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The Theory of κ-like Models of Arithmetic

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Abstract A model $(M, <, \ldots)$ is said to be κ-like if card $M = \kappa$ but for all $a \in M$, card$\{x \in M : x < a\} < \kappa$. In this paper, we shall study sentences true in κ-like models of arithmetic, especially in the cases when κ is singular. In particular, we identify axiom schemes true in such models which are particularly 'natural' from a combinatorial or model-theoretic point of view and investigate the properties of models of these schemes.

1 Introduction A model $(M, <, \ldots)$ is κ-like iff card $M = \kappa$ but for all $a \in M$, card$\{x \in M : x < a\} < \kappa$. In this paper, we shall study κ-like models of arithmetic, especially in the cases when κ is singular. All models here will be nonstandard models of the form $(M, <, 0, 1, +, \cdot, \ldots)$ and will satisfy the theory $I\Delta_0$ of induction on Δ_0 formulas.

It is well known that (with this base theory given) κ-like models of arithmetic where κ is regular must satisfy Peano arithmetic (PA) and that every complete extension of PA has a κ-like model for each uncountable cardinal κ. In fact, we have the following.

Proposition 1.1 (MacDowell-Specker) *Each model M of PA has a proper elementary end-extension N with cardinality* card M.

Corollary 1.2 *If a model M of PA has cardinality $\lambda < \kappa$, then M has a κ-like elementary end-extension N.*

For proofs, see for example Kaye [3].

On the other hand, a κ-like model obviously satisfies PA only if κ is regular. This paper makes a start in understanding the theory of κ-like models for singular κ. The main results here concern model-theoretic properties (in particular with reference to certain types of extension of models) of certain axiom schemes true in all such models.

One interesting aspect of κ-like models is that it is clear that for κ a strong limit (meaning: for all $\lambda < \kappa$, $2^\lambda < \kappa$), each κ-like model must satisfy exponentiation. (To see this, observe that in a model of arithmetic, if $a < b$ and no $x \leqslant b$ satisfies $x = 2^a$, then each $x < b$ gives rise to a unique subset of $\{y : y < a\}$, that is the set of y for which the yth digit in the binary expension of x is 1. It follows that $\mathrm{card}\{x : x < b\} \leqslant 2^{\mathrm{card}\{y:y<a\}}$.) On the other hand, it is not clear whether (in the absense of GCH) there can be κ-like models that do not satisfy exponentiation. A further intriguing aspect of this problem is that studying κ-like models leads us to formulate axioms based on pigeonhole principles and other combinatorial principles clearly related to principles which have proved so problematic in the study of weak systems of arithmetic and complexity theory.

There are many other interesting problems left open, some of them due to the lack of known methods suitable for constructing uncountable models of arithmetic that do *not* satisfy full PA. Indeed it is surprising that these issues have been studied so little, given that the MacDowell-Specker result and κ-like models have been around for some time.

The current paper contains a discussion of the key ideas and problems in this area and a survey of some of the natural axiom schemes one is led to consider when studying κ-like models. It contains some straightforward results and some model-theoretic properties of the axiom schemes discussed. It is intended to be read in conjunction with the parallel paper [5] which contains the proofs of theorems that relate the schemes discussed here with the usual $I\Sigma_n/B\Sigma_n$ hierarchy of subsystems of PA.

2 Notation and previously known results Throughout, $\mathcal{L}$ will be a first-order language with finitely many functions and relations, extending the usual language $\mathcal{L}_{\mathrm{A}} = \{0, 1, +, \cdot, <\}$ of arithmetic. All models will be considered as $\mathcal{L}$-structures, which will be assumed to satisfy at least $I\Delta_0$ and also (where necessary) exponentiation. (Note that some of our results generalize to structures for the language consisting of the order relation alone, but we will not take trouble to point out those that do.) For each a in a model of $I\Delta_0 + \exp$, we define $2_0(a) = a$, and $2_{r+1}(a) = 2^{2_r(a)}$.

We will use the usual subtheories $I\Sigma_n$ and $B\Sigma_m$ of PA [3], Chapter 10. These are finitely axiomatized for $n \geqslant 1$ and $m \geqslant 2$. Note that $B\Sigma_2 \vdash I\Sigma_1 \vdash \exp$ but $B\Sigma_1 \nvdash \exp$. We shall often be interested in models of $I\Delta_0 + \exp$ that satisfy $B\Sigma_n$ but not $I\Sigma_n$ for some n, and we will refer to such as 'models of $B\Sigma_n + \exp + \neg I\Sigma_n$' even though exp is redundant for $n \geqslant 2$.

In the metatheory—ZFC throughout—cardinals are thought of as initial ordinals. Given a model M for $\mathcal{L}$, $\mathcal{L}(M)$ denotes $\mathcal{L}$ expanded by adding a constant symbol for each $a \in M$. For $a \in M$, $<a$ denotes $\{x \in M : x < a\}$, and $\|a\|$ denotes $\mathrm{card}\,({<}a)$. $I_\lambda(M)$ denotes $\{a \in M : \|a\| < \lambda\}$, and for a formula $\varphi(x)$ of $\mathcal{L}(M)$, $\varphi(M)$ denotes $\{x \in M : \varphi(x)\}$. For a model of arithmetic M and a subset $S \subseteq M$, we define $\inf_M(S) = \{x \in M : \forall s \in S\, x < s\}$ and $\sup_M(S) = \{x \in M : \exists s \in S\, x \leqslant s\}$.

