Notre Dame Journal of Formal Logic
Volume 37, Number 1, Winter 1996

A Contingent Russell's Paradox

FRANCESCO ORILIA

Abstract It is shown that two formally consistent type-free second-order systems, due to Cocchiarella, and based on the notion of homogeneous stratification, are subject to a contingent version of Russell's paradox.

In a number of works ([3], [4], [5], etc.), Cocchiarella has presented two interesting systems of nonstandard second-order logic, namely $\lambda\mathbf{HST}*$ and $\mathbf{HST}^*_\lambda$, which can be taken to provide type-free theories of properties and predication. An earlier formulation of a very similar system occurs in Cocchiarella [2]. The systems $\lambda\mathbf{HST}*$ and $\mathbf{HST}^*_\lambda$ have been proven consistent relative to weak Zermelo set theory (cf. [5], p. 231) and have been applied to a wide range of philosophical and linguistic issues, including logicism, natural language nominalization, fiction, and intensional contexts (cf. Chierchia [1], Cocchiarella [6], Landini [9], [10], etc.).[1] In this paper I show that, given a certain quite natural English-to-formal language "translation policy," a contingent version of Russell's paradox is licensed by the logical resources of these two systems. In what follows I shall rely primarily on [5] and thus any chapter, section or page reference below will be with respect to this work, unless otherwise indicated.

The systems $\lambda\mathbf{HST}*$ and $\mathbf{HST}^*_\lambda$ (cf. Ch. V) are nonstandard in that they allow, in Cocchiarella's terminology, for "nominalized predicates." Formally, this is obtained by permitting predicate variables, primitive predicate constants, and λ-abstracts (representing nonprimitive complex predicates and obtained by letting Church's λ-operator bind free indivividual variables in wffs) to occur both as singular terms in argument position and as predicate terms in predicate position. As is well known, Russell's paradox arises, if this is done with complete freedom, while asserting in full generality the so-called principle of λ-conversion.[2] The system $\lambda\mathbf{HST}*$ achieves consistency by imposing a grammatical restriction on the formation of λ-abstracts: only the homogeneously stratified ones[3] are taken to be well-formed (p. 227). The system $\mathbf{HST}^*_\lambda$ has no such grammatical restriction (p. 219), but has a corresponding (although arguably less severe) constraint at the logistic level. In fact, roughly speaking, it allows for λ-conversion, provided that all the terms occurring as arguments of a given λ-abstract stand for values of individual variables

Received October 10, 1995; revised January 16, 1996

(pp. 230 and 223), i.e., are *denoting*. This limits the generality of λ-conversion, since $\mathbf{HST}^*_\lambda$ relies on a logic free of existential presuppositions regarding singular terms (p. 219). In particular, some singular terms—e.g., the nominalized predicate corresponding to Russell's property of non-self-predication—are provably nondenoting. The system $\mathbf{HST}^*_\lambda$, however, grants that at least those λ-abstracts that (i) contain no denoting terms, (ii) are homogeneously stratified, and (iii) are bound to individuals[4] are denoting singular terms (p. 229).

The *denotata*, if any, of nominalized predicates are characterized by Cocchiarella as "intensional objects," and as "concept-correlates," i.e., "real abstract individuals somehow correlated with predicable concepts" (see, e.g., [6], §2).

To derive a contingent Russell's paradox in $\lambda\mathbf{HST}*$ and $\mathbf{HST}^*_\lambda$, in the form in which I shall propose it, it is necessary to assume that such abstract individuals acquire contingent properties by being somehow involved in the mental activities of thinking subjects—activities that are expressible in English by means of psychological verbs such as "to believe," "to desire," "to think," etc. This assumption does not seem problematic, since the systems in question are meant to be applied to the analysis of intensional contexts, including those originating from psychological verbs. In particular, we shall be interested in English expressions such as "to think of ____ at time *t*," "to be intrigued by ____ at time *t*," "to be puzzled by ____ at time *t*," and the like, where the blank is meant to be filled by a nominalized predicate such as "wisdom," or, for that matter, "being a property that does not exemplify itself." Given the rich expressive resources of $\mathbf{HST}^*_\lambda$ and $\lambda\mathbf{HST}*$, the most natural policy is to formally represent such verbal expressions by means of (primitive) dyadic predicates that are meant to express relations that can be true of a thinking subject and an abstract individual.[5]

Suppose that R is one such predicate, corresponding to an intensional verb appropriately indexed to a temporal parameter, say, "to think of ____ at time *t*," where "*t*" stands for a certain specified interval of time. Suppose further that Q^1 is a (primitive) predicate that applies univocally to a certain thinking subject, say, Plato (Q^1 could then be taken to express the property of being the author of *The Republic*). Thus, we shall assume, "$(\exists w)((\forall u)(Q^1(u) \leftrightarrow u = w))$ & $R^2(w, x))$" is an open wff that can be taken to (i) belong to the languages of $\lambda\mathbf{HST}*$ and $\mathbf{HST}^*_\lambda$, (ii) provide a quite natural representation of the English sentential form "there is exactly one individual who is the author of *The Republic*, and such an individual thinks of ____ at time *t*," and (iii) be contingently true of abstract individuals (posited by these systems).

