statements is $\mathrm{c}(A_1)+\mathrm{c}(A_2)+\cdots+\mathrm{c}(A_n)$.

It is generally presumed in logic texts that the atomic statements from which the other statements under study are compounded are logically independent of one another. A more precise account of this matter is imperative here, according to which the members of a set Z of statements of a (finite or infinite) language $\mathcal{L}$ are said to be logically independent under a consequence function G (be it G_1 or G_2) if and only if, for any two disjoint subsets X and Y of Z, $G(X, Y) \neq 1$. The following constraint on G_1 thus requires of the atomic statements of $\mathcal{L}_0$ that they be logically independent under G_1.

Ind_1. If X and Y are sets of atomic statements and $X \cap Y = \varnothing$, then $G_1(X, Y) \neq 1$.[8]

Now for the statements of $\mathcal{L}_0$ that can be compounded from atomic statements by means of '$\neg$' and '$\wedge$'. To accommodate them, we place upon G_1 the following counterparts of Gentzen's *introduction rules (to the left and to the right) for '$\neg$' and '$\wedge$'*, to be known as *the G_1-constraints regarding '$\neg$' and '$\wedge$'*.

$\neg\mathrm{R}_1$. $F_1(X, Y \cup \{\neg A\}) = F_1(\{A\} \cup X, Y)$
$\neg\mathrm{L}_1$. $F_1(\{\neg A\} \cup X, Y) = F_1(X, Y \cup \{A\})$
$\wedge\mathrm{R}_1$. $F_1(X, Y \cup \{A \wedge B\}) = F_1(X, Y \cup \{A\}) + F_1(\{A\} \cup X, Y \cup \{B\})$
$\wedge\mathrm{L}_1$. $F_1(\{A \wedge B\} \cup X, Y) = F_1(\{A, B\} \cup X, Y)$.

Theorem 5.1 *Let the restriction of G_1 to sets of atomic statements of $\mathcal{L}_0$ meet $G_1 1$–$G_1 3$, and Ind_1, and let G_1 meet the G_1-constraints regarding '$\neg$' and '$\wedge$'. Then G_1 meets $G_1 1$–$G_1 3$.*

Proof: That G_1 meets $G_1 2$ is trivial, that it meets the other two constraints is shown by induction on $n = \mathrm{c}(A)+\mathrm{c}(X)+\mathrm{c}(Y)$.
Basis: $n = 0$. Then A, the members of X, and those of Y are atomic statements, and $G_1 2$ and $G_1 3$ are met by assumption.

Inductive step: $n > 0$. (i) That $G_1 1$ is met is immediate by inspection of the constraints for '$\neg$' and '$\wedge$' and the inductive hypothesis.

(ii) For $G_1 3$ there are 6 (not necessarily mutually exclusive) subcases to be considered, namely $\neg B$ being A itself, or a member of X, or one of Y, and $A \wedge B$ likewise being A itself, or a member of X, or one of Y.

Case 1:

Subcase 1 ($\neg B$ is A): Then $F_1(X, Y \cup \{A\})$ equals $F_1(\{B\} \cup X, Y)$ (by $\neg \mathrm{R}_1$) and $F_1(\{A\} \cup X, Y) = F_1(X, Y \cup \{B\})$ (by $\neg \mathrm{L}_1$). Hence $F_1(X, Y \cup \{A\}) + F_1(\{A\} \cup X, Y) = F_1(X, Y)$ by the inductive hypothesis.

Subcase 2 ($\neg B \in X$): Let X' be $X - \{\neg B\}$. Then $F_1(X, Y \cup \{A\}) = F_1(X', Y \cup \{A, B\})$ and $F_1(\{A\} \cup X, Y) = F_1(\{A\} \cup X', Y \cup \{B\})$ (both by $\neg \mathrm{L}_1$). So $F_1(X, Y \cup \{A\}) + F_1(\{A\} \cup X, Y)$ equals $F_1(X', Y \cup \{B\})$ by the inductive hypothesis, and hence equals $F_1(X, Y)$ by $\neg \mathrm{L}_1$.

Subcase 3 ($\neg B \in Y$): Like Subcase 2, but using $\neg \mathrm{R}_1$ in place of $\neg \mathrm{L}_1$.

Case 2:

Subcase 1 ($B \wedge C$ is A): Then $F_1(X, Y \cup \{A\}) = F_1(X, Y \cup \{B\}) + F_1(\{B\} \cup X, Y \cup \{C\})$ (by $\wedge \mathrm{R}_1$) and $F_1(\{A\} \cup X, Y) = F_1(\{B, C\} \cup X, Y)$ (by $\wedge \mathrm{L}_1$). Hence $F_1(X, Y \cup \{A\}) + F_1(\{A\} \cup X, Y)$ equals $F_1(X, Y \cup \{B\}) + F_1(\{B\} \cup X, Y)$ by the inductive hypothesis, and hence equals $F_1(X, Y)$ by the inductive hypothesis again.

Subcases 2 ($B \wedge C \in X$) and 3 ($B \wedge C \in Y$) by similar reasoning. □

The foregoing theorem establishes that, given the constraints for '$\neg$' and '$\wedge$', G_1 is in effect recursively definable from its restriction to finite sets of atomic statements. And by virtue of Theorem 4.1 this result, which parallels Gentzen's Cut-Elimination theorem for propositional logic, includes the case where the arguments of G_1 are infinite sets of statements. Theorem 5.2 will establish that the G_1 functions in this section are indeed a generalization of the consequence relation for propositional logic. The consequence relation for propositional logic being compact, $\models_{G_1}$ is of course defined as in Section 4.

Theorem 5.2 *Let G_1 meet $G_1 1$–$G_1 3$, Ind$_1$, and the G_1-constraints regarding '$\neg$' and '$\wedge$'. Then, $X \models_{G_1} Y$ (i.e., $G_1(X, Y) = 1$ or, equivalently, $F_1(X, Y) = 0$) if and only if $X \models Y$.*

Proof: By induction on $n = \mathrm{c}(X) + \mathrm{c}(Y)$.
Basis: $n = 0$. Then X and Y are sets of atomic statements and $X \models Y$ if and only if $X \cap Y \neq \varnothing$. But if $X \cap Y \neq \varnothing$, then $G_1(X, Y) = 1$ by Lemma 2.2. And if $G_1(X, Y) = 1$, then $X \cap Y \neq \varnothing$ by Ind$_1$. So $X \models Y$ if and only if $G_1(X, Y) = 1$.
Inductive Step: $n > 0$.

Case 1:

Subcase 1 ($\neg A \in X$): Let X' be $X - \{\neg A\}$. Then by $\neg \mathrm{L}_1$ $F_1(X, Y) = 0$ if and only if $F_1(X', Y \cup \{A\}) = 0$, so by the inductive hypothesis if and only if $X' \models Y \cup \{A\}$, so if and only if $X \models Y$.

Subcase 2 ($\neg A \in Y$): Let Y' be $Y - \{\neg A\}$. Then by $\neg R_1$ $F_1(X, Y) = 0$ if and only if $F_1(\{A\} \cup X, Y') = 0$, so by the inductive hypothesis if and only if $\{A\} \cup X \models Y'$, so if and only if $X \models Y$.