For $n \in \mathbb{N}$, the quantifiers $\exists^n$ and $\exists^{\geqslant n}$ denote 'there exist precisely n' and 'there exist at least n'. These are of course first-order.

Many of the results referred to here touch on questions to do with cofinal extensions. Of course, a model M is *cofinal* in N ($M \subseteq_{\mathrm{cf}} N$) if $\forall b \in N\, \exists a \in M\; N \models b \leqslant a$. Cofinal extensions are well understood for models of PA (cf. Gaifman [1]), in particular because of the Splitting Theorem.

Theorem 2.1 (Gaifman's Splitting Theorem) *For $M \subseteq N$, both models of PA, there is a unique K such that $M \subseteq_{\mathrm{cf}} K \subseteq_{\mathrm{e}} N$, and this K satisfies $M \prec K$.*

A proof of this also appears in Kaye [3], Section 7.2. Cofinal extensions are not so well understood for models of theories weaker than PA, but there are some results in the literature (cf. Kaye [2],[4]).

At this point it seems worthwhile to mention three papers that relate to the central issues connected with κ-like models for singular κ. The first, of course, is Keisler's paper [6], where the following is shown.

Theorem 2.2 (Keisler) *For any first-order theory T in signature $(<, \ldots)$, if T has a κ-like model for some strong limit κ, then T has a λ-like model for all singular λ.*

Keisler's paper also contains a useful survey of other results and references on κ-like models in general.

The second paper is the classic one by Kirby and Paris [7] on initial segments. In it (Theorem 7, parts d,e) we find (using my notation defined above, which differs slightly from Kirby and Paris's) the following theorem.

Theorem 2.3 (Kirby-Paris) *Let M be a countable model of PA, and let $I \subseteq_{\mathrm{e}} M$ be strong. Then, for any infinite cardinal λ, there is an extension $K \succ M$ such that $I = \sup_K(I)$, $I_{\lambda^+}(K) = \inf_K(M \setminus I)$, and $\operatorname{card}(I_{\lambda^+}(K)) = \lambda^+$.*

The construction is a beautiful one using indiscernibles. (All that the reader need know about 'strong' initial segments here is that there are arbitrarily large strong cuts in any countable model of PA.) I was able to modify this construction to show the existence of κ-like models of $\Pi_2 - \mathrm{Th}(\mathrm{PA})$ that do not satisfy PA, for all singular κ of cofinality ω [5]. For a while, it seemed an interesting question whether or not $\Pi_2 - \mathrm{Th}(\mathrm{PA})$ is true in all κ-like models, but it turns out (see Theorem 3.20 below) that the answer to this is 'no'.

Finally, I should mention the beautiful paper by Paris and Mills [10]. The main theorem here is the following.

Theorem 2.4 (Paris-Mills) *Let M be a countable model of PA, and let $I \subseteq_{\mathrm{e}} M$. Then: (i) if I is closed under multiplication, there is $N \succ M$ with $I = I_{\omega^+}(N)$ and $\inf\{\|a\| : a \in N, \|a\| > \omega\} = 2^\omega$; (ii) if I is closed under exponentiation and κ is any uncountable cardinal, there is $N \succ M$ with $I = I_{\omega^+}(N)$ and $\inf\{\|a\| : a \in N, \|a\| > \omega\} = \kappa$.*

Unfortunately, PA is used in an essential way here, and there are also serious difficulties extending the result to the uncountable cases, so at the moment it is difficult to see how this may be used to construct interesting κ-like models. On the other hand, Theorem 2.4 does suggest that, as far as the growth-rate of functions is concerned at least, κ-like models need not be closed under any faster-growing function other than exponentiation—even if κ is a singular strong limit—and if κ is not a strong limit, closure under multiplication suffices.

3 New results Here, we shall identify various axiom schemes true in κ-like models, and investigate the model-theoretic properties of these schemes. As a result, we will obtain an interesting family of subtheories of PA, and we will also indicate various inclusions between these theories. Throughout, we shall concentrate on theories with particularly simple and clear axiomatizations or with particularly striking model-theoretic properties.

3.1 CARD, models that look like cardinals The most obvious axiom for κ-like models is the axiom that states that there is no 1–1 map from the model to a proper initial segment. In terms of second-order logic, this is expressed by the axiom CARD_2:

$$\forall X\, \forall a\, (\forall x\, \exists y < a\, \langle x, y\rangle \in X \rightarrow \exists y < a\, \exists^{\geqslant 2} x\, \langle x, y\rangle \in X).$$

If this is considered as a true second-order axiom, we have the following.

Proposition 3.1 *A model M satisfies* CARD_2 *iff it is κ-like for some κ.*

The first-order scheme corresponding to this, CARD, is obtained by letting the second-order variable X range over first-order definable sets (i.e., definable *with* parameters from M). This scheme is provable in PA, but it appears to be very weak and certainly does not obviously characterize the first-order theory of κ-like models.

