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PA(*aa*)

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Abstract The theory PA(*aa*), which is Peano Arithmetic in the context of stationary logic, is shown to be consistent. Moreover, the first-order theory of the class of finitely determinate models of PA(*aa*) is characterized.

1 Introduction It has been known since the work of Gödel in 1931 that incompleteness is ubiquitous, being a property of all sufficiently rich systems such as Peano Arithmetic. This incompleteness phenomenon would not materially affect ordinary mathematical practice if only contrived sentences (for example, Con(PA)) were independent. However, the celebrated Paris-Harrington Theorem of 1977 and many other theorems discovered since then show that there are mathematically interesting sentences in the language of PA that PA does not decide.

One approach to dealing with the incompleteness of PA is to consider PA in the context of logics which are more expressive than first-order logic but retain some of first-order logic's more desirable features, such as compactness. This approach has been taken in the papers Macintyre [7], Morgenstern [9], and Schmerl [10] and [11]. Closely related work involving ZF appears in Kakuda [4] and Kaufmann [6].

In this paper the theory PA(*aa*), which is Peano Arithmetic in the context of stationary logic, is investigated. Stationary logic, which was inspired by the work of Shelah [12], was thoroughly studied in the fundamental paper of Barwise, Kaufmann and Makkai [1], where its completeness and compactness were proved. It will be shown here that PA(*aa*) is consistent. A precise description of the first-order consequences of PA(*aa*) would certainly be hoped for (and was essentially asked for in Remark 4.8 of [6]), but we are able to determine only the first-order theory of the finitely determinate models of PA(*aa*).

Let CA be the second-order theory of arithmetic consisting of PA, the induction axiom, and all instances of the comprehension scheme. Recall that the Paris-Harrington Principle is a consequence of (theories much weaker than) CA. Let (*Det*) denote the scheme in stationary logic for finite determinateness. The main results of this paper are summarized in the following theorem.

Theorem 1.1 *The first-order consequences of PA(aa) + (Det) are precisely the same as the first-order consequences of CA.*

2 Preliminaries Let $\mathcal{L}_{PA}$ be one of the usual languages appropriate for PA. If $\mathcal{L} \supseteq \mathcal{L}_{PA}$ is a larger language, then PA* is derived from PA by adjoining all instances of the induction scheme for $\mathcal{L}$-formulas. We formalize PA* so that there are terms for all Skolem functions. In particular, for each $n < \omega$ there is a term $\langle x_0, x_1, \ldots, x_{n-1}\rangle$ for n-tuple formation.

Now let $\mathcal{L} \supseteq \mathcal{L}_{PA}$ be a countable language and let T $\supseteq$ PA* be a completion of PA*. By a type $p(x)$ over T we mean a maximal set of 1-ary $\mathcal{L}$-formulas consistent with T. A type $p(x)$ is unbounded if it contains the formula $t < x$ for each constant term t. Gaifman [3] defined minimal types and showed that they exist over any T. It follows from Theorem 3.9 of Mills [8] that a type $p(x)$ is minimal iff $p(x)$ is unbounded and for any $(1+n)$-ary $\mathcal{L}$-formula $\varphi(x, y_0, y_1, \ldots, y_{n-1})$ there is a formula $\theta(x)$ in $p(x)$ such that the sentence

$$\begin{aligned} &\forall x\, \exists w[\forall \bar{y}(w < y_0 < y_1 < \cdots < y_{n-1} \wedge \theta(y_0) \wedge \cdots \wedge \theta(y_{n-1}) \longrightarrow \varphi(x, \bar{y}))\\ &\vee \quad \forall \bar{y}(w < y_0 < y_1 < \cdots < y_{n-1} \wedge \theta(y_0) \wedge \cdots \wedge \theta(y_{n-1}) \longrightarrow \neg\varphi(x, \bar{y}))] \end{aligned}$$

is in T.

Gaifman [3] discusses iterated extensions. Consider some completion T $\supseteq$ PA*, and let $\mathbf{M} \models$ T. Fix a minimal type $p(x)$ over T. If $(I, <)$ is a linearly ordered set (for which $M \cap I = \varnothing$), then there is a model $\mathbf{M}(I)$ which is generated by $M \cup I$ such that whenever $a \in M$, $b, c \in I$, and $b \leq c$, then $\mathbf{M}(I) \models a < b \leq c$ and b realizes $p(x)$. The model $\mathbf{M}(I)$, which is unique up to isomorphism over $M \cup I$, is the Ith *canonical iterated extension* of $\mathbf{M}$. The set I is indiscernible in $\mathbf{M}(I)$. We write $J < I$ if $J \subseteq I$ is a proper initial segment of I (that is, $J \neq I$ but possibly $J = \varnothing$). If $J < I$ then $\mathbf{M}(I)$ is an elementary end extension of $\mathbf{M}(J)$. Thus, if $\mathbf{M}$ is countable and $(I, <)$ is ω_1-like, then $\mathbf{M}(I)$ is ω_1-like.

