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A Distinguishable Model Theorem for the Minimal US-Tense Logic

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Abstract A new concept of model for the US-tense logic is introduced, in which ternary relations of betweenness are adjoined to the usual early-later relation. The class of these new models, which contains the class of Kripke models, satisfies, contrary to that, the Distinguishable Model Theorem, in the sense that each model is equivalent to a model in which no two points verify exactly the same formulas.

1 Introduction The Kripke semantics for the "standard" temporal logic (i.e., the propositional logic endowed with two unary operators F—it will be the case that ...— and P—it has been the case that ...) satisfies the following property: given a Kripke model $\mathcal{M} = \langle T, R, V\rangle$, there exists a model $\mathcal{M}'$ equivalent to $\mathcal{M}$ and which is without pairs of equivalent points. Following Segerberg [1], we call a model without equivalent points a *distinguishable* model. The existence of a distinguishable model equivalent to a given model (this statement will be called in the following the "Distinguishable Model Theorem") is a consequence of the Filtration Theorem and has a fundamental role. It is closely connected, as we shall see, with the construction of the Canonical Model and the consequent proof of the Fundamental Theorem; in fact in the Canonical Model each maximal consistent extension of the particular logic under investigation is taken exactly once. Similarly, in the proof of correspondence between temporal structures and temporal algebras, each ultrafilter of the algebra becomes a point of the dual structure, taken exactly once. And, finally, also in showing the equivalence between the Finite Model Property and the Finite Frame Property one makes use of the Distinguishable Model Theorem.

What was said referring to temporal logic can be extended to modal logics with an arbitrary number of unary operators $\Box_i$, $i \in I$, because these logics also satisfy the Distinguishable Model Theorem. Indeed, given a model $\mathcal{M} = \langle T, \{R_i\}_{i\in I}, V\rangle$, the model $\mathcal{M}^0 = \langle T^0, \{R_i^0\}_{i\in I}, V^0\rangle$ (where $T^0 = T/_\equiv$, $[t]R_i^0[u]$ iff there exist a $t' \in [t]$ and a $u' \in [u]$ such that $t' R_i u'$, $V^0(p) = \{[t] : t \in V(p)\}$) is distinguishable and equivalent

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to $\mathcal{M}$. The temporal logic with the binary operators *Until* and *Since* (US-logic) was born as a logic endowed with a Kripke style semantics. The usual frames and models of the US-logic are couples $\langle T, R\rangle$ or triples $\langle T, R, V\rangle$, and the truth-definition of $U(\varphi, \psi)$ is as follows: $t \models U(\varphi, \psi)$ iff there exists t' such that tRt', $t' \models \varphi$ and, for all t'' such that $tRt''Rt'$, $t'' \models \psi$ (analogously for $S(\varphi, \psi)$, referring to R^{-1}). But this semantics, in this case, lacks the Distinguishable Model Theorem and turns out to be highly inadequate.

In §2 we give some examples of Kripke models without US-equivalent distinguishable Kripke models, thus showing that the Distinguishable Model Theorem fails. In §3 we define the notion of e-model, which extends that of Kripke model, and show that the Distinguishable Model Theorem holds for these new models. Finally in §4 we define the notion of US-general e-frame and show that for every US-general e-frame there exists an equivalent distinguishable US-general e-frame.

2 Some examples We refer to two types of languages: the FP-language, that is $\mathcal{L}_{\mathrm{FP}} = \mathrm{PC} \cup \{F, P\}$, where F, P are the unary operators of future and past (as usual G and H stand respectively for $\neg F \neg$ and $\neg P \neg$), and the US-language, that is $\mathcal{L}_{\mathrm{US}} = \mathrm{PC} \cup \{U, S\}$, U, S being binary operators (the definition of truth for $U(\varphi, \psi)$ and $S(\varphi, \psi)$ has been given in the introduction). The symbols $\bot$ and $\top$ stand for any contradiction and any tautology, respectively. Given two points t and t', we write $t \equiv_{\mathrm{FP}} t'$ and $t \equiv_{\mathrm{US}} t'$ to mean that t and t' satisfy the same set of FP-formulas and US-formulas, respectively; analogously for $\mathcal{M} \equiv_{\mathrm{FP}} \mathcal{M}'$ and $\mathcal{M} \equiv_{\mathrm{US}} \mathcal{M}'$. Since, for every point t of every model $\mathcal{M}$, $t \models F\varphi$ iff $t \models U(\varphi, \top)$ and $t \models P\varphi$ iff $t \models S(\varphi, \top)$, the set of FP-formulas can be considered as a subset of the set of US-formulas, and thus $\equiv_{\mathrm{US}}$ implies $\equiv_{\mathrm{FP}}$.

We note the following result.