On the other hand, CARD, like PA, is not axiomatizable by a set of axioms of limited quantifier complexity. More precisely:

Proposition 3.2 *For all $n > 0$ and all nonstandard $M \models$ PA there is $K \prec_{\Sigma_n} M$ satisfying $I\Sigma_{n-1}$ but not satisfying an instance of CARD for a Σ_n-definable set.*

Proof: Let $a \in M$ be nonstandard and let K be the set of Σ_n-definable elements of M, definable using the parameter a. Then $K \prec_{\Sigma_n} M$, and so satisfies $I\Sigma_{n-1}$, and in fact K does not satisfy $B\Sigma_n$, by work of Paris and Kirby (cf. [3], section 10.1). But for all $b \in K$ there is $e \in \mathbb{N}$ such that $K \models \exists u\, \lambda(a, b, e, u)$, where λ is the formula

$$b = u_0 \wedge \mathrm{Sat}_{\Sigma_{n-1}}(e, \langle u, a\rangle) \wedge \forall v\, (\mathrm{Sat}_{\Sigma_{n-1}}(e, \langle v, a\rangle) \rightarrow v \geqslant u)$$

($u = \langle u_0, u_1\rangle$ being the pairing function). So $\exists u\, \lambda$ is Σ_n and

$$K \models \forall b\, \exists! e < a\, \exists u\, \lambda(a, b, e, u),$$

as required. □

The above argument is taken from Paris and Kirby's argument that the model K does not satisfy $B\Sigma_n$. Note that it shows that the scheme $\mathrm{CARD}(\Sigma_n)$ of CARD restricted to Σ_n-definable sets is Π_{n+2} axiomatized but not Σ_{n+2} axiomatized. (We omit the easy details.)

3.2 GPHP, a generalized pigeonhole principle To strengthen the scheme CARD, consider a sort of 'generalized pigeonhole principle', GPHP_2, which is defined to be the second-order axiom,

$$\forall X\, \forall a\, \exists b\, \forall c\, (\forall x < b\, \exists y < a\, \langle x, y, c\rangle \in X \rightarrow \exists y < a\, \exists^{\geqslant 2} x < b\, \langle x, y, c\rangle \in X).$$

Lemma 3.3 *If M is has cardinal κ and $\aleph_0 \leqslant \lambda < \kappa$, then there is $b \in M$ with $\|b\| \geqslant \lambda$.*

Proof: If $\|b\| < \lambda$ for all $b \in M$, let $S \subseteq M$ have cardinality λ, so $S \subseteq_{\mathrm{cf}} M$, hence

$$\kappa = \sum_{b \in S} \|b\| \leqslant \lambda^2 = \lambda,$$

a contradiction. □

Proposition 3.4 *A model M satisfies* GPHP_2 *if and only if it is κ-like for some limit cardinal κ.*

Proof: For $a \in M$ where M is κ-like for some nonsuccessor cardinal κ, let $\lambda = \|a\|^+ < \kappa$. Then we can find $b \in M$ with $\|b\| \geqslant \lambda$. Hence $M \models \mathrm{GPHP}_2$. The converse is just as easy. □

We feel that the first-order theory GPHP obtained from GPHP_2 in the obvious way is particularly attractive. It clearly implies CARD, for instance. However, it will turn out that this theory is still rather weak. For example, one of the results later will imply (rather more than) the conservation result $I\Delta_0 + \exp \equiv_{\Pi_2} I\Delta_0 + \exp + \mathrm{GPHP}$ (even though all axiomatizations of the theory on the right-hand side here are of unbounded complexity).

3.3 COLL, collection The next axiom scheme is probably the best known of all. It is the *collection axiom* and has already been referred to. Again, we give the second-order version first.

$$\forall X\, \forall a\, (\forall x < a\, \exists y\, \langle x, y\rangle \in X \to \exists b\, \forall x < a\, \exists y < b\, \langle x, y\rangle \in X)$$

The following result is well-known, and its proof is obvious.

Proposition 3.5 *A model M satisfies* COLL_2 *iff it is κ-like for some regular κ.*

Modulo $I\Delta_0$, the first-order version COLL of COLL_2 implies the first-order theory GPHP, but for the second-order axioms we have $\mathrm{COLL}_2 \not\models \mathrm{GPHP}_2$. The first-order version CARD of CARD_2 is implied by both COLL and GPHP.

3.4 IPHP, an iterated pigeonhole principle Unfortunately, even GPHP is not obviously strong enough to give the first-order theory of κ-like models, and we apparently need still stronger 'iterated' pigeonhole principles.