Let us now concentrate on $\mathbf{HST}^*_\lambda$. Assume the following abbreviations:

(DF1) $P^1(x) =_{def} (\exists w)((\forall u)(Q^1(u) \leftrightarrow u = w))\ \&\ R^2(w, x));$

(DF2) $G^1 =_{def} [\lambda y \neg(\forall F^1)((\exists z)(z = F^1) \rightarrow \neg(F^1 = y\ \&$
$(\exists x)(P^1(x)\ \&\ \neg F^1(x))))]$.[6]

Since G can be taken to be a well-formed λ-abstract of $\mathbf{HST}^*_\lambda$ and has no free variables, $P(G)$ can in turn be taken to be a sentence of $\mathbf{HST}^*_\lambda$. Moreover, it would seem $P(G)$ may in principle happen to be true. In fact, it can be imagined that Plato thinks of G at time t. Moreover, it can be imagined that in thinking of G at t, Plato happens to think only of G. Accordingly, it can be (quite plausibly) *prima facie* assumed that (A1) below is contingent, and could thus turn out to be true.

(A1) $P(G)$ & $(\forall x)(P(x) \rightarrow x = G)$.

Consider now these additional statements to the effect that Q and R are denoting, as nominalized predicates:

(A2) $(\exists x)(x = Q)$;
(A3) $(\exists x)(x = R)$.

It is quite natural to consider (A2) and (A3) as true. There is apparently no good reason to deny that "being the author of *The Republic*" or "thinking of ____ at time *t*" denote abstract individuals, for seemingly they are, so to speak, "innocuous" predicates, as opposed to noninnocuous predicates such as the one expressing Russell's property of non-self-predication.

I now sketch a proof to the effect that, given (A1), (A2), and (A3), a contradiction is derivable in $\mathbf{HST}^*_\lambda$.

Theorem $\vdash_{\mathbf{HST}^*_\lambda}$ ((A1) & (A2) & (A3)) $\rightarrow$ $(G(G)$ & $\neg G(G))$.

Proof: Recall that $\mathbf{HST}^*_\lambda$ comprises (p. 230) a subsystem $\lambda M*$ granting (pp. 220-221) classical propositional logic and free quantification logic with identity (which for convenience I shall call "standard laws"), as well as the axiom ($\exists/\lambda$-conv$*$). Moreover, $\mathbf{HST}^*_\lambda$ includes (p. 230) the axioms ($\mathbf{CP}^*_\lambda$) (p. 224) and ($\exists$/HSCP$^*_\lambda$) (cf. p. 229).[7] Accordingly, the following demonstration can be reconstructed in $\mathbf{HST}^*_\lambda$.

1. (A1) (assumption)
2. (A2) (assumption)
3. (A3) (assumption)
4. $(\exists x)(x = G) \rightarrow (G(G) \leftrightarrow \neg(\forall F)((\exists z)(z = F) \rightarrow \neg(F = G$ & $(\exists x)(P(x)$ & $\neg F(x)))))$ (by (DF2), ($\exists/\lambda$-conv$*$), and standard laws)
5. $(\exists x)(x = G)$ (from (2) and (3), by ($\exists$/HSCP$^*_\lambda$), since G is homogeneously stratified[8] and bound to individuals[9])
6. $G(G) \leftrightarrow \neg(\forall F)(\exists z(z = F) \rightarrow \neg(F = G$ & $(\exists x)(P(x)$ & $\neg F(x))))$ (from (4) and (5), by standard laws)
7. $G(G) \vee \neg G(G)$ (by standard laws)
8. $G(G)$ (assumption)
9. $\neg(\forall F)((\exists z)(z = F) \rightarrow \neg(F = G$ & $(\exists x)(P(x)$ & $\neg F(x))))$ (from (6) and (8), by standard laws)
10. $(\exists x)(P(x)$ & $\neg G(x))$ (from (9), by standard laws)
11. $\neg G(G)$ (from (10) and (1), by standard laws)
12. $\neg G(G)$ (assumption)
13. $(\forall F)((\exists z)(z = F) \rightarrow \neg(F = G$ & $(\exists x)(P(x)$ & $\neg F(x))))$ (from (12) and (6), by standard laws)
14. $\neg(\exists x)(P(x)$ & $\neg G(x))$ (from (5) and (13), by ($\mathbf{CP}^*_\lambda$) and standard laws, instantiating "F" to "G")[10]
15. $P(G) \rightarrow G(G)$ (from (5) and (14), by standard laws, instantiating *x* to *G*)
16. $G(G)$ (from (1) and (15), by standard laws)
17. ((A1) & (A2) & (A3)) $\rightarrow$ $(G(G)$ & $\neg G(G))$ (from (1), (2), (3), (8), (11), (12), and (16), by standard laws) □