Lemma 2.1 *If $\mathcal{M} \equiv \mathcal{M}'$ (where $\equiv$ is either $\equiv_{\mathrm{FP}}$ or $\equiv_{\mathrm{US}}$) and $T/_{\equiv}$ is finite, then $T/_{\equiv}$ and $T'/_{\equiv}$ have the same cardinality.*

Proof: For each model $\mathcal{M}''$, the schema $\bigvee_{0 \le i, j \le 2^n i \neq j}(\varphi_i \leftrightarrow \varphi_j)$ holds in $\mathcal{M}''$ iff $|T''/_{\equiv}| \le n$. □

Example 2.2 Let us consider the model $\mathcal{M}_1 = \langle T_1, R_1, V_1\rangle$, where $T_1 = \{w_1, w_2\}$, $R_1 = \{(w_1, w_2), (w_2, w_1)\}$, and $V_1(p) = \{w_1, w_2\}$, for every p belonging to the set $\mathbb{P}$ of all propositional letters (see Figure 1). It is immediate to observe that $w_1 \equiv_{\mathrm{US}} w_2$. Therefore $|T/_{\equiv_{\mathrm{US}}}| = 1$, and thus if $\mathcal{M}' \equiv_{\mathrm{US}} \mathcal{M}$ and $\mathcal{M}'$ is distinguishable, then, by Lemma 2.1, $|T'| = 1$, i.e., $T' = \{t'\}$ for some t'. But from $\mathcal{M} \models F\top$ and $\mathcal{M} \models U(\top, \bot)$ it follows that R' can be neither empty (otherwise $t' \not\models F\top$) nor $\{(t', t')\}$ (because from $t'Rt'Rt'$ it follows that $t' \not\models U(\top, \bot)$).

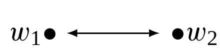

Figure 1

Example 2.3 Here is an example of an infinite model. Let $\mathcal{M}_2 = \langle T_2, R_2, V_2\rangle$ be as follows: $T_2 = \mathbb{Z}$, $R_2 = <$, $V_2(p) = \{2k : k \in \mathbb{Z}\}$ for each $p \in \mathbb{P}$ (see Figure 2). We have that $|T_2/_{\equiv_{\mathrm{US}}}| = 2$. In fact, for each $h, k \in \mathbb{Z}$, $2k \equiv_{\mathrm{US}} 2h$, $2k + 1 \equiv_{\mathrm{US}} 2h + 1$, but

$2k \not\equiv_{US} 2h+1$. Therefore, by Lemma 2.1, if $\mathcal{M}' \equiv_{US} \mathcal{M}_2$ and $\mathcal{M}'$ is distinguishable, then $|T'| = 2$. But $\mathcal{M}' \equiv_{US} \mathcal{M}_2$ implies that $\mathcal{M}' \equiv_{FP} \mathcal{M}_2$, and the only $\mathcal{M}'$ having cardinality 2 and such that $\mathcal{M}' \equiv_{FP} \mathcal{M}_2$ is $\langle T', R', V' \rangle$, with $T' = \{w, v\}$, $R' = T' \times T'$ and $V'(p) = \{w\}$ for each $p \in \mathbb{P}$. But, again, $\mathcal{M}' \not\models U(\top, \bot)$ whereas $\mathcal{M}_2 \models U(\top, \bot)$.

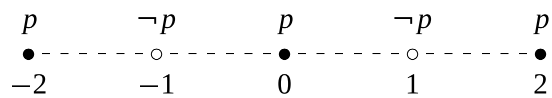

Figure 2

Example 2.4 Let us consider $\mathcal{M}_3 = \langle T_3, R_3, V_3 \rangle$ where $T_3 = \mathbb{R}$, $R_3 =<$ and $V_3(p) = \{x \in \mathbb{R} : 2k \leq x < 2k+1, k \in \mathbb{Z}\}$ for each $p \in \mathbb{P}$ (see Figure 3). The formulas

(φ_1) $p \wedge S(\neg p, \neg p)$
(φ_2) $p \wedge \neg S(\neg p, \neg p)$
(φ_3) $\neg p \wedge S(p, p)$
(φ_4) $\neg p \wedge \neg S(p, p)$

hold respectively at the sets of points

(X_1) $\{x : x = 2k$ for some $k \in \mathbb{Z}\}$
(X_2) $\{x : 2k < x < 2k+1$ for some $k \in \mathbb{Z}\}$
(X_3) $\{x : x = 2k+1$ for some $k \in \mathbb{Z}\}$
(X_4) $\{x : 2k+1 < x < 2k+2$ for some $k \in \mathbb{Z}\}$.

This means that $|T_3/_{\equiv_{US}}| = 4$. Suppose $\mathcal{M}'$ were a model with four points w_1, w_2, w_3, w_4 that is US-equivalent to $\mathcal{M}_3$. Since the φ_i are mutually exclusive, each of the points must verify exactly one of these formulas. Furthermore each of the points must be related to itself and all the others (because for each $i \leq 4$, $\mathcal{M}_3 \models F\varphi_i$). So $\mathcal{M}' \not\models p \to U(p, p)$, whereas $\mathcal{M}_3 \models p \to U(p, p)$.

Note that $\mathcal{M}_3$ and $\mathcal{M}'$ satisfy the same set of variable-free formulas, which is, in both cases, the set of the variable-free formulas true in the frame costituted by a single reflexive point, whereas $\mathcal{M}_3$ is FP-equivalent to the two-points model $\mathcal{M}'$ of Example 2.3.

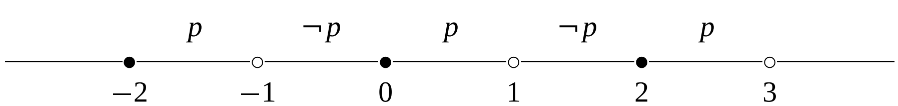

Figure 3

We use the model of Example 2.2 to show this fact: the standard method for obtaining the Canonical Kripke Model for a logic **L**, which consists of defining an appropriate relation over the set of all maximal consistent extensions of **L**, is not, in general, applicable to US-logics. In fact, let **L** be the US-theory of the Kripke frame $\langle T_1, R_1 \rangle$ of the model $\mathcal{M}_1$ (see Figure 1), and let $\mathcal{M} = \langle T, R, V \rangle$ be a Kripke model. Suppose, for a contradiction, that $\mathcal{M}$ has the standard properties of a canonical model for **L**:

1. T is the set of all maximal consistent extensions of **L** (taken once);
2. $w \models \varphi$ iff $\varphi \in w$, for each $w \in T$.