Definition 3.6 Given a set Γ of first-order formulas of $\mathcal{L}$, $X, C \subseteq M$, and $a \in M$, we say that X is $(a, 0, \Gamma)$-*large over C* when

$$\bigwedge_{\varphi\in\Gamma} \forall \bar{c} \in C\ \big(\forall x \in X\, \exists y \leqslant a\, \varphi(x, y, \bar{c}) \to \exists y \leqslant a\, \exists^{\geqslant 2} x \in X\, \varphi(x, y, \bar{c})\big)\,.$$

X is $(a, n+1, \Gamma)$-large over C when

$$\bigwedge_{\psi\in\Gamma} \forall \bar{c} \in C \left(\begin{array}{l} \forall x \in X\, \exists r \leqslant a\, \psi(x, r, \bar{c}) \to \\ \exists r \leqslant a\, \forall d\, \{x \in X : \psi(x, r, \bar{c})\} \text{ is } (a, n, \Gamma)\text{-large over } C \cup \{d\} \end{array} \right).$$

We shall say X is (a, n)-*large over C* if it is (a, n, Γ)-large over C where Γ is the set of all $\mathcal{L}$-formulas. Also, X is *a-large over C* if it is (a, n)-large over C for all n. If ever C is omitted from the notation (as in 'X is n-large') then it is taken to be $\varnothing$.

From this we can derive the first-order axiom scheme IPHP: for all $n \in \mathbb{N}$ and all finite sets Γ of formulas in $\mathcal{L}$,

$$\forall a\, \exists b\, (\{x : x < b\} \text{ is } (a, n, \Gamma)\text{-large over } \varnothing).$$

Lemma 3.7 *If M is a model for $\mathcal{L}$, $a \in M$, $X \subseteq M$ has cardinality* $\operatorname{card} X > \|a\|$, *$\Gamma$ is the set of all $\mathcal{L}$-formulas, $C \subseteq M$, and $n \in \mathbb{N}$, then X is (a, n, Γ)-large over C.*

Proof: If $n = 0$ this is immediate, and if $n \geqslant 1$, $\forall x \in X\, \exists r < a\, \psi(x, r)$ then $X = \bigcup_{r<a} X_r$ where $X_r = \{x \in X : \psi(x, r)\}$, so some X_r has cardinality greater than $\|a\|$. By induction, this X_r is $(a, n-1, \Gamma)$-large over C for any C. □

Once again, it is clear that PA proves all axioms of IPHP. It follows that all κ-like models satisfy IPHP, since the case κ regular is covered by PA, and the case for κ singular is covered by Lemmas 3.3 and 3.7.

We were unable to show that IPHP is indeed an axiomatization of the theory of κ-like models, but it does have some pleasant model-theoretic properties.

Theorem 3.8 *If M is a countable recursively saturated model of IPHP, then for all $a \in M$ there is a proper elementary extension $N \succ M$ such that $\{x \in M : x < a\} = \{x \in N : x < a\}$. Conversely, if M is a model with the property that for all $a \in M$ there is such an extension N, then $M \models$* IPHP.

First we need some lemmas.

Lemma 3.9 *If M is a recursively saturated $\mathcal{L}$-structure, $X \subseteq M$ is definable, $C \subseteq M$ is finite, $a \in M$, and X is (a, n, Γ)-large over C for all n and all finite Γ, then X is a-large over C.*

Proof: We show by induction on n that X is (a, n)-large over C. It is clear to start with that X is $(a, 0)$-large over C.

Now suppose $\bar{c} \in C$ and $\forall x \in X\, \exists r \leqslant a\, \psi(x, r, \bar{c})$. Then

$$\bigwedge_{n,\Gamma} (\exists r < a\, \forall d\, (\psi(M, r, \bar{c}) \text{ is } (a, n, \Gamma)\text{-large over } C \cup \{d\}))$$

whence

$$\exists r < a\, \forall d \bigwedge_{n,\Gamma} (\psi(M, r, \bar{c}) \text{ is } (a, n, \Gamma)\text{-large over } C \cup \{d\})$$

by saturation, and the lemma follows. □

Note too that X $(a, n+1)$-large implies X is (a, n)-large, since we may take $\psi(r, x)$ to be '$r = 0$'.

Lemma 3.10 *If $M \models$* IPHP *is recursively saturated and $a \in M$, then there is $b \in M$ so that $<b$ is a-large.*

Proof: By saturation, and by the last lemma, it suffices to find for each n and each finite Γ an element b such that $<b$ is (a, n, Γ)-large. This follows immediately from the axioms. □

Lemma 3.11 *If M is recursively saturated, $a \in M$, $C \subseteq M$ is finite, $X \subseteq M$ is definable and a-large over C, and $\xi(u, v) \in \mathcal{L}(C)$ then either $X \cap \forall u < a\, \neg\xi(u, M)$ is a-large, or for some $s < a$, $X \cap \xi(s, M)$ is a-large.*

Proof: By saturation, it suffices that for each $n \in \mathbb{N}$ and each finite $\Gamma \subseteq \mathcal{L}$ there is $s < a$ such that

$$X_s = \xi(s, M) \cap X \text{ is } (a, n, \Gamma)\text{-large}$$

or

$$X_a = \forall u < a\, \neg\xi(s, M) \cap X \text{ is } (a, n, \Gamma)\text{-large}.$$

Suppose not. Then

$$\forall x \in X\, \exists s \leqslant a\ [(s < a \wedge \xi(s, x)) \vee (s = a \vee \forall u < a\, \neg\xi(u, x))]$$

but no X_s ($s \leqslant a$) is (a, n, Γ)-large. Hence X is not $(a, n+1, \Gamma \cup \{\psi\})$-large, where ψ is the formula in square brackets above. □

Lemma 3.12 *If M is recursively saturated, $a \in M$, $C \subseteq M$ is finite, $X \subseteq M$ is definable and a-large over C, and $d \in M$, then X is a-large over $C \cup \{d\}$.*

Proof: Just consider $\psi(x, r)$ to be '$r = 0$'. It follows that X is (a, n, Γ)-large over $C \cup \{d\}$ for all n and all finite Γ, which suffices by saturation. □

Proof of Theorem 3.8: By Lemma 3.10 there is $b \in M$ such that $<b$ is a-large. Put $X_0 = <b$, and enumerate all formulas $\varphi(x, y)$ with two free variables as $\varphi_0(x, y), \ldots, \varphi_i(x, y), \ldots$.