Case 2:

Subcase 1 ($A \wedge B \in X$): Let X' be $X - \{A \wedge B\}$. Then by $\wedge L_1$ $F_1(X, Y) = 0$ if and only if $F_1(X' \cup \{A, B\}, Y) = 0$, so by the inductive hypothesis if and only if $X' \cup \{A, B\} \models Y$, so if and only if $X \models Y$.

Subcase 2 ($A \wedge B \in Y$): Let Y' be $Y - \{A \wedge B\}$. Suppose first $F_1(X, Y) = 0$. Then by $\wedge R_1$ $F_1(X, Y' \cup \{A\}) + F_1(\{A\} \cup X, Y' \cup \{B\}) = 0$. So $X \models Y' \cup \{A\}$ and $\{A\} \cup X \models Y' \cup \{B\}$ by $G_1 1$ and the inductive hypothesis. Hence $X \models Y' \cup \{A, B\}$ by *Thinning*, and $X \models Y' \cup \{B\}$ by *Cut*. Hence $X \models Y$. Suppose next $X \models Y$. Then $X \models Y' \cup \{A\}$ and $X \models Y' \cup \{B\}$. So $\{A\} \cup X \models Y' \cup \{B\}$ by *Thinning*. Hence $F_1(X, Y' \cup \{A\}) = 0$ and $F_1(\{A\} \cup X, Y' \cup \{B\}) = 0$ by the inductive hypothesis. So $F_1(X, Y) = 0$ by $\wedge R_1$. □

We next place upon G_2 counterparts of Ind_1 and the constraints above regarding '¬' and '∧'.

Ind_2. If X and Y are sets of atomic statements and $X \cap Y = \varnothing$, then $G_2(X, Y) \neq 1$.

$\neg R_2$. $F_2(X, Y \cup \{\neg A\}) = F_2(\{A\} \cup X, Y) \times G_2(X, \{A\})$

$\neg L_2$. $F_2(\{\neg A\} \cup X, Y) = \begin{cases} F_2(X, Y \cup \{A\})/F_2(X, \{A\}) \text{ if } F_2(X, \{A\}) \neq 0 \\ 0 \text{ otherwise} \end{cases}$

$\wedge R_2$. $F_2(X, Y \cup \{A \wedge B\}) = F_2(X, Y \cup \{A\}) + F_2(\{A\} \cup X, Y \cup \{B\}) \times G_2(X, \{A\})$

$\wedge L_2$. $F_2(\{A \wedge B\} \cup X, Y) = F_2(\{A, B\} \cup X, Y)$.

Theorem 5.3 *Let the restriction of G_2 to sets of atomic statements of $\mathcal{L}_0$ meet $G_2 1$–$G_2 5$, and Ind_2, and let G_2 meet the G_2-constraints regarding '¬' and '∧'. Then G_2 meets $G_2 1$–$G_2 5$.*

Proof: That G_2 meets $G_2 3$ is trivial, that it meets the other four constraints is shown by induction on $n = \text{c}(A) + \text{c}(X) + \text{c}(Y)$.
Basis: $n = 0$. Then A, the members of X, and those of Y are atomic statements, and $G_2 1$, $G_2 2$, $G_2 4$, and $G_2 5$ are met by assumption.
Inductive step: $n > 0$.

(i) That $G_2 1$ and $G_2 2$ are met is immediate by inspection of the constraints for connectives and the inductive hypothesis, except in the case of $\neg L_2$, where in addition the inductive hypothesis for $G_2 4$ has to be invoked so as to yield $F_2(X, Y \cup \{A\}) \leq F_2(X, \{A\})$.

(ii) For $G_2 4$ there are six (not necessarily mutually exclusive) subcases to be considered, namely $\neg B$ being A itself, or a member of X, or one of Y, and $B \wedge C$ likewise being A itself, or a member of X, or one of Y.

Case 1:

Subcase 1 ($\neg B$ *is* A): First assume $F_2(X, \{B\}) = 0$ (i.e., $G_2(X, \{B\}) = 1$). Then $F_2(X, Y \cup \{A\}) = F_2(\{B\} \cup X, Y)$ by $\neg R_2$ and $F_2(\{A\} \cup X, Y) = 0$ by $\neg L_1$. $F_2(X, Y \cup$

$\{B\}) = 0$ by Lemma 3.3(b), which uses the inductive hypothesis for $G_2 1$, $G_2 2$, and $G_2 4$. Hence $F_2(X, Y \cup \{A\}) + F_2(\{A\} \cup X, Y) \times G_2(X, \{A\})$ equals $F_2(\{B\} \cup X, Y)$. And, as $F_2(X, Y \cup \{B\}) = 0$, this equals $F_2(X, Y)$ by the inductive hypothesis for $G_2 4$. Assume next $F_2(X, \{B\}) \neq 0$, $F_2(X, Y \cup \{A\}) = F_2(\{B\} \cup X, Y) \times G_2(X, \{B\})$, $F_2(\{A\} \cup X, Y) = F_2(X, Y \cup \{B\})/F_2(X, \{B\})$, and $G_2(X, \{A\}) = 1 - [F_2(\{B\} \cup X, \varnothing) \times G_2(X, \{B\})]$. But since $F_2(X, \{B\}) \neq 0$, $F_2(X, \varnothing) = 1$ by Lemma 3.6(b), which uses the inductive hypothesis for $G_2 1$, $G_2 2$, $G_2 4$, and $G_2 5$. Hence $1 - [F_2(\{B\} \cup X, \varnothing) \times G_2(X, \{B\})] = F_2(X, \{B\})$ by the inductive hypothesis for $G_2 4$. Hence $F_2(X, Y \cup \{A\}) + F_2(\{A\} \cup X, Y) \times G_2(X, \{A\})$ equals $F_2(\{B\} \cup X, Y) \times G_2(X, \{B\}) + F_2(X, Y \cup \{B\})$, which equals $F_2(X, Y)$ by the inductive hypothesis for $G_2 4$.

