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Editor's Introduction

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This collection of articles on *Models of Arithmetic* is dedicated to the memory of Zygmunt Ratajczyk, who contributed a number of important results to the field, and who unexpectedly died in February 1994.

Manchester, England, is internationally known for its drizzle, the Industrial Revolution and for the development of Models of Arithmetic. There in 1976-77 J.B. Paris and his student L. Kirby gave the first combinatorial independence result for Peano arithmetic (PA), though previous work of G. Wilmers, H. Lessan (both in Manchester), A. Wilkie, H. Friedman, A. Mostowski, S.G. Simpson, G. Takeuti, K. Schütte and the Munich school of proof theory, and a list too long to include of many others certainly contributed to our understanding of arithmetic. The Paris-Harrington independence result[1] heralded a period of intense development in models of arithmetic, where combinatorial independence results were obtained both for fragments of PA and for its second order extensions. Research in fragments of PA has led to questions of computational complexity theory, where techniques of proof theory and combinatorics seem most appropriate. Much of the current research in models of arithmetic now concerns recursively saturated models, the structure of a model's automorphism group, κ-like structures, extension of PA into stationary logic, second order theories, analysis of proof theoretic strength of principles of recursion theory via fragments of PA, etc. The present collection of articles gives a sampling of current research directions.

In R. Murawski's "The contribution of Zygmunt Ratajczyk to the foundations of arithmetic," an overview of Ratajczyk's work in arithmetic is given. For the many logicians who knew him, Zygmunt's absence will be a tragic loss to the community. In H. Kotlarski's "Automorphisms of countable recursively saturated models of PA: a Survey," a summary is presented of results about the structure of the group of automorphisms $Aut(M)$ of countable recursively saturated models M of arithmetic. For instance, $Aut(M)$ has size of the continuum, $Aut(\mathbf{Q}, <)$ and $Aut(M)$ are each mutually embeddable in the other, but *not* isomorphic, and if M is arithmetically saturated, then the open sets of $Aut(M)$ have countable index. In R. Kossak's "Four problems concerning recursively saturated models of arithmetic," several open problems with partial solutions are given. The problems concern a possible converse to Tarski's theorem on undefinability of truth, inductive satisfaction classes, weakly Jónsson models, nonextendible automorphisms of countable recursively saturated models of arithmetic, free cuts, etc. In "Arithmetically saturated models of arithmetic" by R. Kossak and J. Schmerl, several interesting new characterizations for arithmetically saturated models of arithmetic are proved. The principal result is that if M is a countable recursively saturated model of Peano arithmetic, then M is arithmetically saturated exactly when $Aut(M)$ is finitely generated over each of its open subgroups. Moreover, the authors show that the standard system (traces of M-definable sets on the standard model **N**) of a countable arithmetically saturated model of arithmetic is uniquely determined by the lattice of its elementary substructures. As the title indicates, R. Kaye's "The theory of κ-like models of arithmetic" concerns models M of cardinality κ, but for which the cardinality of the set of predecessors of a is less than κ, for each element $a \in M$. Beginning with the pigeonhole principle, its generalizations and iterations, various second order axioms and their related first order schemes are studied. In some cases, the second order axioms are shown to characterize κ-like models, while the first order schemes are compared in strength to Peano arithmetic, and their model theoretic properties are studied. All first order schemes are implied by the theory $IB + exp$, which proves all sentences true in all κ-like models, yet which is known to be Π_2-conservative over the weak theory $I\Delta_0 + exp$ of bounded arithmetic with exponentiation. In J. Schmerl's article "PA(aa)," the theory PA(aa) of Peano arithmetic within the context of stationary logic is shown to be consistent. Here, the quantifier $(aaX)\varphi(X)$ asserts the existence of a closed unbounded set of countable sets A satisfying $\varphi(A)$. More importantly, it is proved that the theories PA$(aa) + (Det)$ and CA have the same first order consequences, where the scheme (Det) of finite determinateness states

$$(aas_1)(aas_2)\cdots(aas_m)(\forall\vec{x})[(aat)\varphi(\vec{x},\vec{s},t) \vee (aat)\neg\varphi(\vec{x},\vec{s},t)]$$

and the theory CA is that of full second order comprehension with the axiom of induction. In "On the strength of Ramsey's theorem" by D. Seetapun and T. Slaman, Seetapun's solution of a long standing open problem in recursion theory and models of arithmetic is presented. In 1972, C.G. Jockusch, Jr. raised the question whether, for a recursive partition of pairs into two classes, the halting problem is recursive in each infinite homogeneous set. Around 1977 J.B. Paris raised the related so-called 2-3 question: does recursive comprehension RCA_0 together with Ramsey's theorem for pairs RT_2 imply arithmetic comprehension ACA_0? For a number of years, the Jockusch/Paris 2-3 problem, though one of the most important in models of arithmetic, resisted attack. In this article, Seetapun's negative solution using forcing on Scott sets is presented, along with Slaman's proof that $RCA_0 + RT_2$ is stronger than RCA_0. Some related results are mentioned. The 2-3 question of J.B. Paris originated from the arithmetic analogue of a characterization of weakly compact cardinals $\kappa \rightarrow (\kappa)^2_2$. Weakly compact cardinals have a number of equivalent characterizations, in terms of the tree property, square-bracket partition relations, total orderings, etc. For instance, regular κ is weakly compact iff for every total ordering on κ, there is a κ-size increasing or decreasing sequence. With the technique of the Seetapun-Slaman paper perhaps the analogous open questions in arithmetic can now be solved.[2]

NOTES

1. See *Handbook of Mathematical Logic*, ed. J. Barwise.
2. Results on square bracket partitions and increasing/decreasing sequences were obtained in 1980-1 in SLNM 834, 890 by the editor.

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