Inductively, given $X_i \subseteq M$, definable and a-large over C_i, where C_i is finite, let C_{i+1} be finite, containing C_i, the parameters from M appearing in $a, \varphi_0, \ldots, \varphi_i$, and any parameters needed to define X_i. Then find $X_{i+1} \subseteq X_i$ a-large over C_{i+1} so that

$$X_{i+1} = X_i \cap \varphi_i(M, r)$$

or

$$X_{i+1} = X_i \cap \forall u < a\, \neg\varphi_i(M, u)$$

using Lemmas 3.11 and 3.12.

At the end of this construction, consider the theory T axiomatized by the complete diagram of M, together with the axioms '$c \in X_i$', where c is a new constant symbol. It is straightforward to check that T is a complete theory in the language $\mathcal{L}(M) \cup \{c\}$, T proves $c > s$ for each $s < a$, and that the set of formulas

$$\{y < a\} \cup \{y \neq s : s < a\}$$

has no support $\psi(y)$ over T. Thus the omitting types theorem applies and we obtain our extension as required.

The converse is easy. If $M \prec N$ and $<a$ is preserved under the extension, then any b for which $<b$ is *not* preserved will be a-large. Indeed, if $x < b$ and $x \notin M$, then for each $y \leqslant a$ and each formula $\varphi(x, y)$ with parameters from M, $N \models \varphi(x, y)$ implies $M \models \exists^{\geqslant 2} z\, \varphi(z, y)$, for otherwise x would be definable over M and hence would be in M. If there are no $b \in M$ for which $<b$ is not preserved by the extension $M \prec N$, then N is an elementary end-extension of M, and so both satisfy PA, hence IPHP. □

3.5 IPHP$_{\mathrm{cf}}$***, a modification of*** IPHP If, in Theorem 3.8, we also want $N \succ_{\mathrm{cf}} M$ we proceed in a similar way but want to also omit the type

$$\{x > b : b \in M\}.$$

In the proof of the theorem this means, given a definable X_i and $\varphi(x, y)$ (which may have parameters from M or C) we want to find a definable subset X_i of the form

$$X_i \cap \forall y\, \neg\varphi(M, y)$$

or

$$X_i \cap \exists y < b\, \varphi(M, y)$$

for some $b \in M$. We modify the definition of the last section to: X is $(a, n+1, \Gamma)$-large$_{\mathrm{cf}}$ over C iff

$$\bigwedge_{\psi\in\Gamma} \forall \bar{c} \in C \left(\begin{array}{l} \forall x \in X\, \exists r \leqslant a\, \psi(x, r, \bar{c}) \to \\ \exists r \leqslant a\, \forall d\, \{x \in X : \psi(x, r, \bar{c})\} \text{ is } (a, n, \Gamma)\text{-large}' \text{ over } C \cup \{d\} \end{array} \right)$$

and X is (a, n, Γ)-large$'$ over C iff

$$\bigwedge_{\psi\in\Gamma} \forall \bar{c} \in C \left(\begin{array}{l} \{x \in X : \forall y\, \neg\psi(x, y, \bar{c})\} \text{ is } (a, n, \Gamma)\text{-large}_{\mathrm{cf}} \text{ over } C \vee \\ \exists b\, \{x \in X : \exists y\ < b \psi(x, y, \bar{c})\} \text{ is } (a, n, \Gamma)\text{-large}_{\mathrm{cf}} \text{ over } C \cup \{b\} \end{array} \right).$$

X is $(a, n, 0)$-large$_{\mathrm{cf}}$ over C iff it is $(a, n, 0)$-large over C. The modification of Theorem 3.8 is then the following.

Theorem 3.13 *Let M be countable and recursively saturated and satisfy the theory* IPHP$_{\mathrm{cf}}$. *Then for all $a \in M$ there is $N \succ_{\mathrm{cf}} M$ such that $\{x \in M : x < a\} = \{x \in N : x < a\}$. Conversely, if for all $a \in M$ there is such an N, then $M \models$* IPHP$_{\mathrm{cf}}$.

In fact, this strengthened pigeonhole principle is still part of the theory of κ-like models.

Theorem 3.14 *If M is κ-like for some infinite cardinal κ, then $M \models$* IPHP$_{\mathrm{cf}}$.