Then, since $\mathcal{M}_1 \models \mathbf{L}$, we have, by (1), that $w = \{\varphi : w_1 \models \varphi\} \in T$. Let $\mathcal{M}'$ be the submodel of $\mathcal{M}$ generated by w. We show that all points in $\mathcal{M}'$ are equivalent. In fact, for each $p \in \mathbb{P}$ and each $n \in \mathbb{N}$, $w_1 \models p \wedge G^n p$ and hence, by (2), $w \models p \wedge G^n p$. So for every $u \in T'$ and $p \in \mathbb{P}$, $u \models p$. The inductive steps for Boolean connectives are obvious. Let $u \in T'$ and $u \models U(\varphi, \psi)$; then there exists a $u' \in T'$ such that $u' \models \varphi$ and so, by the inductive hypothesis, all the points of T' satisfy φ. Let v be any point of T'. Since $\mathrm{L} \vdash U(\top, \bot)$ we have that there exists a $v' \in T'$ such that vRv' and $\{z : vRzRv'\} = \varnothing$. Then from $v' \models \varphi$ it follows that $v \models U(\varphi, \psi)$. So all the points of T' are equivalent. Moreover from vRv' and $\{z : vRzRv'\} = \varnothing$ it follows that $v \neq v'$; therefore the maximal set $\{\varphi : v \models \varphi\}$ is contained in $\mathcal{M}$ more than once.

3 e-models and the Distinguishable Model Theorem

Definition 3.1 An *e-model* $\mathcal{A}$ is a four-tuple $\langle A, R, \beta, V\rangle$ where $\langle A, R, V\rangle$ is a Kripke model and β is a function from R into $\mathcal{P}(\mathcal{P}(A))$ such that for all $(x, y) \in R$

1. $\beta(x, y) \neq \varnothing$
2. if $X \in \beta(x, y)$, then $X \subseteq \{z : xRzRy\}$.

In other words, $\beta(x, y)$ is a nonempty set of sets of points between x and y (notice that the case $\beta(x, y) = \{\varnothing\}$ is not ruled out). The definition of "truth" of a formula φ at a point w of an e-model is the standard truth-definition as regards propositional variables and Boolean connectives, and the truth of $U(\varphi, \psi)$ and $S(\varphi, \psi)$ is defined as follows :

3. $w \models U(\varphi, \psi)$ iff there exists a point v such that wRv and $v \models \varphi$, and there exists $Z \in \beta(w, v)$ such that $u \models \psi$, for each $u \in Z$
4. $w \models S(\varphi, \psi)$ iff there exists a point v such that vRw and $v \models \varphi$, and there exists $Z \in \beta(v, w)$ such that $u \models \psi$, for each $u \in Z$.

Intuitively, a formula $U(\varphi, \psi)$ is true at a point x of an e-model if, for some y in the future of x, φ is true at y and ψ is true for enough points between x and y. Considering that $F\varphi = U(\varphi, \top)$ and $P\varphi = S(\varphi, \top)$, we see that the definitions of truth for F and P coincide with the usual ones. If for every $(x, y) \in R$, $|\beta(x, y)| = 1$ (that is $\beta(x, y) = \{Z\}$) then $\mathcal{A}$ is said a *simple* e-model. In particular, if for each $(x, y) \in R$, $\beta(x, y) = \{\{z : xRzRy\}\}$, then the above definitions 3.1.3 and 3.1.4 reduce to the standard case, and so $\langle A, R, \beta, V\rangle \equiv_{\mathrm{US}} \langle A, R, V\rangle$. Therefore we may consider the class of Kripke models as a subclass of the class of e-models.

Theorem 3.2 *For each* US-*formula* φ, φ *is true in every Kripke model iff* φ *is true in every* e-*model.*

Proof: Let $\mathrm{TL}_{\mathrm{US}} = \{\varphi : \langle T, R, V\rangle \models \varphi$, for each $\langle T, R, V\rangle\}$ and $X = \{\varphi : \langle A, R, \beta, V\rangle \models \varphi$, for each $\langle A, R, \beta, V\rangle\}$. Since, as already observed, each Kripke model is (equivalent to) an e-model, we have that $X \subseteq \mathrm{TL}_{\mathrm{US}}$. For the converse, we know (see Xu [2]) that $\mathrm{TL}_{\mathrm{US}}$ is the set of theorems of the system whose axioms are:

(0) tautologies
(1) $G(p \to q) \to (U(p,r) \to U(q,r)) \wedge (U(r,p) \to U(r,q))$
(2) $H(p \to q) \to (S(p,r) \to S(q,r)) \wedge (S(r,p) \to S(r,q))$
(3) $p \wedge U(q,r) \to U(q \wedge S(p,r), r)$
(4) $p \wedge S(q,r) \to S(q \wedge U(p,r), r)$

and whose rules of inference are Uniform Substitution, Modus Ponens, $\varphi / G\varphi$ and $\varphi / H\varphi$. It is now a matter of routine to verify that each e-model satisfies these axioms and is closed under these rules, and so $\mathrm{TL}_{\mathrm{US}} \subseteq X$. □

From Definition 3.1 we have the following.