Subcase 2 ($\neg B \in X$): Let X' be $X - \{\neg B\}$. Assume first that $F_2(X', \{B\}) = 0$, in which case $F_2(\{A\} \cup X', \{B\}) = 0$ as well by Lemma 3.3(b), which uses the inductive hypothesis for $G_2 1$, $G_2 2$, and $G_2 4$. Then $F_2(X, Y \cup \{A\})$, $F_2(\{A\} \cup X, Y)$, and $F_2(X, Y)$ all equal 0 by $\neg L_2$. So $G_2 4$. Assume next that $F_2(X', \{B\}) \neq 0$, but that $F_2(\{A\} \cup X', \{B\}) = 0$. Then $F_2(\{A\} \cup X', Y \cup \{B\}) = 0$ by Lemma 3.3(b), which uses the inductive hypothesis for $G_2 1$, $G_2 2$, and $G_2 4$. Then, by $\neg L_2$, $F_2(X, Y \cup \{A\}) = F_2(X', Y \cup \{A, B\})/F_2(X', \{B\})$ and $F_2(\{A\} \cup X, Y) = 0$. Hence $F_2(X, Y \cup \{A\}) + F_2(\{A\} \cup X, Y) \times G_2(X, \{A\})$ equals $F_2(X', Y \cup \{A, B\})/F_2(X', \{B\})$, which equals $[F_2(X', Y \cup \{A, B\}) + F_2(\{A\} \cup X', Y \cup \{B\}) \times G_2(X', \{A\})]/F_2(X', \{B\})$, which equals $F_2(X', Y \cup \{B\})/F_2(X', \{B\})$ by the inductive hypothesis, and hence $F_2(X, Y)$ by $\neg L_2$. Assume finally that $F_2(\{A\} \cup X', \{B\}) \neq 0$, and so $F_2(X', \{B\}) \neq 0$ as well, by Lemma 3.3(b), which uses the inductive hypothesis for $G_2 1$, $G_2 2$, and $G_2 4$. Then $F_2(X, Y \cup \{A\}) = F_2(X', Y \cup \{A, B\})/F_2(X', \{B\})$, $F_1(\{A\} \cup X, Y) = F_1(\{A\} \cup X', Y \cup \{B\})/F_2(\{A\} \cup X', \{B\})$, and $G_2(X, \{A\}) = 1 - [F_2(X', \{A, B\})/F_2(X', \{B\})]$ (all by $\neg L_2$). Then $G_2(X, \{A\})$ equals $[F_2(X', \{B\}) - F_2(X', \{A, B\})]/F_2(X', \{B\})$, which equals $[G_2(X, \{A\}) \times F_2(\{A\} \cup X', \{B\})]/F_2(X', \{B\})$ by the inductive hypothesis for $G_2 4$. Hence $F_2(X, Y \cup \{A\}) + F_2(\{A\} \cup X, Y) \times G_2(X, \{A\})$ equals $[F_2(X', Y \cup \{A, B\}) + F_1(\{A\} \cup X', Y \cup \{B\}) \times G_2(X, \{A\})]/F_2(X', \{B\})$, which equals $F_2(X', Y \cup \{B\})/F_2(X', \{B\})$ by the inductive hypothesis, and hence equals $F_2(X, Y)$ by $\neg L_2$.

Subcase 3 ($\neg B \in Y$): Let Y' be $Y - \{\neg B\}$. Then $F_2(X, Y \cup \{A\}) = F_2(\{B\} \cup X, Y' \cup \{A\}) \times G_2(X, \{B\})$ and $F_2(\{A\} \cup X, Y) = F_2(\{A, B\} \cup X, Y') \times G_2(\{A\} \cup X, \{B\})$ by $\neg R_2$. $G_2(\{A\} \cup X, \{B\}) \times G_2(X, \{A\}) = G_2(\{B\} \cup X, \{A\}) \times G_2(X, \{B\})$ by Lemma 3.7, which uses the inductive hypothesis for $G_2 4$. So $F_2(X, Y \cup \{A\}) + F_2(\{A\} \cup X, Y) \times G_2(X, \{A\})$ equals $[F_2(\{B\} \cup X, Y' \cup \{A\}) + F_2(\{A, B\} \cup X, Y') \times G_2(\{B\} \cup X, \{A\})] \times G_2(X, \{B\})$, so by the inductive hypothesis $F_2(\{B\} \cup X, Y') \times G_2(X, \{B\})$, and hence $F_2(X, Y)$ by $\neg R_2$.

Case 2:

Subcase 1 ($B \wedge C$ is A): Then $F_2(X, Y \cup \{A\}) = F_2(X, Y \cup \{B\}) + F_2(\{B\} \cup X, Y \cup \{C\}) \times G_2(X, \{B\})$ (by $\wedge R_2$) and $F_2(\{A\} \cup X, Y) = F_2(\{B, C\} \cup X, Y)$ (by $\wedge L_2$), and $G_2(X, \{A\}) = 1 - [F_2(X, \{B\}) + F_2(\{B\} \cup X, \{C\}) \times G_2(X, \{B\})$ (by $\wedge R_2$). Hence $F_2(X, Y \cup \{A\}) + F_2(\{A\} \cup X, Y) \times G_2(X, \{A\})$ equals $F_2(X, Y \cup \{B\}) + F_2(\{B\} \cup X, Y) \times G(X, \{B\})$ by the inductive hypothesis, and hence equals $F_2(X, Y)$ by the inductive hypothesis again.

Subcase 2 ($B \wedge C \in X$): Let $X' = X - \{B \wedge C\}$. Then $F_2(X, Y \cup \{A\}) = F_2(\{B, C\} \cup X', Y \cup \{A\})$, $F_2(\{A\} \cup X, Y) = F_2(\{A, B, C\} \cup X', Y)$, and $G_2(X, \{A\}) = G_2(\{B, C\} \cup X', \{A\})$ (all by $\wedge L_2$). So $F_2(X, Y \cup \{A\}) + F_2(\{A\} \cup X, Y) \times G_2(X, \{A\})$ equals $F_2(\{B, C\} \cup X', Y \cup \{A\}) + F_2(\{A, B, C\} \cup X', Y) \times G_2(\{B, C\} \cup X', \{A\})$, which equals $F_2(\{B, C\} \cup X', Y)$ by the inductive hypothesis, and hence equals $F_2(X, Y)$ by $\neg L_2$.

Subcase 3 ($B \wedge C \in Y$): By similar reasoning.

(iii) For $G_2 5$ there are four (not necessarily mutually exclusive) subcases to be considered, namely $\neg B$ being A or a member of X, and $B \wedge C$ being A or a member of X. So assume $G_2(X, \{A\}) = 0$.

Case 1:

Subcase 1 ($\neg B$ is A): Then $F_2(\{B\} \cup X, \varnothing) \times G_2(X, \{B\}) = 1$ by $\neg R_2$. Hence $G_2(X, \{B\}) = 1$ by the inductive hypothesis for $G_2 1$ and $G_2 2$. So $F_2(\{\neg B\} \cup X, \varnothing) = 0$ by $\neg L_2$. Hence $G_2(\{A\} \cup X, \varnothing) = 1$.

Subcase 2 ($\neg B \in X$): Let $X' = X - \{\neg B\}$. It follows that $F_2(X', \{B\}) \neq 0$ and $F_2(X', \{A, B\}) = F_2(X', \{B\})$ by $\neg L_2$. But $F_2(X', \{B\}) = F_2(X', \{A, B\}) + F_2(\{A\} \cup X', \{B\}) \times G_2(X', \{A\})$ by the inductive hypothesis for $G_2 4$. So $F_2(\{A\} \cup X', \{B\}) \times G_2(X', \{A\}) = 0$. If $F_2(\{A\} \cup X', \{B\}) = 0$, then $F_2(\{A, \neg B\} \cup X', \varnothing) = 0$ by $\neg L_2$. If $G_2(X', \{A\}) = 0$, then $G_2(\{A\} \cup X', \varnothing) = 1$ by the inductive hypothesis for $G_2 5$, and $G_2(\{A, \neg B\} \cup X', \varnothing) = 1$ by Lemma 3.5(b), which uses the inductive hypothesis for $G_2 2$, $G_2 4$, and $G_2 5$. Hence $G_2(\{A\} \cup X, \varnothing) = 1$ in either case.