Proof: The case of κ regular is covered by PA and the Splitting Theorem. Let $\mathrm{cf}\,\kappa = \lambda$, suppose $a \in M$, and without loss of generality suppose $\|a\| > \lambda$. Let $b \in M$ with $\|b\| > \|a\|$, and suppose $X \subseteq {<}b$ has cardinality $> \|a\|$. Now suppose $\varphi(x, y)$ is a formula with parameters from M. We show that either

$$Y = \{x \in X : \forall y\, \neg\varphi(x, y)\}$$

or

$$X_c = \{x \in X : \forall y < c\, \varphi(x, y)\}$$

for some $c \in M$ has cardinality greater than $\|a\|$. But if this is false then choosing a cofinal sequence c_i $(i < \lambda)$ in M, we have

$$X = \bigcup_{i<\lambda} X_{c_i} \cup Y$$

so $\mathrm{card}\, X \leqslant \lambda\|a\| = \|a\|$. □

3.6 INDISC, indiscernibles The two main lemmas Keisler [6] used to prove Theorem 2.2 are the following.

Lemma 3.15 (Keisler) *Suppose K is an $\mathcal{L}^+$-structure with a linear order $<$, where $\mathcal{L}^+$ is an expansion of $\mathcal{L}$, and K has the property that the reduct of any $\mathcal{L}^+$-substructure of K to $\mathcal{L}$ is elementary in K as $\mathcal{L}$-structures (for example, it might be that $\mathcal{L}^+ = \mathcal{L}_{\mathrm{Sk}}$ and $K \models T_{\mathrm{Sk}}(\mathcal{L})$), and suppose that K has elements c_{ij} for $i < \lambda = \mathrm{cf}(\kappa) < \kappa$ and $j < \kappa_i < \kappa$ where $\kappa = \sum_{i<\lambda} \kappa_i$ satisfying*

a. $c_{ij} < c_{kl}$ *for all* $i < k$ *or* $i = k$ *and* $j < l$
b. $\tau(c_{i_1 j_1}, \dots, c_{i_n j_n}) < c_{ij}$ *all* $i_1, j_1, \dots, i_n, j_n, i, j$ *with* $i_1, \dots, i_n < i$

for all $\mathcal{L}^+$-terms τ in the arguments shown, and

c. $\tau(c_{i_1 j_1}, \dots, c_{i_n j_n}) < c_{ij} \to \tau(c_{i_1 j_1}, \dots, c_{i_n j_n}) = \tau(c_{i_1 l_1}, \dots, c_{i_n l_n})$

for all $\bar{\imath}, \bar{\jmath}, \bar{l}$ with $i < i_1, \dots, i_n$ and $\langle c_{i_1 j_1}, \dots, c_{i_n j_n}\rangle$, $\langle c_{i_1 l_1}, \dots, c_{i_n l_n}\rangle$ both increasing, and all $\mathcal{L}^+$-terms τ which here may contain elements c_{kl} for $k \leqslant i$. Then the $\mathcal{L}^+$-substructure of K generated by the c_{ij} is a κ-like substructure of K elementary for $\mathcal{L}$.

Strictly, Keisler worked with the full Skolemized language, but the lemma is true by the same proof, and here we are interested only in the $\mathcal{L}$-theory of such models, so it makes no difference.

The other lemma of Keisler's uses partition properties to get models M of the form required for Lemma 3.15.

Lemma 3.16 (Keisler) *Let M be a $\mathcal{L}$-structure linearly ordered by some $<$ in $\mathcal{L}$. Suppose M is μ-like, where μ is a strong limit. Then the set of sentences (a), (b) and (c) in Lemma 3.15 in new constants c_{ij} ($i < \lambda$, $j < \kappa_i$) is consistent with* $\mathrm{Th}(M)$.

Theorem 2.2 follows by applying Lemma 3.16 to the Skolemization of a μ-like model M, and applying the compactness theorem and Lemma 3.15 to get a κ-like model elementary equivalent to M.

Notice that, for theories of arithmetic, we can conclude that the existence of indiscernibles as in the last two lemmas implies that the model is closed under exponentiation. This follows from Keisler's theorem and remarks already made, but it is easy to obtain a direct proof too.

It is certainly possible—indeed, straightforward—to abstract directly from these lemmas of Keisler's a list of first-order sentences that axiomatize the $\mathcal{L}_{\mathrm{A}}$-theory INDISC of κ-like models of arithmetic for strong limit cardinals κ, but by doing so we would not be going very far beyond what is already clear from Lemmas 3.15 and 3.16, and so we probably wouldn't learn much.

On the other hand, we can obtain an interesting and elegantly axiomatized first-order theory IB + exp which implies that of κ-like models for singular κ. To understand the next definition, recall that by the usual truth predicates in arithmetic, $I\Sigma_n$ and $B\Sigma_{n+1}$ are finitely axiomatized for all $n \geqslant 1$.

Definition 3.17 The theory IB is axiomatized by $B\Sigma_1$ together with the sentences

$$I\Sigma_n \to B\Sigma_{n+1}$$

for all $n \in \mathbb{N}$

Clearly, PA $\vdash$ IB. However, we may ignore models of PA when trying to prove first-order consequences of IB, as the next lemma shows.

Lemma 3.18 *If σ is a first-order sentence true in each model of $B\Sigma_n + \neg I\Sigma_n$ for each $n \geqslant 1$, then* $\mathrm{IB} \vdash \sigma$.