Remark 3.3 Let $\mathcal{A}$ be an e-model, $(x, y) \in R$, and let Z_1, Z_2 be subsets of A such that $Z_1, Z_2 \in \beta(x, y)$ and $Z_1 \subseteq Z_2$. Then Z_2 is without influence, i.e., $\mathcal{A}$ is US-equivalent to $\mathcal{A}' = \langle A, R, \beta', V \rangle$ where $\beta'(x, y) = \beta(x, y) - \{Z_2\}$ and $\beta'(u, v) = \beta(u, v)$ for $(u, v) \neq (x, y)$.

Because of this fact, it is enough to consider only the minimal sets of $\beta(x, y)$, and so if $\beta(x, y)$ has a least set B, then we can reduce $\beta(x, y)$ to $\{B\}$. Therefore, if for each $(x, y) \in R$ $\beta(x, y)$ has a least element, then $\mathcal{A}$ is (equivalent to) a simple e-model.

Definition 3.4 Let $\mathcal{A} = \langle A, R, \beta, V \rangle$ be an e-model, let Σ be a set of US-formulas closed under subformulas, and, for every $w \in A$, let $[w]_\Sigma = \{w' \in A : \forall \varphi \in \Sigma, w' \models \varphi \text{ iff } w \models \varphi\}$. An e-model $\mathcal{A}_\Sigma = \langle A_\Sigma, R_\Sigma, \beta_\Sigma, V_\Sigma \rangle$ is a *filtration of* $\mathcal{A}$ *through* Σ if

1. $\mathcal{A}_\Sigma = \{[w]_\Sigma : w \in A\}$
2. for each $p \in \Sigma$, $V_\Sigma(p) = \{[w]_\Sigma : w \in V(p)\}$
3. (a) wRv implies that $[w]_\Sigma R_\Sigma [v]_\Sigma$
   (b) $Z \in \beta(w, v)$ implies that $\{[u]_\Sigma : u \in Z\} \in \beta_\Sigma([w]_\Sigma, [v]_\Sigma)$
4. $\{[u]_\Sigma : u \in Z\} \in \beta_\Sigma([w]_\Sigma, [v]_\Sigma)$ implies that
   (a) for each formula $U(\varphi, \psi) \in \Sigma$, if $w \not\models U(\varphi, \psi)$ then either $v \not\models \varphi$ or there exists $u \in Z$ such that $u \not\models \psi$
   (b) for each formula $S(\varphi, \psi) \in \Sigma$, if $v \not\models S(\varphi, \psi)$ then either $w \not\models \varphi$ or there exists $u \in Z$ such that $u \not\models \psi$.

Observe that from condition (4-a) it follows:

4-a′. $[w]_\Sigma R_\Sigma [v]_\Sigma$ implies that, for all formulas $F\varphi \in \Sigma$, if $w \not\models F\varphi$ then $v \not\models \varphi$.

Suppose in fact that $[w]_\Sigma R_\Sigma [v]_\Sigma$ and $w \not\models F\varphi$ (i.e., $w \not\models U(\varphi, \top)$). By definition of e-model, from $[w]_\Sigma R_\Sigma [v]_\Sigma$ it follows that there exists $Z_\Sigma \in \beta_\Sigma([w]_\Sigma, [v]_\Sigma)$. So, from (4-a) together with $w \not\models U(\varphi, \top)$ and $u \models \top$, we obtain $v \not\models \varphi$. An analogous condition for P follows from (4-b).

Lemma 3.5 *Given an* e-*model* $\mathcal{A}$ *and a set* Σ *of* US-*formulas which is closed under subformulas, the class of the filtrations of* $\mathcal{A}$ *through* Σ *is not empty.*

Proof: We show that there exists at least the so called "finest filtration." Let R_Σ and β_Σ be defined as follows.

1. $[w]_\Sigma R_\Sigma [v]_\Sigma$ iff there exist $w' \in [w]_\Sigma$ and $v' \in [v]_\Sigma$ such that $w' R v'$;
2. $\{[u]_\Sigma : u \in Z\} \in \beta_\Sigma([w]_\Sigma, [v]_\Sigma)$ iff there exist $w' \in [w]_\Sigma, v' \in [v]_\Sigma$ and $Z' \subseteq A$ such that $w' R v'$, $Z' \in \beta(w', v')$ and $\{[u]_\Sigma : u \in Z'\} = \{[u]_\Sigma : u \in Z\}$.

We show that if A_Σ and V_Σ are as in Definition 3.4.1–2 with $V_\Sigma(p) = \varnothing$ for each $p \notin \Sigma$, then the four-tuple $\mathcal{A}_\Sigma = \langle A_\Sigma, R_\Sigma, \beta_\Sigma, V_\Sigma\rangle$ is an e-model. First notice that $\beta_\Sigma([w]_\Sigma, [v]_\Sigma) \neq \varnothing$, for every $([w]_\Sigma, [v]_\Sigma) \in R_\Sigma$. In fact $[w]_\Sigma R_\Sigma [v]_\Sigma$ implies that $w' R v'$ for some $w' \in [w]_\Sigma$ and $v' \in [v]_\Sigma$; and since $\mathcal{A}$ is an e-model, we have that $\beta(w', v') \neq \varnothing$ and hence, by (2), $\beta_\Sigma([w]_\Sigma, [v]_\Sigma) \neq \varnothing$. Moreover, $\{[u]_\Sigma : u \in Z\} \in \beta_\Sigma([w]_\Sigma, [v]_\Sigma)$ implies, by (2), that there exist w', v', and Z' such that $w' \in [w]_\Sigma$, $v' \in [v]_\Sigma$, $Z' \in \beta(w', v')$ and $\{[u]_\Sigma : u \in Z\} = \{[u]_\Sigma : u \in Z'\}$. Since $Z' \subseteq \{u : w' R u R v'\}$ by (1) we obtain that $\{[u]_\Sigma : u \in Z\} \subseteq \{[u]_\Sigma : [w]_\Sigma R_\Sigma [u]_\Sigma R_\Sigma [v]_\Sigma\}$. $\mathcal{A}_\Sigma$ is therefore an e-model. Finally, the proof that $\mathcal{A}_\Sigma$ is a filtration of $\mathcal{A}$ is straightforward. □