We will need second-order structures of the form $(\mathbf{M}, \mathcal{X})$, where $\mathbf{M} \models$ PA* and $\mathcal{X} \subseteq \wp(M)$. If $X \in \mathcal{X}$ and $m \in M$, then $(X)_m = \{a \in M : \langle m, a\rangle \in X\}$. The second-order $\mathcal{L}_{PA}$-theory CA consists of PA, the induction axiom, and the scheme of comprehension axioms

$$\exists X \forall x(x \in X \longleftrightarrow \varphi(x)),$$

where $\varphi(x)$ is a second-order formula which can have undisplayed first or second-order free variables other than X. The second-order $\mathcal{L}_{PA}$-theory AC consists of CA plus the scheme of choice axioms

$$\forall x \exists X \varphi(X, x) \longrightarrow \exists X \forall x \varphi((X)_x, x),$$

where $\varphi(X, x)$ may have additional undisplayed free variables. It is well known that if $(\mathbf{M}, \mathcal{X}) \models$ CA, then there is $\mathcal{X}_0 \subseteq \mathcal{X}$ such that $(\mathbf{M}, \mathcal{X}_0) \models CA + AC$. For languages $\mathcal{L} \supseteq \mathcal{L}_{PA}$, we define the second-order $\mathcal{L}$-theories CA* and AC* in an analogous way.

If $\mathbf{M} \models$ PA*, then $Def(\mathbf{M})$ is the set of parametrically definable subsets of M, and $Def_0(\mathbf{M})$ is the set of subsets of M which are definable without parameters.

Stationary logic is an extension of first-order logic formed by adjoining the second-order "almost all" quantifier aa. We give a brief description of it here. For any set A let $\wp_{\omega_1}(A)$ be the set of countable subsets of A. A set $X \subseteq \wp_{\omega_1}(A)$ is closed iff whenever $s_0 \subseteq s_1 \subseteq s_2 \subseteq \cdots$ is an increasing sequence of elements of X, then $\bigcup\{s_n : n < \omega\} \in X$; and X is unbounded iff whenever $t \in \wp_{\omega_1}(A)$, then

$t \subseteq s \in X$ for some s. To define the satisfaction relation $\mathbf{A} \models \theta$, where θ is a formula of stationary logic, adjoin to the usual clauses in the definition of satisfaction the following clause: $\mathbf{A} \models aas\varphi$ iff there is a closed, unbounded $X \subseteq \wp_{\omega_1}(A)$ such that for each $s \in X, \mathbf{A} \models \varphi$. In other words, $\mathbf{A} \models aas\varphi$ iff for almost all countable $s \subseteq A, \mathbf{A} \models \varphi$. By convention, all models for stationary logic are models of the sentence $aas\, \exists x(x \notin s)$, and hence are uncountable. Among the valid sentences of stationary logic are those of the Diagonal Intersection Scheme:

$$\forall x\, aas\varphi(x, s) \longrightarrow aas\, \forall x \in s\varphi(x, s).$$

The scheme of finite determinateness, denoted by (Det) and introduced by Kaufmann [5], is the set of sentences $aas_0\, aas_1 \ldots aas_{m-1}\, \forall\bar{x}[aat\varphi(\bar{x}, \bar{s},\ t) \vee aat\neg\varphi(\bar{x}, \bar{s}, t)]$, where $\varphi(\bar{x}, \bar{s},\ t)$ is a formula in which only the displayed variables occur freely. Models of (Det) are said to be *finitely determinate*.

Eklof and Mekler [2] studied finitely determinate models and obtained, among other things, a useful criterion for finite determinateness. Let $\mathbf{A}$ be a structure having cardinality $\aleph_1$. A *filtration* for $\mathbf{A}$ is a sequence $\langle \mathbf{A}_\alpha : \alpha < \omega_1 \rangle$ of countable substructures of $\mathbf{A}$ such that:

1. if $\alpha < \beta < \omega_1$ then $\mathbf{A}_\alpha \prec \mathbf{A}_\beta \prec \mathbf{A}$;
2. if $\lambda < \omega_1$ is a limit ordinal, then $\mathbf{A}_\lambda = \bigcup\{\mathbf{A}_\alpha : \alpha < \lambda\}$;
3. $\mathbf{A} = \bigcup\{\mathbf{A}_\alpha : \alpha < \omega_1\}$.

The following is Theorem 1.3(1) of [2].

Theorem 2.1 (The Eklof-Mekler Criterion) *Suppose* $\mathbf{A}$ *has cardinality* $\aleph_1$. *Then* $\mathbf{A}$ *is finitely determinate iff there is a filtration* $\langle \mathbf{A}_\alpha : \alpha < \omega_1 \rangle$ *for* $\mathbf{A}$ *such that whenever* $k \leq n < \omega, r < \omega, \alpha_0 < \alpha_1 < \cdots < \alpha_n < \omega_1, \beta_0 < \beta_1 < \cdots < \beta_n < \omega_1$ *and* $c_0, c_1, \ldots, c_r \in A_{\alpha_k} \cap A_{\beta_k}$ *are such that* $\alpha_j = \beta_j$ *for* $j < k$*, then*

$$(\mathbf{A},\ A_{\alpha_0},\ A_{\alpha_1},\ \ldots,\ A_{\alpha_n},\ c_0,\ c_1,\ \ldots,\ c_r) \equiv$$
$$(\mathbf{A},\ A_{\beta_0},\ A_{\beta_1},\ \ldots,\ A_{\beta_n},\ c_0,\ c_1,\ \ldots,\ c_r).$$

The theory PA(aa) is the $\mathcal{L}_{PA}$-theory for stationary logic consisting of PA together with all instances of the induction scheme:

$$\varphi(0) \wedge \forall x(\varphi(x) \longrightarrow \varphi(x+1)) \longrightarrow \forall x\varphi(x),$$

where $\varphi(x)$ is a formula of stationary logic which may have undisplayed free first-order variables but which has no free second-order variables. The purpose of this paper is to prove that PA(aa) + (Det) and CA have precisely the same first-order consequences.