Case 2:

Subcase 1 ($B \wedge C$ is A): Then $1 = F_2(X, \{B\}) + F_2(\{B\} \cup X, \{C\}) \times G_2(X, \{B\})$ by $\wedge R_2$, i.e., $G_2(X, \{B\}) \times G_2(\{B\} \cup X, \{C\}) = 0$. If $G_2(\{B\} \cup X, \{C\}) = 0$, then $G_2(\{B, C\} \cup X, \varnothing) = 1$ by the inductive hypothesis for $G_2 5$. If $G_2(X, \{B\}) = 0$, then $G_2(\{B\} \cup X, \varnothing) = 1$ by the inductive hypothesis for $G_2 5$, and $G_2(\{B, C\} \cup X, \varnothing) = 1$ by Lemma 3.5(b), which uses the inductive hypothesis for $G_2 2$, $G_2 4$, and $G_2 5$. So $G_2(\{A\} \cup X, \varnothing) = 1$ by $\wedge L_2$ in either case.

Subcase 2 ($B \wedge C \in X$): Let X' be $X - \{B \wedge C\}$. Then $G_2(\{B, C\} \cup X', \{A\}) = 0$ by $\wedge L_2$. So $G_2(\{A, B, C\} \cup X', \varnothing) = 1$ by the inductive hypothesis for $G_2 5$. Hence $G_2(\{A\} \cup X, \varnothing) = 1$ by $\neg L_2$ again. □

The foregoing theorem is of course the counterpart for G_2 functions of Theorem 5.1. It shows that, given the constraints for '$\neg$' and '$\wedge$', G_2 is in effect recursively definable from its restriction to finite sets of atomic statements. And by virtue of Theorem 4.2 this result, which again parallels Gentzen's Cut-Elimination theorem for propositional logic, includes the case where the arguments of G_2 are infinite sets of statements. The next theorem will establish that the G_2 functions of this section are indeed a generalization of the consequence relation for propositional logic. Said consequence relation being compact, $\models_{G_2}$ is of course defined as in Section 4.

Theorem 5.4 *Let G_2 meet $G_2 1$–$G_2 5$, Ind_2, and the G_2-constraints regarding '$\neg$' and '$\wedge$'. Then, $X \models_{G_2} Y$ (i.e., $G_2(X, Y) = 1$ or, equivalently, $F_2(X, Y) = 0$) if and only if $X \models Y$.*

Proof: By induction on $n = \mathrm{c}(X) + \mathrm{c}(Y)$.
Basis: $n = 0$. Then X and Y are sets of atomic statements and $X \models Y$ if and only if $X \cap Y \neq \varnothing$. But if $X \cap Y \neq \varnothing$, then $G_2(X, Y) = 1$ by Lemma 3.4. And if $G_2(X, Y) = 1$, then $X \cap Y \neq \varnothing$ by Ind_2. So $X \models Y$ if and only if $G_2(X, Y) = 1$.
Inductive Step: $n > 0$.

Case 1:

Subcase 1 ($\neg A \in X$): Let $X' = X - \{\neg A\}$. Suppose first that $F_2(X', \{A\}) = 0$. Then $F_2(X, Y) = 0$ by $\neg\mathrm{L}_2$. Suppose next that $F_2(X', \{A\}) \neq 0$. Then $F_2(X, Y) = 0$ if and only if $F_2(X', Y \cup \{A\})/F_2(X', \{A\}) = 0$ (by $\neg\mathrm{L}_2$), so by the inductive hypothesis if and only if $X' \models Y \cup \{A\}$, so if and only if $X \models Y$.

Subcase 2 ($\neg A \in Y$): Let $Y' = Y - \{\neg A\}$. Assume $F_2(X, Y) = 0$. Then $F_2(\{A\} \cup X, Y') \times G_2(X, \{A\}) = 0$ (by $\neg\mathrm{R}_2$). Hence $F_2(\{A\} \cup X, Y') = 0$ or $G_2(X, \{A\}) = 0$. But if $G_2(X, \{A\}) = 0$, then $G_2(\{A\} \cup X, \varnothing) = 1$ by $G_2 5$, and $G_2(\{A\} \cup X, Y') = 1$ by Lemma 3.3(c). And, if $F_2(\{A\} \cup X, Y') = 0$, also $G_2(\{A\} \cup X, Y') = 1$. Hence by the inductive hypothesis, $\{A\} \cup X \models Y'$ in either case. So $X \models Y$. Conversely, assume $X \models Y$. Then $\{A\} \cup X \models Y'$, and by the inductive hypothesis $F_2(\{A\} \cup X, Y') = 0$. So $F_2(X, Y) = 0$ by $\neg\mathrm{L}_2$.

Case 2:

Subcase 1 ($A \wedge B \in X$): Let $X' = X - \{A \wedge B\}$. Then $F_2(X, Y) = 0$ if and only if $F_2(X' \cup \{A, B\}, Y) = 0$ (by $\wedge\mathrm{L}_2$), so by the inductive hypothesis if and only if $X' \cup \{A, B\} \models Y$, so if and only if $X \models Y$.

Subcase 2 ($A \wedge B \in Y$): Let $Y' = Y - \{A \wedge B\}$. Suppose $F_2(X, Y) = 0$. Then $F_2(X, Y' \cup \{A\}) + F_2(\{A\} \cup X, Y' \cup \{B\}) \times G_2(X, \{A\}) = 0$ by $\wedge\mathrm{R}_2$. If $G_2(X, \{A\}) = 0$, then $G_2(\{A\} \cup X, \varnothing) = 1$ by $G_2 5$, and hence $G_2(\{A\} \cup X, Y' \cup \{B\}) = 1$ by Lemma 3.3(c). If $G_2(X, \{A\}) \neq 0$, then both $F_2(X, Y' \cup \{A\}) = 0$ and $F_2(\{A\} \cup X, Y' \cup \{B\}) = 0$ by $G_2 2$. So in either case, $X \models Y' \cup \{A\}$ and $\{A\} \cup X \models Y' \cup \{B\}$ by the inductive hypothesis. Hence $X \models Y' \cup \{A, B\}$ by *Thinning*, and $X \models Y' \cup \{B\}$ by *Cut*. Hence $X \models Y$. On the other hand, assume $X \models Y$. Then $X \models Y' \cup \{A\}$ and $X \models Y' \cup \{B\}$. So $\{A\} \cup X \models Y' \cup \{B\}$ by *Thinning*. Hence $F_2(X, Y' \cup \{A\}) = 0$ and $F_2(\{A\} \cup X, Y' \cup \{B\}) = 0$ by the inductive hypothesis. So $F_2(X, Y) = 0$ by $\wedge\mathrm{R}_2$. □

6 Relative probabilities (and hence absolute probabilities, the latter being but restrictions of the former to a tautology as second argument) have been interpreted in a variety of ways: *subjectively*, as when $P(A, B)$ is taken to be the degree to which A is credible in light of B, and *objectively*, as when $P(A, B)$ is taken to be the degree to which A is confirmed by B. Carnap, the objectivist par excellence on this matter, understood $P(A, B)$ the latter way. Our final item of business in this paper is to take Carnap's position one step further and show that degrees of confirmation are but degrees of logical consequence in disguise. This will be done in two steps, as we deal *first* with the absolute probability functions associated in Roeper and Leblanc [9] with Carnap and then with the relative ones which are associated there with him.