Theorem 3.6 (Filtration Theorem for e-models) *If $\mathcal{A}_\Sigma$ is a filtration of the* e-*model $\mathcal{A}$ through Σ, then for each $w \in A$ and $\varphi \in \Sigma$, $w \models \varphi$ iff $[w]_\Sigma \models \varphi$.*

Proof: The only nonstandard steps of the proof (by induction on the construction of φ) are the inductive steps for the operators U and S. Suppose $w \models U(\varphi, \psi)$; then there exists a point v such that wRv and $v \models \varphi$, and there exists $Z \in \beta(w, v)$ such that, for each $u \in Z, u \models \psi$. By Definition 3.4.3 and the inductive hypothesis we obtain that $[w]_\Sigma R_\Sigma [v]_\Sigma$, $[v]_\Sigma \models \varphi$, $[u]_\Sigma \models \psi$ for each $u \in Z$, and $\{[u]_\Sigma : u \in Z\} \in \beta_\Sigma([w]_\Sigma, [v]_\Sigma)$, and thus $[w]_\Sigma \models U(\varphi, \psi)$. For the converse, suppose that $[w]_\Sigma \models U(\varphi, \psi)$, let $[v]_\Sigma$ be such that $[w]_\Sigma R_\Sigma [v]_\Sigma$ and $[v]_\Sigma \models \varphi$, and let $\{[u]_\Sigma : u \in Z\} \in \beta_\Sigma([w]_\Sigma, [v]_\Sigma)$ be such that $[u]_\Sigma \models \psi$ for each $u \in Z$. By the inductive hypothesis, $v \models \varphi$ and $u \models \psi$ for each $u \in Z$ and so, by condition 3.4.4, $w \models U(\varphi, \psi)$. The case of S is analogous. □

Now, the Distinguishable Model Theorem for e-models immediately follows from Lemma 3.5 and the Filtration Theorem, simply taking Σ to be the set of all US-formulas.

Theorem 3.7 (Distinguishable Model Theorem) *For each* e-*model $\mathcal{A}$ there exists a distinguishable* e-*model $\mathcal{A}'$ such that $\mathcal{A} \equiv_{\mathrm{US}} \mathcal{A}'$.*

Considering that Kripke models are (equivalent to) particular e-models, we obtain the following.

Corollary 3.8 *For every Kripke model $\mathcal{M}$ there exists a distinguishable* e-*model $\mathcal{A}$ such that $\mathcal{M} \equiv_{\mathrm{US}} \mathcal{A}$.*

As an example, we determine the distinguishable e-models equivalent to the Kripke models of Examples 2.2–2.4.

As regards Example 2.2, the e-model equivalent to $\mathcal{M}_1$ is $\mathcal{A}_1 = \langle A, R_1', \beta_1, V_1'\rangle$ such that $A = \{a\}$, $R_1' = \{(a, a)\}$, $V_1'(p) = \{a\}$ for every $p \in \mathbb{P}$, and $\beta_1(a, a) = \{\varnothing\}$. If $\mathcal{M}_1$ is considered as an e-model, i.e., endowed with the function $\beta(w_1, w_2) = \{u : w_1 R_1 u R_1 w_2\} = \{\varnothing\}$ and $\beta(w_2, w_1) = \{u : w_2 R_1 u R_1 w_1\} = \{\varnothing\}$, then $\mathcal{A}_1$ is the finest filtration of $\mathcal{M}_1$ through Wff_{US}; in fact $[w_1] = [w_2] = a$, whereas $w_1 R_1 w_2$ implies that $aR_1'a$, and $\beta(w_1, w_2) = \beta(w_2, w_1) = \{\varnothing\}$ implies that $\beta_1(a, a) = \{\varnothing\}$. As expected, $\mathcal{A}_1$ is not a Kripke model; in fact $\{u : aR_1'uR_1'a\} = \{a\}$ but $\beta_1(a, a) = \{\varnothing\}$. Nevertheless, $\mathcal{A}_1$ is a simple e-model.

As regards Example 2.3, the finest filtration of $\mathcal{M}_2$ is $\mathcal{A}_2 = \langle A_2, R_2', \beta_2, V_2'\rangle$, where $A_2 = \{[0], [1]\}$, $R_2' = A_2 \times A_2$, $V_2'(p) = \{[0]\}$ for every $p \in \mathbb{P}$, $\beta_2([0], [1]) =$

$\beta_2([1],[0]) = \{\varnothing, \{[0],[1]\}\}$, $\beta_2([0],[0]) = \{\{[1]\},\{[0],[1]\}\}$, and $\beta_2([1],[1]) = \{\{[0]\},\{[0],[1]\}\}$. By Remark 3.3, β_2 can be equivalently defined as follows : $\beta_2([0],[1]) = \beta_2([1],[0]) = \{\varnothing\}$, $\beta_2([0],[0]) = \{\{[1]\}\}$, $\beta_2([1],[1]) = \{\{[0]\}\}$; thus we obtain that $\mathcal{A}_2$ is (equivalent to) a simple e-model (not a Kripke model, because for each $x, y \in A_2$, $\{u : xR_2'uR_2'y\} = A_2$).