3 Properties of models of PA(aa)

In this section we determine some properties of models of PA(aa).

Proposition 3.1 *If* $\mathbf{N} \models PA(aa)$*, then* $\mathbf{N}$ *is* ω_1*-like.*

Proof: Suppose $\mathbf{N} \models \mathrm{PA}(aa)$, and consider the formula $\varphi(x) = aas\forall y(y < x \longrightarrow y \in s)$. Clearly, $\mathbf{N} \models \varphi(0) \wedge \forall x(\varphi(x) \longrightarrow \varphi(x+1))$, so that $\mathbf{N} \models \forall x\varphi(x)$. Thus, for each $a \in N$ the set $\{y \in N : y < a\}$ is countable. Each proper initial segment of $\mathbf{N}$ is countable, but $\mathbf{N}$ is uncountable; therefore $\mathbf{N}$ is ω_1-like.

Suppose that $\mathbf{N} \models \mathrm{PA}$ is ω_1-like. Then almost all countable $s \subseteq N$ are cuts of $\mathbf{N}$ such that $\mathbf{N}|s \prec \mathbf{N}$. A subset $X \subseteq s$ is *coded* if $X = s \cap Y$ for some $Y \in Def(\mathbf{N})$. Let $\mathcal{X}(s)$ be the set of coded subsets of s, and let $s^* = (\mathbf{N}|S, \mathcal{X}(s))$. Notice that if $\varphi(\bar{y})$ is a second-order formula with only the variables $y_0,\ y_1,\ \ldots,\ y_{n-1}$ free, then there is an $\mathcal{L}_{PA}(aa)$-formula $\theta(s,\ \bar{y})$ having no set quantifiers such that for any elementary cut s of $\mathbf{N}$ (and thus for almost all countable $s \subseteq \mathbf{N}$) and all $a_0,\ a_1,\ \ldots,\ a_{n-1} \in s$, $s^* \models \varphi(\bar{a})$ iff $(\mathbf{N},\ s) \models \theta(s,\ \bar{a})$. □

The sentences of CA that occur as instances of the comprehension scheme involve a formula $\varphi(x)$ which is allowed to have undisplayed free first and second-order variables. Let $\Lambda-$CA be the theory consisting of PA together with all instances of the comprehension scheme in which the formula $\varphi(x)$ does not have free second-order variables. The following lemma is a consequence of Theorem 1.5 of [10].

Lemma 3.2 *CA and $\Lambda-$CA have the same first-order consequences.*

The next lemma, with Lemma 3.2, has (the easy) half of the Theorem 1.1 as an immediate consequence.

Lemma 3.3 *Suppose $\mathbf{N} \models PA(aa) + (Det)$. Then for almost all $s \subseteq N, s^* \models \Lambda-CA$.*

Proof: Since $\Lambda-$CA contains only countably many sentences, it suffices to show that for any second-order formula $\varphi(x,\ \bar{y})$ (with only the variables $x,\ y_0,\ y_1,\ \ldots,\ y_{n-1}$ free) and for almost all countable $s \subseteq N$,

$$s^* \models \forall\bar{y}\exists X\,\forall x(x \in X \longleftrightarrow \varphi(x,\ \bar{y})).$$

Let $\theta(s,\ \bar{y})$ be an $\mathcal{L}_{PA}(aa)$-formula such that for almost all countable $s \subseteq N$ and all $a_0,\ a_1,\ \ldots, a_{n-1} \in s$;

$$s^* \models \exists X\,\forall x(x \in X \longleftrightarrow \varphi(x,\ \bar{a})) \quad \text{iff} \quad \mathbf{N} \models \theta(s,\ \bar{a}).$$

To obtain a contradiction, we now assume that

$$\mathbf{N}\ \models\ \neg aas\forall y_0,\ y_1,\ \ldots,\ y_{n-1} \in s\,\theta(s,\ \bar{y}).$$