Proof: Let $M \models \mathrm{IB}$. Then either $M \models B\Sigma_n + \neg I\Sigma_n$ for some n, hence $M \models \sigma$, or $M \models \mathrm{PA}$. In this second case, we may assume (by taking an elementary extension) that M is nonstandard. Let $a \in M \setminus \mathbb{N}$, and take $n \in \mathbb{N}$ large compared to the complexity of σ. Then by a standard construction due to Paris and Kirby [9] and Lessan [8] (see Kaye [3], Section 10.2) $I = I^n(M, a) \prec_{\Sigma_{n-1}} M$ and $I \models B\Sigma_n + \neg I\Sigma_n$, so $I \models \sigma$ and $M \models \sigma$. □

The next lemma also follows from the standard construction of models of $B\Sigma_n + \neg I\Sigma_n$ and results due to Paris, Kirby and Lessan on $I^{n+1}(M, a)$.

Lemma 3.19 *For all $n \in \mathbb{N}$, $I\Sigma_n + \mathrm{IB}$ is Π_{n+2}-conservative over $I\Sigma_n$. Similarly,* $\mathrm{IB} + \exp$ *is Π_2-conservative over $I\Sigma_n + \exp$, and $B\Sigma_{n+1} + \mathrm{IB}(+\exp)$ is conservative over $B\Sigma_{n+1}(+\exp)$ for sentences of the form $\sigma \vee \tau$ where σ and τ are Π_{n+2} and Σ_{n+2} sentences respectively.*

Proof: Let $a \in M \models I\Sigma_n + \varphi(a)$ be recursively saturated, where φ is Π_{n+1}. Consider $I = I^{n+1}(M, a) \prec_{\Sigma_n} M$ ([3], Theorem 10.7). Then $I \models B\Sigma_{n+1} + \neg I\Sigma_{n+1} + \varphi(a)$ ([3], Theorem 10.10), hence $\mathrm{IB} \nvdash \forall x\, \neg\varphi(x)$, as required. The case for $B\Sigma_{n+1}$ is similar except if $M \models B\Sigma_{n+1}$ then ([3], Theorem 10.8) we have in addition that $I^{n+1}(M, a) \models \Pi_{n+2} - \mathrm{Th}(M, a)$, so if both $\neg\sigma$ and $\neg\tau$ are true in M then they are true in $I^{n+1}(M, a)$ also. □

The main theorem in the companion paper to this one [5] is the following.

Theorem 3.20 *Let κ be singular. The theory* $\mathrm{IB} + \exp$ *proves all sentences true in all κ-like models. In particular, it proves the theory of κ-like models for singular strong limit cardinals κ.*

Note that these results mean that the theory of κ-like models for singular κ has rather low 'consistency strength' and that all the theories considered in this paper are actually Π_2-conservative over $I\Delta_0 + \exp$.

3.7 TREEIND, tree-indiscernibles This section describes an attempt to abstract the important features behind the construction in Theorem 2.4 (i) and is included to suggest avenues for future work.

Here, 'tree' means a full binary tree of height equal to some ordinal α. That is, the trees we are considering are sets of functions $T_\alpha = \{\tau\colon \beta \to \{0, 1\} : \beta < \alpha\}$. This set has a *length* function $\mathrm{len}(\tau) = \beta$, where $\tau\colon \beta \to \{0, 1\}$, and two successor functions $S_0(\tau) = \tau^\frown 0$ and $S_1(\tau) = \tau^\frown 1$.

We say σ is *a successor* of τ if $\tau = \sigma \restriction \beta$ where $\beta = \mathrm{len}\,\tau \leqslant \mathrm{len}\,\sigma$. More generally, $\bar\sigma$ is a successor of $\bar\tau$ if each element of the tuple $\bar\tau = \tau_1, \ldots, \tau_n$ has the same length β, each σ_i in $\bar\sigma = \sigma_1, \ldots, \sigma_n$, has the same length $\gamma \geqslant \beta$, and σ_i is a successor of τ_i for each i.

Definition 3.21 A *set of tree-indiscernibles* (of length α) in a model M is a set $\{x_\tau : \tau \in T_\alpha\} \subseteq M$ such that (i) for each $\beta < \alpha$, $\mathrm{len}\,\tau = \mathrm{len}\,\sigma = \beta$ and $\tau \neq \sigma$ implies $x_\tau \neq x_\sigma$, and (ii) $\bar{\tau} = \tau_1, \dots, \tau_n$, $\bar{\sigma} = \sigma_1, \dots, \sigma_n$ with $\mathrm{len}(\tau_i) = \beta$ for all i, $\mathrm{len}(\sigma_i) = \gamma \geqslant \beta$ all i and $\beta > \ulcorner\varphi\urcorner$, if $\bar{\sigma}$ is a successor of $\bar{\tau}$ then

$$M \models \varphi(x_{\tau_1}, \dots, x_{\tau_n}) \iff \varphi(x_{\sigma_1}, \dots, x_{\sigma_n}).$$

This definition admits the usual variations: 'tree-indiscernible for $\varphi_1, \dots, \varphi_n$' means we are only interested in indiscernibility for those formulas listed, and 'over A' means that parameters $\bar{a}$ from the set A are allowed in the formulas φ.