Finally, the distinguishable e-model equivalent to $\mathcal{M}_3$ and obtained by the finest filtration is $\mathcal{A}_3 = \langle A_3, R_3', \beta_3, V_3'\rangle$, where $A_3 = \{a, b, c, d\}$ with $a = [0]$, $b = [\frac{1}{2}]$, $c = [1]$, $d = [\frac{3}{2}]$. Then each world is related to itself and every other world, that is $R_3 = A_3 \times A_3$, $V_3'(p) = \{a, b\}$. Finally,

$$\beta(a,a) = \beta(a,d) = \beta(b,a) = \beta(b,d) = \{\{b,c,d\}, A_3\}$$
$$\beta(a,b) = \beta(a,c) = \beta(b,b) = \beta(b,c) = \{\{b\}, A_3\}$$
$$\beta(c,a) = \beta(c,d) = \beta(d,a) = \beta(d,d) = \{\{d\}, A_3\}$$
$$\beta(c,b) = \beta(c,c) = \beta(d,b) = \beta(d,c) = \{\{d,a,b\}, A_3\}.$$

By erasing A_3 from each set we will obtain a simple e-model.

Therefore the three examples 2.2–2.4, introduced at the beginning to show the nonexistence of equivalent distinguishable Kripke models, have equivalent distinguishable simple e-models. But the class of simple e-models does not satisfy the Distinguishable Model Theorem, as is shown in the next result.

Theorem 3.9 *There exists a model $\mathcal{M}$ such that for every distinguishable simple* e-*model $\mathcal{A}$, $\mathcal{M} \not\equiv_{US} \mathcal{A}$.*

Proof: Consider the Kripke model $\mathcal{M}_4$, where $T_4 = \{w, w', v, v', u, u', t, t'\}$, R_4 is as shown in Figure 4 (in Figures 4–7 below, the symbol TR means that the relation is transitive), and V_4 is any valuation such that $\{p : v \in V_4(p)\} \neq \{p : u \in V_4(p)\}$ and, for each $x \in \{w, v, u, t\}$, $\{p : x \in V_4(p)\} = \{p : x' \in V_4(p)\}$, for example let $V_4(p) = \{w, w', v, v', t, t'\}$ for each $p \in \mathbb{P}$. It is a matter of routine to verify that, for each $x, y \in \{w, v, u, t\}$, $x \equiv_{US} x'$ and $x \not\equiv_{US} y$ if $x \neq y$. Since, clearly, Lemma 2.1 holds also for e-models, we obtain that every distinguishable e-model $\mathcal{A}$ equivalent to $\mathcal{M}_4$ must contain exactly four points. Moreover, since $\mathcal{A} \equiv_{US} \mathcal{M}_4$ implies $\mathcal{A} \equiv_{FP} \mathcal{M}_4$ and since the FP-theory of a point does not depend upon β, we obtain that $\langle A, R, V\rangle$ has to be as follows (see Figure 5): $A = \{a_x : x \in \{w, v, u, t\}\}$, $R = \{(a_x, a_y) : x_0 R_4 y_0$ for some $x_0 \in \{x, x'\}$, $y_0 \in \{y, y'\}\}$ and $a_x \in V(p)$ iff $x \in V_4(p)$. This is, in fact, the only model having four points which is FP-equivalent to $\mathcal{M}$. But there is no β such that $\langle A, R, \beta, V\rangle$ is a simple e-model US-equivalent to $\mathcal{M}_4$. Suppose, for instance, that $\beta(a_w, a_t) = \{\{a_v, a_u\}\}$, and consider the formula $\varphi = (G^3\bot \wedge \neg G^2\bot) \rightarrow U(G\bot, p_0)$. We have that $\mathcal{M}_4 \models \varphi$, because the only points which satisfy $G^3\bot \wedge \neg G^2\bot$ are w and w', the only points which satisfy $G\bot$ are t and t', and from $\{z : wR_4zR_4t\} = \{v\}$ and $\{z : w'R_4zR_4t'\} = \{v'\}$ it follows that w and w' satisfy $U(G\bot, p_0)$. On the other hand $\mathcal{A} \not\models \varphi$ because $a_u \models \neg p_0$. Therefore $\beta(a_w, a_t)$ cannot be $\{\{a_v, a_u\}\}$. In a similar way it can be shown that, for each subset X of A, each e-model $\langle A, R, \beta, V\rangle$ such that $\beta(a_w, a_t) = \{X\}$ is not US-equivalent to $\mathcal{M}$ (for instance, if $X = \{a_v\}$, then $\langle A, R, \beta, V\rangle \models \varphi$, but $\langle A, R, \beta, V\rangle \not\models \varphi_{[p_0/\neg p_0]}$, whereas $\mathcal{M}_4 \models \varphi_{[p_0/\neg p_0]}$). Therefore there is not a distinguishable simple e-model US-equivalent to $\mathcal{M}_4$. □

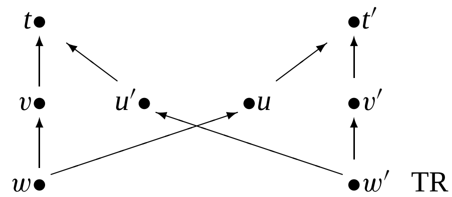

Figure 4

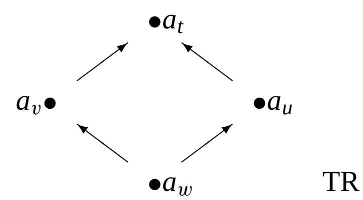

Figure 5

We observe that the distinguishable e-model US-equivalent to $\mathcal{M}_4$ that we obtain by the finest filtration is such that $\beta(a_w, a_t) = \{\{a_v\}, \{a_u\}\}$, and it is not simple. In a similar way, we can show the following.