By the Diagonal Intersection Scheme,

$$\mathbf{N}\ \models\ \exists\bar{y}\neg aas\,\theta(s,\ \bar{y}),$$

and then by (Det),

$$\mathbf{N}\ \models\ \exists\bar{y}\,aas\neg\theta(s,\ \bar{y}).$$

Then let $a_0,\ a_1,\ \ldots,\ a_{n-1} \in N$ be such that

$$\mathbf{N}\ \models\ aas\neg\theta(s,\ \bar{a}).$$

Let $B = \{x \in N : \mathbf{N} \models aas[s^* \models \varphi(x, \bar{a})]\}$, so $\mathbf{N} \models \forall x(x \in B \longleftrightarrow aas[s^* \models \varphi(x, \bar{a})])$. Then by (Det), $\mathbf{N} \models \forall x\, aas\, (x \in B \longleftrightarrow [s^* \models \varphi(x, \bar{a})])$, and therefore, by the Diagonal Intersection Scheme, $\mathbf{N} \models aas\, \forall x \in s(x \in B \longleftrightarrow [s^* \models \varphi(x, \bar{a})])$. But $B \cap s \in \mathcal{X}(s)$ for each s, so that for almost all countable $s \subseteq N$, $s^* \models \exists X \forall x(x \in X \longleftrightarrow \varphi(x, \bar{a}))$. This is a contradiction. $\square$

Corollary 3.4 *If $\mathbf{N} \models PA(aa) + (Det)$, then there is $(\mathbf{M}, \mathcal{X}) \models CA$ such that $\mathbf{M} \equiv \mathbf{N}$.*

4 Constructing the models The aim of this section is to construct models of $\mathrm{PA}(aa) + (Det)$. More specifically, for any model $\mathbf{M} \models \mathrm{PA}$ whose theory is consistent with the first-order consequences of CA, we will construct $\mathbf{N} \equiv \mathbf{M}$ such that $\mathbf{N} \models \mathrm{PA}(aa) + (Det)$. The existence of such a model $\mathbf{N}$, together with Corollary 3.4, constitutes the proof of Theorem 1.1.

It can be assumed, without loss of generality, that the model $\mathbf{M}$ with which we start is countable. Moreover, it can be assumed that $(\mathbf{M}, \mathcal{X}) \models \mathrm{CA}$ for some countable $\mathcal{X}$. As mentioned in Section 2, it can further be assumed that $(\mathbf{M}, \mathcal{X}) \models \mathrm{CA}{+}\mathrm{AC}$.

Consider some countable $(\mathbf{M}, \mathcal{X}) \models \mathrm{CA}{+}\mathrm{AC}$, which is to be fixed for the remainder of this section. Let $\mathcal{X} = \{A_0, A_1, A_2, \ldots\}$ and let $\mathbf{M}^* = (\mathbf{M}, A_0, A_1, A_2, \ldots)$, which is an $\mathcal{L}$-structure for some $\mathcal{L} \supseteq \mathcal{L}_{PA}$. Clearly $\mathbf{M}^* \models \mathrm{PA}^*$ and $Def(\mathbf{M}^*) = Def_0(\mathbf{M}^*) = \mathcal{X}$. Consider some minimal type $p(x)$ over $Th(\mathbf{M}^*)$, which is also to be fixed for the remainder of this section. Throughout this section we will be taking canonical iterated extensions tacity understood to be relative to this minimal type.

Lemma 4.1 *Suppose that $(I, <)$ is a linearly ordered set and that $n < \omega$, and let $I_0 < I_1 < \cdots < I_{n-1} < I$. Let $J \subseteq I$ be such that if we set $J_i = J \cap I_i$ for $i < n$, then $J_0 < J_1 < \cdots < J_{n-1} < J$. Then*

$$
\begin{aligned}
&(\mathbf{M}^*(J), M^*(J_0),\ M^*(J_1), \ldots, M^*(J_{n-1})) \\
&\prec (\mathbf{M}^*(I), M^*(I_0), M^*(I_1), \ldots, M^*(I_{n-1})).
\end{aligned}
$$

Proof: It will be convenient to let $I_n = I$ and $J_n = J$. We introduce the following notation: if $F \subseteq I$, then $\mathbf{M}'(F) = (\mathbf{M}^*(F), M^*(F \cap I_0), M^*(F \cap I_1), \ldots, M^*(F \cap I_n))$. With this notation, our objective is to prove that $\mathbf{M}'(J) \prec \mathbf{M}'(I)$.

Observe that it suffices to prove the lemma just for finite I. For, suppose that we have done so, and then let $F_0 \subseteq J$ be finite such that $F_0 \cap (I_{i+1} \backslash I_i) \neq \varnothing$ for each $i < n$. Consider the directed system

$$D_I = \{\mathbf{M}'(F) : F_0 \subseteq F \subseteq I \text{ and } F \text{ is finite}\},$$

which is directed by extension. Let D_J be its directed subsystem consisting of those $\mathbf{M}'(F)$ in D_I for which $F \subseteq J$. Using that the lemma has been proved for finite I, we see that D_I and D_J are directed by elementary extension and their unions are respectively $\mathbf{M}'(I)$ and $\mathbf{M}'(J)$. Then, for any $\mathbf{M}'(F)$ in D_I we see that $\mathbf{M}'(F \cap J) \in D_J$ and that $\mathbf{M}'(F \cap J) \prec \mathbf{M}'(F) \prec \mathbf{M}'(I)$ and $\mathbf{M}'(F \cap J) \prec \mathbf{M}'(J)$, from which it easily follows that $\mathbf{M}'(J) \prec \mathbf{M}'(I)$.