Paris and Mills [10] constructed a special kind of indiscernible set of this form.

Theorem 3.22 *Let $M \models$ PA be countable and nonstandard, and let $I \subseteq_e M$ be closed under multiplication. Then there is a family of tree-indiscernibles $\{x_\tau : \tau \in T_\omega\}$ cofinal in I such that*

> *for all Skolem terms t there is $a_t > I$ in M such that for all $\bar{\tau} = \tau_1, \dots, \tau_n$ with $\mathrm{len}(\tau_i) = \beta > \ulcorner t\urcorner$ and successor $\bar{\sigma} = \sigma_1, \dots, \sigma_n$ of $\bar{\tau}$ with $\mathrm{len}(\sigma_i) = \gamma \geqslant \beta$ all i either*
>
> $$t(x_{\tau_1}, \dots, x_{\tau_n}) = t(x_{\sigma_1}, \dots, x_{\sigma_n})$$
>
> *or*
>
> $$t(x_{\tau_1}, \dots, x_{\tau_n}) > a_t.$$

This, together with a compactness argument, gives part of their main result, stated as Theorem 2.4 (i) above.

It would seem that this could be made to give a useful way of building κ-like models that do not satisfy exp (albeit, ones which do satisfy a strong enough pigeonhole principle to make the combinatorics in Paris and Mills' work go through), but there are still many serious problems in getting the analogues of (b) and (c) in Lemma 3.15 to work, and so making progress in this direction would appear to be rather difficult.

4 Summary and open problems In Section 3 above, we identified several axiom schemes in the language $\mathcal{L}_\mathrm{A}$ with implications

$$\mathrm{IB} + \exp \Rightarrow \mathrm{INDISC} \Rightarrow \mathrm{IPHP_{cf}} \Rightarrow \mathrm{IPHP} \Rightarrow \mathrm{GPHP} \Rightarrow \mathrm{CARD}. \tag{‡}$$

All of these schemes are consequences of PA, but by Proposition 3.2 none of these schemes is finitely axiomatized, or indeed axiomatized by sentences of bounded quantifier complexity. (Strictly, the scheme INDISC was not written down, but it axiomatizes precisely the theory of κ-like models for singular strong limits κ; that this theory is recursively axiomatized and an axiomatization can in principle be written down follows directly from Keisler's results.)

Of course, by Theorem 3.20 and Lemma 3.19 we have the following (which as we have seen is simple in the 'strongest' case of IB + exp, but seems much more interesting in the other cases).

Theorem 4.1 *For all $n \in \mathbb{N}$, and for each theory T in (‡), $T + I\Sigma_n + \exp$ is Π_{n+2}-conservative over $I\Sigma_n + \exp$, and $T + B\Sigma_{n+1} + \exp$ is conservative over $B\Sigma_{n+1} + \exp$ for sentences of the form $\sigma \vee \tau$ where σ and τ are Π_{n+2} and Σ_{n+2} sentences respectively.*

The main family of open problems is the following.

Problem 4.2 Do any of the implications in (‡) reverse?

In particular,

Problem 4.3 Is IB + exp an axiomatization of the theory of κ-like models for singular stong limit cardinals κ?

A positive solution to this last problem would be a particularly elegant way of describing the theory of such κ-like models.

There are many other more technical problems with some bearing on these questions concerning the model-theoretic constructions we have used here. For example, can the extension constructed in Theorem 3.8 be cofinal? (This is essentially the same as asking whether the implication $\mathrm{IPHP}_{\mathrm{cf}} \Rightarrow \mathrm{IPHP}$ reverses.)

Similarly, we can ask whether the extension methods in Sections 3.4 and 3.5 can be iterated through uncountable cardinalities.

Concerning these iterations, there are even some interesting problems at the countable level. Firstly, can the extension N be chosen to be countable and recursively saturated? This would seem to be a natural way to iterate the construction, but note too that if a is nonstandard, then $\mathrm{SSy}\, M = \mathrm{SSy}\, N$, so if M and N were recursively saturated and countable, they would be isomorphic. Thus this first question can be regarded as asking about elementary submodels of M. This raises a second possibility: can the extension N be isomorphic to *M over the set $<a$*? In other words, can $(M, x)_{x<a} \cong (N, x)_{x<a}$ with $<a = \{x \in M : x < a\} = \{x \in N : x < a\}$ and the isomorphism being identity on $<a$? However, note that by the usual trick with binary representations of numbers less than 2^a, such N can only be a proper extension of M if 2^a exists in M, so exponentiation may again turn out to be important here.

The last family of problems concerns where and when and how the hypothesis that the model is closed under exponentiation can be omitted from the arguments. Certainly, the problem of constructing κ-like models takes a different flavor altogether if κ is not a strong limit and GCH is false. However, none of IB, $\mathrm{IPHP}_{\mathrm{cf}}$, IPHP, GPHP, and CARD obviously imply exp, and all of these (except IB) are obviously true in all κ-like models. The methods used by Paris and Mills in proving Theorem 2.4 part (i) (see for example Section 3.7 above) seem particularly relevant for these issues, but the area is still largely unexplored.

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