More specifically, let $\mathcal{L}_0$ again be the propositional language of Section 5, let X and Y be finite (rather than finite or infinite) sets of statements of $\mathcal{L}_0$, let $\bigwedge X$ and $\bigvee Y$

respectively be the conjunction of the statements in X and the disjunction of those in Y, when X and Y are not empty, let $\bigwedge \varnothing$ be a fixed but arbitrary contradiction and $\bigvee \varnothing$ be its negation, and let P be a unary function on the statements of $\mathcal{L}_0$. We shall *first* show that

> *If $P(A)$ is understood as $G_1(\varnothing, \{A\})$ (or, equivalently, as $F_1(\{A\}, \varnothing)$), then P constitutes an absolute probability function in Carnap's sense (= Theorem 6.2),*

and

> *If $G_1(X, Y)$ is understood as $P(\bigwedge X \supset \bigvee Y)$, where P is an absolute probability function in Carnap's sense, then G_1 meets constraints $G_1$1–$G_1$3 plus the G_1-constraints in Section 5 relating to '¬' and '∧' (= Theorem 6.3).*

This done, let $\mathcal{L}_0$, X, Y, $\bigwedge X$, $\bigvee Y$, $\bigwedge \varnothing$, and $\bigvee \varnothing$ be as before, but let P this time be a binary function on $\mathcal{L}_0$. We shall next show that:

> *If $P(A, B)$ is understood as $G_2(\{B\}, \{A\})$, then P constitutes a relative probability function in Carnap's sense (= Theorem 6.8),*

and

> *If $G_2(X, Y)$ is understood as $P(\bigvee Y, \bigwedge X)$, where P is a relative probability function in Carnap's sense, then G_2 meets constraints $G_2$1–$G_2$5 plus the G_2-constraints in Section 5 relating to '¬' and '∧' (= Theorem 6.10).*

The constraints placed on Carnap's absolute probability functions are the following six, simplifications of the constraints that Popper [7] places on his own functions, plus a seventh one which is characteristic of those among Popper's functions that are Carnap ones.

A1. $0 \leq P(A)$ (Non-Negativity)
A2. $P(\neg(A \wedge \neg A)) = 1$ (Normality)
A3. $P(A) + P(\neg A) = 1$ (Addition)
A4. $P(A \wedge B) + P(A \wedge \neg B) = P(A)$ (Special Addition)
A5. $P(A \wedge B) \leq P(B \wedge A)$ (Commutation)
A6. $P(A \wedge (B \wedge C)) \leq P((A \wedge B) \wedge C)$ (Association)
AC. If $P(A) = 1$, then $\varnothing \models \{A\}$.

As for the constraints on Carnap's relative probability functions, they are the following seven, simplifications of the constraints that Popper [7] places on his own functions, plus an eighth one which is characteristic of those among Popper's functions that are Carnap ones.

B1. For at least one A and at least one B, $P(A, B) \neq 1$ (Existence)
B2. $0 \leq P(A, B)$ (Non-Negativity)
B3. $P(A, A) = 1$ (Normality)
B4. If $P(C, B) \neq 1$ for at least one C,[9] then $P(A, B) + P(\neg A, B) = 1$ (Addition)
B5. $P(A \wedge B, C) = P(A, B \wedge C) \times P(B, C)$ (Multiplication)
B6. $P(A \wedge B, C) \leq P(B \wedge A, C)$ (Commutation)
B7. $P(A, B \wedge C) \leq P(A, C \wedge B)$ (Commutation)
BC. If $P(A, B) = 1$, then $\{B\} \models \{A\}$.

Unlike those in Carnap [1] the present functions are total ones, with $P(A, B)$ equal to 1 when B is a contradiction. But Carnap himself, though he did not officially count contradictions as second arguments of P, saw no objection to its being done, and that has been our practice in all of our papers since Leblanc and Roeper [5].[10]

By the way, it is shown in Leblanc [4] that, (i) given the so-called *autonomous* constraints A1–A6,

$$\text{If } \varnothing \models \{A \equiv A'\}, \text{ then } P(A) = P(A'),$$

and, (ii) given the *autonomous* ones B1–B7,

$$\text{If } \varnothing \models \{A \equiv A'\}, \text{ then } P(A, B) = P(A', B)$$

and

$$\text{If } \varnothing \models \{B \equiv B'\}, \text{ then } P(A, B) = P(A, B').$$

In the proofs that follow we shall avail ourselves of these results and interchange without further ado arguments—be they of absolute probability functions or of relative ones—that are logically equivalent.

Lemma 6.1 $G_1(\varnothing, \{A\}) = F_1(\{A\}, \varnothing)$.

Proof: $F_1(\varnothing, \{A\}) + F_1(\{A\}, \varnothing) = F_1(\varnothing, \varnothing)$ by $G_1 3$. But $F_1(\varnothing, \varnothing) = 1$ by $G_1 2$. Hence Lemma 6.1. □

Theorem 6.2 *If $P(A)$ is understood as $G_1(\varnothing, \{A\})$, where G_1 meets $G_1 1$–$G_1 3$, Ind_1, and the G_1-constraints regarding '$\neg$' and '$\wedge$', then P constitutes an absolute probability function in Carnap's sense.*

Proof: (1) $0 \leq G_1(\varnothing, \{A\})$ by Lemma 2.4(c). So P meets A1.

(2) $F_1(\varnothing, \{\neg(A \wedge \neg A)\}) = F_1(\{A \wedge \neg A\}, \varnothing)$ by $\neg R_1$, which equals $F_1(\{A, \neg A\}, \varnothing)$ by $\wedge L_1$, and $F_1(\{A\}, \{A\})$ by $\neg L_1$. But $G_1(\{A\}, \{A\}) = 1$, and hence $F_1(\{A\}, \{A\}) = 0$, by Lemma 2.2. So $G_1(\varnothing, \{\neg(A \wedge \neg A)\}) = 1$. So P meets A2.

(3) $F_1(\varnothing, \{\neg A\}) = F_1(\{A\}, \varnothing)$ by $\neg R_1$. $F_1(\varnothing, \varnothing) = F_1(\varnothing, \{A\}) + F(\{A\}, \varnothing)$ by $G_1 3$. But $F_1(\varnothing, \varnothing) = 1$ by $G_1 2$. So P meets A3.

(4) $F_1(\{A \wedge B\}, \varnothing) = F_1(\{A, B\}, \varnothing)$ by $\wedge L_1$. $F_1(\{A \wedge \neg B\}, \varnothing) = F_1(\{A\}, \{B\})$ by $\wedge L_1$ and $\neg L_1$. But $F_1(\{A\}, \{B\}) + F_1(\{A, B\}, \varnothing) = F_1(\{A\}, \varnothing)$ by $G_1 3$. So P meets A4 by Lemma 6.1.