Theorem 3.10 *For every n, there exists a Kripke model $\mathcal{M}$ such that for every distinguishable* e-*model $\mathcal{A}$, if $\mathcal{M} \equiv_{\mathrm{US}} \mathcal{A}$ then $\mathcal{A}$ contains two points a, b such that $|\beta(a, b)| > n$.*

Proof: Let us consider the model $\mathcal{M} = \langle T, R, V\rangle$, where $T = \{t_k : k \le n+1\} \cup \{w_i : i \le n+1\} \cup \{v_{i,k} : i, k \le n+1\}$, $R = \{(w_s, t_r) : s, r \le n+1\} \cup \{(w_i, v_{i,k}) : i, k \le n+1\} \cup \{(v_{i,k}, t_k) : i, k \le n+1\}$, and V satisfies the following conditions (we write $V^*(x)$ instead of $\{p : x \in V(p)\}$):

1. $V^*(t_i) = V^*(t_j)$, $V^*(w_i) = V^*(w_j)$, for $i, j \le n+1$
2. $V^*(v_{1,1}) = V^*(v_{2,2}) = \cdots = V^*(v_{n,n}) = V^*(v_{n+1,n+1})$
   $V^*(v_{1,2}) = V^*(v_{2,3}) = \cdots = V^*(v_{n,n+1}) = V^*(v_{n+1,1})$
   and, in general,
   $V^*(v_{1,k}) = V^*(v_{2,k\oplus 1}) = \cdots = V^*(v_{n,k\oplus(n-1)}) = V^*(v_{n+1,k\oplus n})$,
   where $\oplus$ denotes the sum modulo $n+1$
3. $V^*(x) \ne V^*(y)$ in the other cases.

(In Figure 6 we take $n+1 = 3$.)

As for the model $\mathcal{M}_4$ of the previous example, we have that $x \equiv_{\mathrm{US}} y$ iff $V^*(x) = V^*(y)$ and thus the equivalence classes are $t = \{t_k : k \le n+1\}$, $w = \{w_k : k \le n+1\}$, and, for each $k \le n+1$, $v_k = \{v_{i,k\oplus(i-1)} : i \le n+1\}$. Therefore each distinguishable e-model $\mathcal{A}$ equivalent to $\mathcal{M}$ is as in Figure 7 as regards R, whereas regarding β we have that, for each $k \le n+1$, $\{v_k\} \in \beta(w, t)$. □

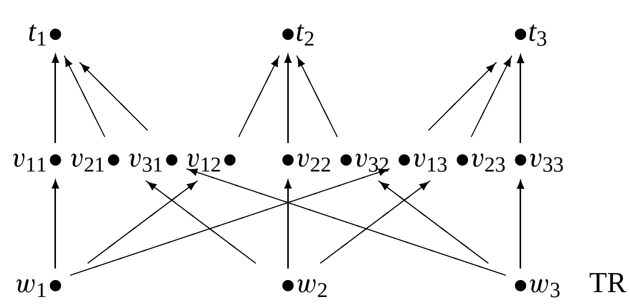

Figure 6

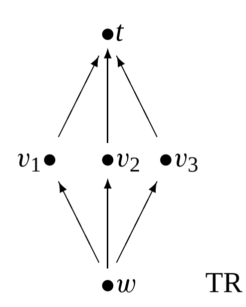

Figure 7

These last examples of e-models seem to suggest an interpretation of $\beta(x, y)$ in terms of *possible* paths. But this picture is in conflict with other examples. For instance, in $\mathcal{A}_2$ we have that $\beta_2([0],[0])$ contains [1] and not [0], but we have $[0]R[0]R[0]$. In this case $\beta(x, y)$ seems to be the set of *admissible* paths. But also this picture is wrong. In fact, if we consider the model $\langle A, R, V\rangle$ of Figure 5 but with $\beta(x, y) = \{\{z : xRzRy\}\}$, we have that $\beta(a_w, a_t) = \{\{a_u, a_v\}\}$, and so in this case β is not a set of paths but the singleton of their union.

