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The Contribution of Zygmunt Ratajczyk to the Foundations of Arithmetic

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In February 1994 Dr. Zygmunt Ratajczyk died unexpectedly and prematurely. He was born in 1949, and studied mathematics at the University of Warsaw where he received his MSc. in 1972 under the supervision of Professor Andrzej Mostowski. He then became an assistant to Mostowski in the Department of Foundations of Mathematics at the University of Warsaw, receiving his PhD. in 1978. His thesis, written under the supervision of Professor Wiktor Marek, was devoted to expandability of models of set theory to models of the theory of classes. From 1979 he served as an assistant professor (*adiunkt*) at Warsaw University.

The scientific interests of Zygmunt Ratajczyk were very broad. His published works can be divided into two groups: papers devoted to the problem of expandability of models of ZF to models of KM (namely his [1]–[4]) and papers devoted to the foundations of arithmetic (namely his [5]–[11], and [13]–[14]). In the latter his interests in proof theory are visible. At the end of his life he also became interested in computational complexity (see for example his [12]).

His first paper in the foundations of arithmetic was "Satisfaction classes and combinatorial sentences independent from PA" where the theory PA(S) was studied. PA(S) is an extension of Peano arithmetic PA obtained by adding a binary predicate S and axioms stating that S is a full inductive satisfaction class. In [5] Ratajczyk considered the problem of the axiomatizability of the theory $\mathrm{PA}^{PA(S)} = \{\varphi \in L(\mathrm{PA}) : \mathrm{PA}(S) \vdash \varphi\}$ and constructed a recursive set of sentences (a modification of Ramsey's Theorem) Φ such that Δ_0-PA$+\Phi$ is an axiomatization of $\mathrm{PA}^{PA(S)}$ where Δ_0-PA is PA with induction restricted to bounded formulas only. As a corollary an example of a combinatorial sentence independent of PA(S) was given. A combinatorial criterion for the existence of full inductive satisfaction classes was also obtained.

The latter problem was studied by Ratajczyk together with Kotlarski in the subsequent papers "Inductive full satisfaction classes" and "More on induction in the language with a satisfaction class." In [10] a system of ω-logic was constructed and there were given necessary and sufficient conditions—in the language of consistency of this system or its fragments—for the existence of full inductive and full Σ_m-inductive satisfaction classes ($m \in \omega$) over countable recursively saturated models of PA. In this way new axiomatizations of the theories $\mathrm{PA}^{PA(S)}$ and $\mathrm{PA}^{\Sigma_m - PA(S)}$ were also obtained. In [11] a similar problem was considered but this time transfinite induction instead of ω-logic was used.

Another group of Ratajczyk's results concerns fragments of arithmetic, more exactly $I\Sigma_n (I^- \Sigma_n)$ i.e., Σ_n induction with (without) parameters.[1] In "A combinatorial analysis of functions provably recursive in $I\Sigma_n$" a combinatorial analysis of functions provably recursive in $I\Sigma_n$ was given. In particular a simple proof of the estimation of the growth of those functions was provided. In "Functions provably total in $I^- \Sigma_n$" one finds a theorem on the estimation of the growth of functions provably total in $I^- \Sigma_n$ in the terms of Hardy's Hierarchy (it is a generalization of a result of Adamowicz and Bigorajska for functions provably total in $I^- \Sigma_1$).

The last two papers of Ratajczyk devoted to arithmetic were "Arithmetical transfinite induction and hierarchies of functions" and "Subsystems of true arithmetic and hierarchies of functions." They constituted his *Habilitationsschrift*—unfortunately the habilitation procedure was not finished due to the unexpected death of Ratajczyk just two days before the final exam. They belong to the new domain called combinatorial proof theory. This grew up from works of Paris, Kirby, Harrington, Solovay and Ketonen on independent sentences and is connected with fast growing functions. In [13] and [14] Ratajczyk considered the theory $I(\alpha)$ obtained from PA by adding the axiom of transfinite induction up to the ordinal α and proved among other things:

1. the sentence $\forall x \exists y H_{\varepsilon_{\alpha+1}}(x) = y$ and the sentence PH_α are independent of $I(< \varepsilon_\alpha)$, where $H_{\varepsilon_{\alpha+1}}$ denotes Hardy's Function and PH_α is a combinatorial principle expressing the αth iteration of the Paris-Harrington principle;
2. a generalization of Wainer's Theorem for $I(< \varepsilon_\alpha)$; and,
3. that $I(\varepsilon_\alpha, S)$ is logically equivalent with respect to arithmetical sentences to the theory $I(< \varepsilon_{\varepsilon_{\alpha+1}})$.

One should also mention Ratajczyk's abstract "Traces of models on initial segments" where Paris's problem concerning semiregular cuts was solved and a theorem stating that any countable model of RCA_0 has a cofinal extension to a model of WKL_0 was proved. The latter theorem now plays an important role in the reverse mathematics (it was obtained independently by Harrington).

Zygmunt Ratajczyk was a deep and subtle mathematician who, with mastery, used sophisticated and technically complex methods, in particular combinatorial and proof-theoretic ones. Walking always along his own paths and being immune from actual trends and fashions he hesitated to publish his results, looking endlessly for their improvement. Only after being forced by his colleagues or the rigor of the university machinery did he decide to do it. A good man, open to others, he never exposed either his results or his person.

NOTE

1. Editor's note: $I^-\Sigma_n$ is also known as $I\Sigma_n^-$ in the literature, and Kaye, Paris, and Dimitracopoulos ("On parameter-free induction schemas," *The Journal of Symbolic Logic*, vol. 53 (1988), pp. 1082–1097) showed that $I\Sigma_n^-$ and $I^-\Sigma_n$ have the same provably recursive functions.

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