(5) $F_1(\{A \wedge B\}, \varnothing) = F_1(\{B \wedge A\}, \varnothing)$ by $\wedge L_1$. So P meets A5 by Lemma 6.1.

(6) $F_1(\{A \wedge (B \wedge C)\}, \varnothing) = F_1(\{(A \wedge B) \wedge C\}, \varnothing)$ by $\wedge L_1$. So P meets A6 by Lemma 6.1.

(7) Assume $P(A) = 1$, i.e., $G_1(\varnothing, \{A\}) = 1$. Then $\varnothing \models \{A\}$ by Theorem 5.2. So P meets AC. □

Theorem 6.3 *If $G_1(X, Y)$ is understood as $P(\bigwedge X \supset \bigvee Y)$,[11] where P is an absolute probability function in Carnap's sense, then G_1 meets $G_1 1$–$G_1 3$, Ind_1, plus the G_1-constraints regarding '$\neg$' and '$\wedge$'.*

Proof: Note first that $F_1(X, Y) = P(\bigwedge X \wedge \neg \bigvee Y)$ by A3.

(1) $P(\bigwedge X \supset \bigvee Y) \leq 1$ by A1 and A3. So G_1 meets $G_1 1$.

(2) $P(\bigwedge \varnothing \supset \bigvee \varnothing)$ equals $P(\bigvee \varnothing)$, which equals 0 by A2 and A3. So G_1 meets $G_1 2$.

(3) $P((\bigwedge X \wedge \neg \bigvee Y) \wedge \neg A) + P((\bigwedge X \wedge \neg \bigvee Y) \wedge A) = P(\bigwedge X \wedge \neg \bigvee Y)$ by A4. So $P(\bigwedge X \wedge \neg \bigvee (Y \cup \{A\})) + P(\bigwedge(\{A\} \cup X) \wedge \neg \bigvee Y) = P(\bigwedge X \wedge \neg \bigvee Y)$, and G_1 meets $G_1 3$.

(4) Suppose X and Y are sets of atomic statements and $X \cap Y = \varnothing$. Then $\varnothing \not\models \bigwedge X \supset \bigvee Y$. Hence by AC $P(\bigwedge X \supset \bigvee Y) \neq 1$. So $G_1(X, Y) \neq 1$, and G_1 meets Ind_1.

(5) $P(\bigwedge X \wedge \neg(\bigvee (Y \cup \{\neg A\}))$ equals $P(\bigwedge X \wedge \neg(\bigvee Y \vee \neg A))$, which equals $P(\bigwedge X \wedge (\neg \bigvee Y \wedge A))$, which in turn equals $P(\bigwedge(\{A\} \cup X) \wedge \neg \bigvee Y)$. So G_1 meets $\neg\text{R}_1$.

(6) $P(\bigwedge(X \cup \{\neg A\}) \wedge \neg \bigvee Y)$ equals $P((\bigwedge X \wedge \neg A) \wedge \neg \bigvee Y)$, which in turn equals $P(\bigwedge X \wedge \neg \bigvee (Y \cup \{A\})$. So G_1 meets $\neg\text{L}_1$.

(7) $P(\bigwedge X \wedge \neg \bigvee (Y \cup \{A \wedge B\})$ equals $P((\bigwedge X \wedge \neg \bigvee Y) \wedge \neg(A \wedge B))$, which by A4 equals $P(\bigwedge X \wedge \neg \bigvee Y) - P((\bigwedge X \wedge \neg \bigvee Y) \wedge (A \wedge B))$, which by A4 again equals $P((\bigwedge X \wedge \neg \bigvee Y) \wedge A) + P((\bigwedge X \wedge \neg \bigvee Y) \wedge \neg A) - P((\bigwedge X \wedge \neg \bigvee Y) \wedge (A \wedge B))$, which equals $P((\bigwedge X \wedge \neg \bigvee Y) \wedge \neg A) + P((\bigwedge X \wedge \neg \bigvee Y) \wedge A) - P(((\bigwedge X \wedge \neg \bigvee Y) \wedge A) \wedge B)$, which by A4 once more equals $P((\bigwedge X \wedge \neg \bigvee Y) \wedge \neg A) + P(((\bigwedge X \wedge \neg \bigvee Y) \wedge A) \wedge \neg B)$, which equals $P(\bigwedge X \wedge \neg \bigvee (Y \cup \{A\}) + P(\bigwedge(\{A\} \cup X) \wedge \neg \bigvee (Y \cup \{B\})$. So G_1 meets $\wedge\text{R}_1$.

(8) $P(\bigwedge(X \cup \{A \wedge B\}) \wedge \neg \bigvee Y)$ equals $P(\bigwedge(X \cup \{A, B\}) \wedge \neg \bigvee Y)$ by definition. So G_1 meets $\wedge\text{L}_1$. □

Lemma 6.4 *(a) Given $G_2 1$–$G_2 5$, $\neg R_2$ and $\neg L_2$ are equivalent to*

$$\neg_2\text{:} \quad F_2(X, Y \cup \{\neg A\}) = F_2(X, Y) - F_2(X, Y \cup \{A\});$$

and (b) given $G_2 1$–$G_2 5$ and $\wedge L_2$, $\wedge R_2$ is equivalent to

$$\wedge_2\text{:} \quad G_2(X, \{A \wedge B\}) = G_2(X, \{A\}) \times G_2(\{A\} \cup X, \{B\}).$$

Proof: (a) Assume first $\neg\text{R}_2$. Then $\neg_2$ by $G_2 4$. Assume next $\neg_2$. Then $\neg\text{R}_2$ by $G_2 4$. Moreover, $F_2(X, Y) = F_2(X, Y \cup \{\neg A\}) + F_2(\{\neg A\} \cup X, Y) \times G_2(X, \{\neg A\})$ by $G_2 4$. But $F_2(X, Y \cup \{\neg A\}) = F_2(X, Y) - F_2(X, Y \cup \{A\})$, and $G_2(X, \{\neg A\}) = G_2(X, \varnothing) + F_2(X, \{A\})$, both by $\neg_2$. Hence $F_2(\{\neg A\} \cup X, Y) \times [G_2(X, \varnothing) + F_2(X, \{A\})] = F_2(X, Y \cup \{A\})$. If $F_2(X, \{A\}) = 0$ and $G_2(X, \varnothing) = 0$, then $G_2(X, \{\neg A\}) = 0$ and so $G_2(\{\neg A\} \cup X, Y) = 1$, i.e., $F_2(\{\neg A\} \cup X, Y) = 0$, by $G_2 5$ and Lemma 3.3(c). If $F_2(X, \{A\}) = 0$ but $G_2(X, \varnothing) \neq 0$, then $F_2(X, Y \cup \{A\}) = 0$ by Lemma 3.3(c), and so $F_2(\{\neg A\} \cup X, Y) = 0$. And if $F_2(X, \{A\}) \neq 0$, then $G_2(X, \varnothing) = 0$ by Lemma 3.6(a) and Lemma 3.3(c). So $F_2(\{\neg A\} \cup X, Y) = F_2(X, Y \cup \{A\}) / F_2(X, \{A\})$. Hence $\neg\text{L}_2$.