4 General e-frames What we have said up to now referring to models can be extended to general frames. We recall that a general frame for $\mathcal{L}_{\mathrm{FP}}$ (or FP-frame) is a triple $\langle T, R, \Pi\rangle$ where $\langle T, R\rangle$ is a frame and Π is a nonempty set of subsets of T closed under Boolean operations and under the operators π_F and π_P, where $\pi_F(X) = \{w : \exists v \in X, wRv\}$ and $\pi_P(X) = \{w : \exists v \in X, vRw\}$. Analogously, a triple $\langle T, R, \Pi\rangle$ is a US-general frame if Π is closed under Boolean operations and the binary operations π_U and π_S, where $\pi_U(Y, Z) = \{x \in T : \exists y \in Y, xRy \text{ and } \{w : xRwRy\} \subseteq Z\}$ and $\pi_S(Y, Z) = \{x \in T : \exists y \in Y, yRx \text{ and } \{w : yRwRx\} \subseteq Z\}$. Clearly, the underlying idea in the definitions of π_U and π_S is the following: if Y and Z are the sets of all the points at which the formulas φ and ψ, respectively, hold, then $\pi_U(Y, Z)$ and $\pi_S(Y, Z)$ are the sets of points at which the formulas $U(\varphi, \psi)$ and $S(\varphi, \psi)$, respectively, hold. From the fact that $\pi_F(X) = \pi_U(X, T)$ and $\pi_P(X) = \pi_S(X, T)$ it follows that a US-general frame is an FP-general frame. Moreover $\langle T, R, \mathcal{P}(T)\rangle$ is a US-general frame, and thus Kripke frames can be considered as particular cases of US-general frames. $\mathcal{M} = \langle T, R, V\rangle$ is a model over $\mathcal{F} = \langle T, R, \Pi\rangle$ if for every $p \in \mathbb{P}$, $V(p) \in \Pi$. The link between models and general frames lies in the fact that, on the one hand, if $\mathcal{M}$ is a model over $\mathcal{F}$, then, for each φ, $\{w : (\mathcal{M}, w) \models \varphi\} \in \Pi$, and, on the other hand, given a model $\langle T, R, V\rangle$, the triple $\langle T, R, \Pi_V\rangle$, where $\Pi_V = \{X \subseteq T : \exists \varphi, X = \{w : (\mathcal{M}, w) \models \varphi\}\}$, is a general frame. A general frame is said to be *distinguishable* if, for each $w, v \in T$, there exists $X \in \Pi$ such that $w \in X$ and $v \notin X$. It is immediate to observe that $\mathcal{M}$ is distinguishable iff $\langle T, R, \Pi_V\rangle$ is distinguishable and that no distinguishable model can be defined over a nondistinguishable general frame. The usual proof that, for every model, there exists an FP-equivalent distinguishable model can be immediatly converted into a proof of the fact that, among the FP-general frames, for every general frame there exists an FP-equivalent distinguishable frame. On the other hand, each example among Examples 2.2–2.4 and Theorem 3.9 provides an example of a US-general frame without any US-equivalent distinguishable frame; in these cases it is in fact sufficient to consider the Π generated by those sets containing points with the same valuation. So, from Example 2.2, we obtain that the US-general frame $\langle T_1, R_1, \Pi_1\rangle$, where $\Pi_1 = \{\varnothing, T_1\}$, is without any US-equivalent distinguishable frame; and the same happens for $\langle T_2, R_2, \Pi_2\rangle$, where $\Pi_2 = \{\varnothing, \mathbb{Z}, \{2k : k \in \mathbb{Z}\}, \{2k+1 : k \in \mathbb{Z}\}\}$, for $\langle T_3, R_3, \Pi_3\rangle$, where $\Pi_3 = \{\varnothing, \mathbb{R}, \{x : 2k \leq x < 2k+1, k \in \mathbb{Z}\}, \{x : 2k+1 \leq x < 2k+2, k \in \mathbb{Z}\}\}$, and for $\langle T_4, R_4, \Pi_4\rangle$ (see Theorem 3.9), where Π_4 contains $\varnothing$ and all possible unions of the sets $\{x, x'\}$ for $x \in \{w, v, u, t\}$. As observed in the introduction, the fact that there exist general frames without any US-equivalent distinguishable frame says that this concept is not the appropriate concept for a general treatment of the US-logics. In fact these structures cannot be interpreted as dual spaces of those algebras obtained by adding to Boolean algebras two binary operators u and s satisfying the identities that correspond to the theorems in $\mathrm{TL}_{\mathrm{US}}$ (i.e., the minimal US-logic, see [2]).

So, in analogy with the case of models, we give the following definition of an e-frame.

Definition 4.1 A *US-general e-frame* is a four-tuple $\langle A, R, \beta, \Pi\rangle$, where $\langle A, R, \Pi\rangle$ is a US-general frame and β is defined as in Definition 3.1.

A four-tuple $\langle A, R, \beta, V\rangle$ is an e-model over $\langle A, R, \beta, \Pi\rangle$ iff, for every $p \in \mathbb{P}$, $V(p) \in \Pi$; on the other hand, if $\langle A, R, \beta, V\rangle$ is an e-model, then $\langle A, R, \beta, \Pi_V\rangle$ is a US-general e-frame. Therefore we have the following result.

Theorem 4.2

1. *For each* US-*formula* φ, φ *is true in every Kripke frame iff* φ *is true in every* US-*general* e-*frame;*
2. *for every* US-*general frame there exists an equivalent distinguishable* US-*general* e-*frame.*

Proof:

1. Both conditions are equivalent to $\varphi \in \mathrm{TL}_{\mathrm{US}}$ (see Theorem 3.2).
2. Let $\langle T, R, \Pi\rangle$ be a US-general frame and, for every $w \in T$, let $[w]_\Pi = \{v : v \in X \quad \text{iff} \quad w \in X \text{ for every } X \in \Pi\}$. Let us consider the four-tuple $\langle T', R', \beta', \Pi'\rangle$, where $T' = \{[w]_\Pi : w \in T\}$, R' and β' are defined as in 1–2 in the proof of Lemma 3.5, and $\Pi' = \{\{[w]_\Pi : w \in X\} : X \in \Pi\}$. From the proof of Lemma 3.5 and Theorem 3.6 we obtain that this is a US-general e-frame and $\langle T, R, \Pi\rangle \equiv_{\mathrm{US}} \langle T', R', \beta', \Pi'\rangle$. □

□

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