Furthermore, it clearly suffices to prove the lemma just for finite I and J for which $|I| = |J| + 1 = n + 1 \geq 2$. So let $I = \{a_0, a_1, \ldots, a_n\}$, where $a_0 < a_1 < \cdots a_n$, and let $m \leq n$ be such that $J = I \backslash \{a_m\}$. For each $i \leq n$ let

$$I_i = \begin{cases} \{a_0, a_1, \ldots, a_{i-1}\} & \text{if } i < m, \\ \{a_0, a_1, \ldots, a_i\} & \text{if } i > m, \end{cases}$$

and for $i = m$ there are two possibilities: either $I_m = \{a_0, a_1, \ldots, a_{m-1}\}$ or $I_m = \{a_0, a_1, \ldots, a_m\}$. Then $J_i = I_i \backslash \{a_m\}$ for each $i \leq n$. Let $h : n+1 \to n+2$ be such that $I_i = \{a_0, a_1, \ldots, a_{h(i)-1}\}$ for each $i \leq n$.

Let $\mathcal{L}'$ be the language appropriate for $\mathbf{M}'(I)$; specifically, let $\mathcal{L}' = \mathcal{L} \cup \{U_0, U_1, \ldots, U_n\}$, where $M^*(I_i)$ is the interpretation of U_i in $\mathbf{M}'(I)$, and $M^*(J_i)$ is its interpretation in $\mathbf{M}'(J)$.

In order to prove that $\mathbf{M}'(J) \prec \mathbf{M}'(I)$, we consider an arbitrary $\mathcal{L}'$-formula $\varphi(v_0, v_1, \ldots, v_n)$, with free variables $v_0, v_1, \ldots, v_n$, intending to prove: if $\mathbf{M}'(I) \models \varphi(\bar{a})$, then there is $b \in M^*(J)$ such that $\mathbf{M}'(J) \models \varphi(a_0, a_1, \ldots, a_{m-1}, b, a_{m+1}, \ldots, a_n)$.

Without loss of generality, we can assume that $\varphi(\bar{v})$ has the form $(Q_0 x_0 \in U_{s_0})(Q_1 x_1 \in U_{s_1}) \ldots (Q_t x_t \in U_{s_t}) \psi(\bar{x}, \bar{v})$, where for each $j \leq t$, Q_j is one of the quantifiers $\exists$ or $\forall$ and $s_j \leq n$, and $\psi(\bar{v})$ is an $\mathcal{L}$-formula.

We are about to define a second-order formula $\sigma(\bar{v})$ for which we will need some second-order variables. For each $j < \omega$, we will use the second-order variable X_j to range over functions. First, obtain the formula $\psi'(\bar{X}, \bar{v})$ by replacing each free occurrence of x_j in $\psi(\bar{x}, \bar{v})$ by

$$\mathcal{X}_j(\langle v_0, v_1, \ldots, v_{h(s_j)-1}\rangle).$$

Then let $\sigma(\bar{v})$ be the formula $(Q_0 X_0)(Q_1 X_1) \ldots (Q_t X_t) \psi'(\bar{X}, \bar{v})$

Let $\mathcal{X}_0 = Def_0(\mathbf{M}^*(I))$. Think of each $X \in \mathcal{X}_0$ as a function; that is, $X(a) = b$ iff either b is the unique element for which $\langle a, b\rangle \in X$ or else there is no such unique element and $b = 0$. Notice that for any $b \in M^*(I)$, $b \in M^*(I_j)$ iff there is $X \in \mathcal{X}_0$ such that

$$b = \mathcal{X}(\langle a_0, a_1, \ldots, a_{h(s_j)-1}\rangle).$$

Therefore, it is clear that $\mathbf{M}'(I) \models \varphi(\bar{a})$ iff $(\mathbf{M}^*(I), \mathcal{X}_0) \models \sigma(\bar{a})$.

Since $(\mathbf{M}, \mathcal{X}) \models \text{CA}$, there is $B \in \mathcal{X}$ such that $(\mathbf{M}, \mathcal{X}) \models \forall \bar{v}(\langle v_0, v_1, \ldots, v_n\rangle \in B \leftrightarrow \sigma(\bar{v}))$. Then, since $p(x)$ is a minimal type over $Th(\mathbf{M}^*)$, there is a formula $\theta(x)$ in $p(x)$ such that either

$$\mathbf{M}^* \models \forall \bar{v}(\theta^<(\bar{v}) \longrightarrow \langle v_0, v_1, \ldots, v_n\rangle \in B),$$

or

$$\mathbf{M}^* \models \forall \bar{v}(\theta^<(\bar{v}) \longrightarrow \langle v_0, v_1, \ldots, v_n\rangle \notin B),$$

where we have let $\theta^<(\bar{v})$ be an abbreviation for $v_0 < v_1 < \cdots < v_n \wedge \theta(v_0) \wedge \theta(v_1) \wedge \cdots \wedge \theta(v_n)$.