(b) Assume first $\wedge\text{R}_2$. Then $F_2(X, \{A \wedge B\}) = F_2(X, \{A\}) + F_2(\{A\} \cup X, \{B\}) \times G_2(X, \{A\})$, i.e., $G_2(X, \{A \wedge B\}) = G_2(X, \{A\}) \times G_2(\{A\} \cup X, \{B\})$. Hence $\wedge_2$. Assume next $\wedge_2$ and $\wedge\text{L}_2$. Then $F_2(X, Y) = F_2(X, Y \cup \{A \wedge B\}) + F_2(\{A, B\} \cup X, Y) \times G_2(X, \{A\}) \times G_2(\{A\} \cup X, \{B\})$ by $G_2 4$, $\wedge\text{L}_2$, and $\wedge_2$. But $F_2(\{A, B\} \cup X, Y) \times G_2(\{A\} \cup X, \{B\}) = F_2(\{A\} \cup X, Y) - F_2(\{A\} \cup X, Y \cup \{B\})$, and $F_2(X, Y) = F_2(X, Y \cup \{A\}) + F_2(\{A\} \cup X, Y) \times G_2(X, \{A\})$, both by $G_2 4$. So $F_2(X, Y \cup \{A \wedge B\}) = F_2(X, Y \cup \{A\}) + F(\{A\} \cup X, Y \cup \{B\}) \times G_2(X, \{A\})$. Hence $\wedge\text{R}_2$. □

So constraint $\neg_2$ will hereafter do duty for $\neg R_2$ and $\neg L_2$, and constraint $\wedge_2$ do duty for $\wedge R_2$. As a result, the G_2-constraints regarding '$\neg$' and '$\wedge$' will be $\neg_2$, $\wedge_2$, and $\wedge L_2$.

Lemma 6.5 *If* $G_2(X, \varnothing) = 0$, *then* $G_2(X, \{A\}) + G_2(X, \{\neg A\}) = 1$.

Proof: By $\neg_2$ and $G_2 3$. □

Lemma 6.6 $G_2(X, \{A, A \wedge B\}) = G_2(X, \{A\})$.

Proof: $F_2(X, \{A, A \wedge B\}) = F_2(X, \{A\}) + F_2(\{A\} \cup X, \{A, B\}) \times G_2(X, \{A\})$ by $\wedge R_2$. But $F_2(\{A\} \cup X, \{A, B\}) = 0$ by Lemma 3.4. Hence Lemma 6.6. □

Lemma 6.7 $G_2(X, \{A \wedge B\}) = G_2(X, \{B \wedge A\})$.

Proof: By $\wedge_2$ and Lemma 3.7. □

Theorem 6.8 *If* $P(A, B)$ *is understood as* $G_2(\{B\}, \{A\})$, *where* G_2 *meets* $G_2 1$–$G_2 5$, Ind_2, *and the* G_2*-constraints regarding '*$\neg$*' and '*$\wedge$*', then P constitutes a relative probability function in Carnap's sense.*

Proof: (1) Let A and B be distinct atomic statements. Then $G_2(\{B\}, \{A\}) \neq 1$ by Ind_2. So P meets B1.

(2) $0 \leq G_2(\{B\}, \{A\})$ by $G_2 1$. So P meets B2.

(3) $G_2(\{A\}, \{A\}) = 1$ by Lemma 3.4. So P meets B3.

(4) Suppose $G_2(\{B\}, \{C\}) \neq 1$ for some C in $\mathcal{L}_0$. Then $G_2(\{B\}, \varnothing) = 0$ by Lemma 3.6(b). Hence $G_2(\{B\}, \{A\}) + G_2(\{B\}, \{\neg A\}) = 1$ by Lemma 6.5. So P meets B4.

(5) $G_2(\{C\}, \{A \wedge B\}) = G_2(\{C\}, \{B \wedge A\})$ by Lemma 6.7. So $G_2(\{C\}, \{A \wedge B\}) = G_2(\{C\}, \{B\}) \times G_2(\{B, C\}, \{A\})$ by $\wedge_2$. So $G_2(\{C\}, \{A \wedge B\}) = G_2(\{B \wedge C\}, \{A\}) \times G_2(\{C\}, \{B\})$ by $\wedge L_2$. So P meets B5.

(6) $G_2(\{C\}, \{A \wedge B\}) = G_2(\{C\}, \{B \wedge A\})$ by Lemma 6.7. So P meets B6.

(7) $G_2(\{B \wedge C\}, \{A\}) = G_2(\{C \wedge B\}, \{A\})$ by $\wedge L_2$. So P meets B7.

(8) Assume $P(A, B) = 1$, i.e., $G_2(\{B\}, \{A\}) = 1$. Then $\{B\} \models \{A\}$ by Theorem 5.4. So P meets BC. □

The following lemma lists, without proof,[12] well-known results about Carnap relative probability functions.

Lemma 6.9 *Let* P *be a relative probability function in Carnap's sense. Then*
(a) $P(A, B) \leq 1$;
(b) $P(\bigvee \varnothing, \bigwedge \varnothing) = 0$;
(c) $P(A \vee B, C) = P(A, C) + P(B, C) - P(A \wedge B, C)$;
(d) If $P(A, B) = 0$, *then* $P(\bigvee \varnothing, A \wedge B) = 1$;
(e) If B *is* P*-abnormal, then* $P(B, \bigwedge \varnothing) = 0$.

Theorem 6.10 *If* $G_2(X, Y)$ *is understood as* $P(\bigvee Y, \bigwedge X)$, *where* P *is a relative probability function in Carnap's sense, then* G_2 *meets constraints* $G_2 1$–$G_2 5$, Ind_2, *and the* G_2*-constraints regarding '*$\neg$*' and '*$\wedge$*'.*

Proof: (1) $0 \leq P(\bigvee Y, \bigwedge X)$ by B2. So G_2 meets G_21.

(2) $P(\bigvee Y, \bigwedge X) \leq 1$ by Lemma 6.9(a). So G_2 meets G_22.

(3) $P(\bigvee \varnothing, \bigwedge \varnothing) = 0$ by Lemma 6.9(b). So G_2 meets G_23.

(4) By the definition of F_2 and that of G_2, $F_2(X, Y \cup \{A\})$ equals $1 - P(\bigvee Y \vee A, \bigwedge X)$, which by Lemma 6.9(c) equals $1 - P(\bigvee Y, \bigwedge X) - P(A, \bigwedge X) + P(\bigvee Y \wedge A, \bigwedge X)$, which by B5 equals $1 - P(\bigvee Y, \bigwedge X) - P(A, \bigwedge X) + P(\bigvee Y, A \wedge \bigwedge X) \times P(A, \bigwedge X)$, which by the definition of F_2 and that of G_2 equals $F_2(X, Y) - F_2(\{A\} \cup X, Y) \times G_2(X, \{A\})$. So G_2 meets G_24.

(5) Suppose $G_2(X, \{A\}) = 0$, i.e., $P(V\{A\}, \bigwedge X) = 0$, i.e., $P(A, \bigwedge X) = 0$. Then $P(\bigvee \varnothing, \bigwedge(\{A\} \cup X)) = 1$ by Lemma 6.9(d). So G_2 meets G_25.

(6) Suppose X and Y are sets of atomic statements and $X \cap Y = \varnothing$. Then $\{\bigwedge X\} \not\models \{\bigvee Y\}$. Hence by BC $P(\bigvee Y, \bigwedge X) \neq 1$. So $G_2(X, Y) \neq 1$ and G_2 meets Ind_2.

(7) Suppose $\bigwedge X$ P-normal. Then $1 - P(\bigvee(Y \cup \{\neg A\}), \bigwedge X) = 1 - P(\bigvee Y \vee \neg A, \bigwedge X)$, which by Lemma 6.9(c) equals $1 - [P(\bigvee Y, \bigwedge X) + P(\neg A, \bigwedge X) - P(\bigvee Y \wedge \neg A, \bigwedge X)]$, which by B4 equals $1 - P(\bigvee Y, \bigwedge X) - 1 + P(A, \bigwedge X) + P(\bigvee Y, \bigwedge X) - P(\bigvee Y \wedge A, \bigwedge X)$, which by Lemma 6.9(c) again equals $[1 - P(\bigvee Y, \bigwedge X] - [1 - P(\bigvee Y \vee A, \bigwedge X)]$, which equals $[1 - P(\bigvee Y, \bigwedge X)] - [1 - P(\bigvee(Y \cup \{A\}), \bigwedge X)]$. But $1 - P(\bigvee(Y \cup \{A\}), \bigwedge X) = [1 - P(\bigvee Y, \bigwedge X)] - [1 - P(\bigvee(Y \cup \{A\}), \bigwedge X)$ by definition when $\bigwedge X$ is P-abnormal. So G_2 meets $\neg_2$.

(8) $P(A \wedge B, \bigwedge X)$ equals $P(B, \bigwedge X \wedge A) \times P(A, \bigwedge X)$ by B5, B6, and B7, which of course equals $P(B, \bigwedge(\{A\} \cup X)) \times P(A, \bigwedge X)$. So G_2 meets $\wedge_2$.

(9) Since $\bigwedge(X \cup \{A \wedge B\})$ and $\bigwedge(X \cup \{A, B\})$ are the same, $P(\bigvee Y, \bigwedge(X \cup \{A \wedge B\})$ cannot but equal $P(\bigvee Y, \bigwedge(X \cup \{A, B\})$. So G_2 meets $\wedge \mathrm{L}_2$. □

Degrees of logical consequence in the sense represented by G_2 functions can also be introduced for languages whose consequence relation is not compact. The corresponding relative probability functions turn out to be those we associate with Rényi in [8], and as a result they too prove to be degrees of logical consequence in disguise.[13]

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NOTES

1. Gentzen's extant contributions to logic and metamathematics [3], edited and translated by Szabo, were published in 1969 by North-Holland. A second edition, with a new introduction and minor changes, is to appear shortly.

2. Note that for brevity's sake we use '$\mathcal{L}$' here and on later occasions to refer to the set of the statements of $\mathcal{L}$.

3. The reader will often have to think of 'F_1' and '$1 - G_1$', and of course of 'G_1' and '$1 - F_1$', as interchangeable, a warning also in order when '2' as well as '1' will be subscripted to 'F' and 'G'.

   Constraints G_11–G_13 are independent of one another. This can be shown by (i) using a language with just one statement, A, hence with just two sets of statements ($\varnothing$ and $\{A\}$), and (ii) constructing for each of the three constraints a function which does not meet that constraint but meets the other two.

	$G_1 1$	$G_1 2$	$G_1 3$
$G_1(\varnothing, \varnothing) =$	0	1	0
$G_1(\varnothing, \{A\}) =$	2	1	0
$G_1(\{A\}, \varnothing) =$	−1	1	0
$G_1(\{A\}, \{A\}) =$	1	1	0

4. $G_2 5$ is the G_2-counterpart of a constraint placed on the relative probability functions that we associate in Roeper-Leblanc [9] with Kolmogorov. It is the definition of $X \models Y$ as $G_2(X, Y) = 1$ that isolates here those among Kolmogorov's functions that are Carnap ones.

   Constraints $G_2 1$–$G_2 5$ are independent of one another, as can be shown by using the same language as in Note 3 and constructing for each of the five constraints a function which does not meet that constraint but meets the other four.

	$G_2 1$	$G_2 2$	$G_2 3$	$G_2 4$	$G_2 5$
$G_2(\varnothing, \varnothing) =$	0	0	1	0	0
$G_2(\varnothing, \{A\}) =$	−1	2	1	1/2	0
$G_2(\{A\}, \varnothing) =$	0	0	1	1/2	0
$G_2(\{A\}, \{A\}) =$	1	1	1	1	1

5. To anticipate, Y in $G_2(X, Y)$ is disjunctively understood, whereas in $H(X, Y)$ Y is conjunctively understood; $G_2(X, Y)$ amounts to $P(\bigvee Y, \bigwedge X)$, whereas $H(X, Y)$ amounts to $P(\bigwedge Y, \bigwedge X)$.
6. For finite X, Y, and Z, R2′–R3′ are consequences of R2–R3, as is obvious in the first case and easily verified in the second. The resulting R3′ is called *Cut for Sets* in Shoesmith and Smiley [10], p. 29.
7. Recall that 'F_2' and '$1 - G_2$' are interchangeable. So $G_2 4$ is in effect a constraint on G_2.
8. The notion of logical independence used here is a generalization of one of Moore's (cf. [6]), who took two statements A and B to be logically independent of each other if neither of A and $\neg A$ has either of B and $\neg B$ as a logical consequence. Wittgenstein [11], Proposition 4.27, may have been the first explicitly to require of atomic statements that they be logically independent of one another in Moore's sense. For more on the matter see the authors' "Of A and B being logically independent of each other and of their having no common factual content."
9. Hereafter we shall say that B is P-abnormal when $P(C, B) = 1$ for every C.
10. The total functions we associate here with Carnap can be defined in terms of his absolute ones thus:

$$P(A, B) = \begin{cases} P(A \wedge B, \bigwedge \varnothing)/P(B, \bigwedge \varnothing) \text{ if } P(B, \bigwedge \varnothing) \neq 0 \\ \quad \text{(i.e., if } \{B\} \not\models \varnothing) \\ 1 \text{ otherwise,} \end{cases}$$

    a definition which parallels Lemma 3.11. Since $P(A)$ is tantamount to $P(A, \bigwedge \varnothing)$, Carnap's absolute probability functions thus stand in one-to-one correspondence with his relative ones, the way the G_1 functions and G_2 ones do.
11. Note that with $G_1(X, Y)$ thus defined, $F_1(X, Y)$ is the same as $P(\bigwedge X \wedge \neg \bigvee Y)$, a fact of which we shall make use in the proof of the theorem.

12. Proof of (e) is in Roeper and Leblanc [9], p. 13, where it appears as Theorem 3.6.

13. Shown in Roeper-Leblanc [9] is that Carnap's relative probability functions are all Rényi ones, but of course not vice-versa.

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