We will show that if $\mathbf{M}'(I) \models \varphi(\bar{a})$, then the first alternative holds. Suppose, to the contrary, that $\mathbf{M}'(I) \models \varphi(\bar{a})$ and $\mathbf{M}^* \models \forall \bar{v}(\theta^<(\bar{v}) \longrightarrow \langle v_0, v_1, \ldots, v_n\rangle \notin B)$. Then $(\mathbf{M}, \mathcal{X}) \models \forall \bar{v}(\theta^<(\bar{v}) \to \neg\sigma(\bar{v}))$. Now construct a sentence γ as follows: first, obtain $\psi''(\bar{X}, \bar{v})$ from $\psi'(\bar{X}, \bar{v})$ by replacing each occurrence of X_j, where Q_j is $\forall$, by $(X_j)_{\bar{v}}$. Then define γ to be the sentence

$$\gamma = (Q_0 X_0)(Q_1 X_1)\ldots(Q_t X_t)[\exists \bar{v}(\theta^{<}(\bar{v}) \wedge \psi''(\bar{X}, \bar{v}))].$$

It is easy to see that

$$AC^* \vdash \forall \bar{v}(\theta^{<}(\bar{v}) \longrightarrow \neg\sigma(\bar{v})) \longleftrightarrow \neg\gamma$$

and

$$\vdash \neg\gamma \longrightarrow \forall \bar{v}(\theta^{<}(\bar{v}) \longrightarrow \neg\sigma(\bar{v})).$$

It follows from $(\mathbf{M}^*(I), \mathcal{X}_0) \models \sigma(\bar{a})$ that $(\mathbf{M}^*(I), \mathcal{X}_0) \models \gamma$, from which it easily follows that $(\mathbf{M}^*, \mathcal{X}) \models \gamma$. Then, since $(\mathbf{M}^*, \mathcal{X}) \models AC^*$, it follows that $(\mathbf{M}^*, \mathcal{X}) \models \exists \bar{v}(\theta^{<}(\bar{v}) \wedge \sigma(\bar{v}))$, which is a contradiction. This proves $\mathbf{M}^* \models \forall \bar{v}(\theta^{<}(\bar{v}) \longrightarrow \langle v_0, v_1, \ldots, v_n\rangle \in B)$.

To complete the proof of the lemma, assume $\mathbf{M}'(I) \models \varphi(\bar{a})$, intending to prove $\mathbf{M}'(J) \models \varphi(a_0, a_1, \ldots, a_{m-1}, b, a_{m+1}, \ldots, a_n)$ for some $b \in M'(J)$.

Let $t_1(v)$ be the Skolem term which is "the largest element x for which $x < v \wedge \theta(x)$" and let $t_2(v)$ be the Skolem term which is "the smallest element x for which $x > v \wedge \theta(x)$." We will use $t_1(v)$ if $I_m = \{a_0, a_1, \ldots, a_{m-1}\}$, and $t_2(v)$ if $I_m = \{a_0, a_1, \ldots, a_m\}$. Since the two cases are so similar, we will consider only the second one. So, assume that $I_m = \{a_0, a_1, \ldots, a_m\}$.

Let $\theta'(x) \in p(x)$ be such that $\mathbf{M}^* \models \forall x(\theta'(x) \longrightarrow \theta(x)) \wedge \forall xy\exists z(\theta'(x) \wedge \theta'(y) \wedge x < y \longrightarrow x < z < y \wedge \theta(z))$. Clearly there is such a formula $\theta'(x)$; either the formula asserting that x is the kth element for which $\theta(x)$ for some even k, or the formula asserting the same thing for odd k, will work. Let $b \in M^*(I)$ be such that $\mathbf{M}^*(I) \models b = t_2(a_{m-1})$. (If $m = 0$, then let b be the smallest such that $\mathbf{M}^*(I) \models \theta(b)$.) Notice that $b \in M^*(J_m) \subseteq M^*(J)$. We claim that $\mathbf{M}^*(J) \models \varphi(a_0, a_1, \ldots, a_{m-1}, b, a_{m+1}, \ldots, a_n)$.

First, observe that $\mathbf{M}^*(I) \models a_0 < a_1 < \cdots < a_n \wedge \theta'(a_0) \wedge \cdots \wedge \theta'(a_n)$. Therefore, $\mathbf{M}^*(I) \models a_0 < a_1 < \cdots < a_{m-1} < b < a_{m+1} < \cdots < a_n \wedge \theta(a_0) \wedge \cdots \wedge \theta(a_{m-1}) \wedge \theta(b) \wedge \theta(a_{m+1}) \wedge \cdots \wedge \theta(a_n)$. Next, let $\sigma'(v_0, v_1, \ldots, v_{m-1}, v_{m+1}, \ldots, v_n)$ be the formula obtained from $\sigma(\bar{v})$ by replacing each free occurrence of the variable v_m with $t_2(v_{m-1})$. Then, as we saw before, $\mathbf{M}^* \models \forall \bar{v}(\theta'(v_0) \wedge \cdots \wedge \theta'(v_{m-1}) \wedge \theta'(v_{m+1}) \wedge \cdots \wedge \theta'(v_n) \wedge v_0 < v_1 < \cdots < v_{m-1} < v_{m+1} < \cdots < v_n \longrightarrow \sigma'(v_0, v_1, \ldots, v_{m-1}, v_{m+1}, \ldots, v_n))$. Thus, it easily follows that $\mathbf{M}^*(J) \models \varphi(a_0, \ldots, a_{m-1}, b, a_{m+1}, \ldots, a_n)$. $\square$

Corollary 4.2 *If $(I, <)$ is ω_1-like, then $\mathbf{M}^*(I)$ is finitely determinate.*

Proof: Since $(I, <)$ is ω_1-like, it has a filtration $\langle I_\alpha : \alpha < \omega_1\rangle$ such that $I_\alpha < I_\beta < I$ whenever $\alpha < \beta < \omega_1$. Then $\langle \mathbf{M}^*(I_\alpha) : \alpha < \omega_1\rangle$ is a filtration for $\mathbf{M}^*(I)$. We will show that this filtration satisfies the Eklof-Mekler Criterion.

Let $k, n, r, \alpha_0, \alpha_1, \ldots, \alpha_n, \beta_0, \beta_1, \ldots \beta_n, c_0, c_1, \ldots, c_r$ be as in that criterion. Pick some

$$a_j \in I_{\alpha_{j+1}} \backslash I_{\alpha_j}$$

and

$$b_j \in I_{\beta_{j+1}} \backslash I_{\beta_j}$$

for $j = k, k+1, \ldots, n-1$ and pick

$$a_n \in I \backslash I_{\alpha_n}$$

and

$$b_n \in I \backslash I_{\beta_n}.$$

Let

$$J = (I_{\alpha_k} \cap I_{\beta_k}) \cup \{a_k, a_{k+1}, \ldots, a_n\}$$

and

$$K = (I_{\alpha_k} \cap I_{\beta_k}) \cup \{b_k, b_{k+1}, \ldots, b_n\}.$$

Then Lemma 4.1 implies that $\mathbf{M}'(J) \prec \mathbf{M}'(I)$ and $\mathbf{M}'(K) \prec \mathbf{M}'(I)$. On the other hand, there is an isomorphism $f : \mathbf{M}^*(J) \longrightarrow \mathbf{M}^*(K)$ which is the identity on

$$M^*(I_{\alpha_k} \cap I_{\beta_k})$$

and which maps

$$M^*(J) \cap I_{\alpha_i}$$

onto

$$M^*(K) \cap I_{\beta_i}$$

for $i \leq n$. Therefore $(\mathbf{M}'(J), c_0, c_1, \ldots, c_r) \equiv (\mathbf{M}'(K), c_0, c_1, \ldots, c_r)$, so the Eklof-Mekler Criterion is verified. $\square$

Remark 4.3 In some situations it is much easier to prove that $\mathbf{M}^*(I)$ is finitely determinate. For example, if $(I, <)$ has order type $\eta \cdot \omega_1$ (where η is the order type of the rationals $(\mathfrak{Q}, <)$), then in the Eklof-Mekler Criterion we can easily get an isomorphism, not just elementary equivalence, thereby getting the stronger property of $\mathbf{M}^*(I)$ which might be referred to as “fully determinate.” For this conclusion we do not even need that $(\mathbf{M}^*, Def(\mathbf{M}^*)) \models$ CA+AC, but only that $\mathbf{M}^* \models \mathrm{PA}^*$. However, Lemma 4.1, its proof, and the ideas contained therein will be useful in the proof of the next lemma.

Lemma 4.4 *Suppose that $(I, <)$ is a linearly ordered set and that $n < \omega$, and let $I_0 < I_1 < \cdots < I_n = I$. Let $\mathbf{M}'(I) = (\mathbf{M}^*(I), M^*(I_0), M^*(I_1), \ldots, M^*(I_n))$. Suppose $D \in Def(\mathbf{M}'(I))$. Then $D \cap M^*(I_0) \in Def(\mathbf{M}^*(I_0))$.*

Proof: By Lemma 4.1 we can assume that $I = \{a_1, a_2, \ldots, a_n\}$, where $a_1 < a_2 < \cdots < a_n$ and $I_j = \{a_1, a_2, \ldots, a_j\}$ for $j \leq n$. Let $\varphi(v_0, v_1, \ldots, v_n)$ be an $\mathcal{L}$-formula and suppose that $D = \{b \in M'(I) : \mathbf{M}'(I) \models \varphi(b, \bar{a})\}$ and, without loss of generality, that $D \subseteq M^* = M^*(I_0)$. As in the proof of Lemma 4.1, we can assume that $\varphi(\bar{v})$ has the form

$$(Q_0 x_0 \in U_{r_0})(Q_1 x_1 \in U_{r_1}) \ldots (Q_t x_t \in U_{r_t}) \psi(\bar{x}\, \bar{v}),$$

where for each $j \leq t$, Q_j is one of the quantifiers $\exists$ or $\forall$ and $s_j \leq n$, and $\psi(\bar{x}, \bar{v})$ is an $\mathcal{L}$-formula. Similar to what was done in the proof of Lemma 4.1, obtain the formula $\psi'(\bar{X}, \bar{v})$ by replacing each free occurrence of x_j in $\psi(\bar{x}, \bar{v})$ by $X_j(\langle v_1, v_2, \ldots, v_j\rangle)$. Then define

$$\sigma(\bar{v}) = (Q_0 X_0)(Q_1 X_1) \ldots (Q_t X_t) \psi'(\bar{X}, \bar{v}).$$

Then for each $b \in M^*$

$$\mathbf{M}'(I) \models \varphi(b, \bar{a}) \text{ iff } (\mathbf{M}^*(I), Def_0(\mathbf{M}^*(I)) \models \sigma(b, \bar{a}).$$

There is $B \in \mathcal{X}$ such that

$$(\mathbf{M}, \mathcal{X}) \models \forall \bar{v}(\langle v_0, v_1, \ldots, v_n \rangle \in B \longleftrightarrow \sigma(\bar{v})).$$

Since $p(x)$ is a minimal type over $Th(\mathbf{M}^*)$, there is a formula $\theta(x)$ in $p(x)$ such that

$$\begin{aligned} \mathbf{M}^* \models\ & \forall v_0 \exists w[\forall v_1 v_2 \ldots v_n(w < v_1 \wedge \theta^<(v_1, \ldots, v_n) \longrightarrow \langle v_0, v_1, \ldots, v_n \rangle \in B) \\ \vee\ & \forall v_1 v_2 \ldots v_n(w < v_1 \wedge \theta^<(v_1, \ldots, v_n) \longrightarrow \langle v_0, v_1, \ldots, v_n \rangle \notin B)]. \end{aligned}$$

Obtain $\psi''(\bar{X}, \bar{v})$ from $\psi'(\bar{X}, \bar{v})$ by replacing each occurrence of X_j,where Q_j is $\forall$, by

$$(\mathcal{X}_j)_{\langle v_1, v_2, \ldots, v_n \rangle}.$$

Then define

$$\gamma(v_0) = (Q_0 X_0)(Q_1 X_1) \ldots (Q_t X_t)[\exists v_1, \ldots, v_n(\theta^<(v_1, \ldots, v_n) \wedge \psi''(\bar{X}, \bar{v}))].$$

It is easy to see that

$$AC^* \vdash \forall v_0[\forall v_1 \ldots v_n(\theta^<(v_1, \ldots, v_n) \longrightarrow \neg\sigma(\bar{v})) \longleftrightarrow \neg\gamma(v_0)]$$

and

$$\vdash \forall v_0[\neg\gamma(v_0) \longrightarrow \forall v_1 \ldots v_n(\theta^<(v_1, \ldots, v_n) \longrightarrow \neg\sigma(\bar{v}))].$$

Thus, if $b \in D$, then

$$\mathbf{M}^* \models \exists w \forall v_1 \ldots v_n(w < v_1 \wedge \theta^<(v_1, \ldots, v_n) \longrightarrow \langle b, \bar{v} \rangle \in B)$$

and if $b \in M^* \backslash D$, then

$$\mathbf{M}^* \models \exists w \forall v_1 \ldots v_n(w < v_1 \wedge \theta^<(v_1, \ldots, v_n) \longrightarrow \langle b, \bar{v} \rangle \notin B)$$

Therefore, the formula

$$\exists w \forall v_1 \ldots v_n(w < v_1 \wedge \theta^<(v_1, \ldots, v_n) \longrightarrow \langle x, \bar{v} \rangle \in B)$$

defines D in $\mathbf{M}^*$. □

The following theorem completes the proof of Theorem 1.1.

Theorem 4.5 *Suppose* $(\mathbf{M}, \mathcal{X}) \models$ *CA+AC is countable,* $(I, <)$ *is* ω_1*-like, and* $\mathbf{N} = \mathbf{M}^*(I) \restriction \mathcal{L}_{PA}$*. Then* $\mathbf{N} \models$ *PA(aa)+(Det).*

Proof: By Corollary 4.2, $\mathbf{M}^*(I)$ is finitely determinate, so clearly $\mathbf{N}$ also is. We now prove that $\mathbf{N} \models$ PA(aa). Suppose $a \in N$ and $\varphi(x, y)$ is an $\mathcal{L}_{PA}(aa)$ formula in which the only free variables are the first-order variables x and y. Suppose $D = \{b \in N : \mathbf{N} \models \varphi(b, a)\}$ is such that $0 \in D$ and that $x + 1 \in D$ whenever $x \in D$. Since $\mathbf{N}$ is finitely determinate, we can assume that $\varphi(x, y)$ has the form $aas_0\, aas_1 \ldots aas_{n-1}\psi(\bar{s}, x, y)$, where $\psi(\bar{s}, x, y)$ has no second-order quantifiers. By Lemma 4.4, whenever $I_0 < I_1 < \cdots < I_n = I$, there is a formula in the language of $\mathbf{M}^*$ with parameters from $M^*(I_0)$ defining $\{b \in M^*(I_0) : \mathbf{N} \models \psi(M^*(I_0), \ldots, M^*(I_{n-1}), b, a)\}$. It is clear from Lemma 4.1 that this defining formula depends only on I_0. By Fodor's Lemma, there is a single formula $\theta(x)$ such that for a stationary set of I_0, whenever $I_0 < I_1 < \cdots < I_n = I$, then $\theta(x)$ defines $\{b \in M^*(I_0) : \mathbf{N} \models \psi(M^*(I_0), \ldots, M^*(I_{n-1}), b, a)\}$ in $\mathbf{M}^*(I_0)$. Then it easily follows that $\theta(x)$ defines D in $\mathbf{M}^*(I)$. But $\mathbf{M}^*(I) \models \mathrm{PA}^*$, so $D = M^*(I) = N$. $\square$

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