This proof generates a contradiction from self-predication, on the plausible assumption that (i) the innocuous predicates Q and R are denoting and (ii) a certain *prima facie* contingent proposition happens to be true. It can then be taken to provide a contingent version of Russell's paradox. In light of this result,[11] (A1), (A2), and (A3) cannot all be true, unless some of the principles constituting $\mathbf{HST}^*_\lambda$ are rejected. This gives rise to three main options, which I shall now briefly discuss.

According to *option 1*, we should reject either (A2) or (A3). But then, since a reasonable, principled way for classifying apparently innocuous predicates as either denoting or nondenoting, as the case may be, neither has been proposed nor is (in all likelihood) forthcoming, it would seem that all we can do to decide such matters is to rely on empirical considerations (such as whether or not a sentence like (A1) happens to be true). This is not desirable. Indeed, if we followed this line, the aforementioned applications of $\mathbf{HST}^*_\lambda$ would be undermined, for many instances of λ-conversion that are therein taken for granted could no longer be generally assumed.

According to *option 2*, we should take the above proof as showing that (A1) is, contrary to its *prima facie* contingent character, impossible after all. Let us for convenience label the two conjuncts of (A1) as (A1a) and (A1b), respectively. We could perhaps deny with some plausibility the *prima facie* contingent character of (A1a), if G were nondenoting. For then, roughly speaking, one could argue that Plato, in his mental tokening of G (or rather of its Greek counterpart), would not succeed in directing his mental activity on an abstract individual. But once we reject option 1, we are forced to assume that G is denoting. It is not clear what other reasons there could be to deny that (A1a) is contingent. Once we grant the contingency of (A1a), the simplest way of rejecting *a priori* the contingent character of (A1) is to commit oneself to the following claim: if (A1a) happens to be true, then (A1b) *must* be false (and therefore at t Plato must be thinking of something other than G).[12] This is certainly more plausible than merely denying the contingent character of either (A1a) or (A1b), but it still hard to swallow. Of course, intuitions to the effect that a certain proposition is contingent may be defeasible. Yet, it seems to me that, whatever motivations there are in favor of the logical grammar and the logical principles that characterize $\mathbf{HST}^*_\lambda$, they can hardly override those in favor of the fact that it is possible that at t Plato thinks of G (is intrigued by G', is puzzled by G''),[13] without thinking of (being intrigued by, being puzzled by) something else, say, wisdom or horsehood (even though Plato is known to have devoted some of his time to these *ideas*).

According to *option 3*, the logical grammar and/or logical principles of the system $\mathbf{HST}^*_\lambda$ should be reconsidered. Perhaps free logic, homogeneous stratification, and boundness to individuals do not provide the most appropriate way of asserting instances of λ-conversion, without accepting it in full generality.

Let us now turn our attention to $\lambda\mathbf{HST}*$. Assume the following abbreviation:

(DF3) $H =_{def} [\lambda y(\exists F)(F = y \mathbin{\&} (\exists x)(P(x) \mathbin{\&} \neg F(x)))]$.

Clearly, H is homogeneously stratified[14] and therefore is a well-formed λ-abstract of the language of $\lambda\mathbf{HST}*$. Consider now (A1′), which is like (A1), except that H replaces G. Clearly, (A1′) is a sentence of $\lambda\mathbf{HST}*$ that (at least *prima facie*) may happen to be true. Since $\lambda\mathbf{HST}*$ allows for full λ-conversion,[15] it follows that $\vdash_{\mathbf{HST}^*_\lambda} H(H) \leftrightarrow (\exists F)(F = H \mathbin{\&} (\exists x)(P(x) \mathbin{\&} \neg F(x)))$. Therefore, by classical quantification logic with identity, which λ**HST**$*$ fully endorses,[16] $\vdash_{\mathbf{HST}^*_\lambda}$ (A1$'$) $\rightarrow$ $(H(H)\ \&\ \neg H(H))$.[17] The proof does not involve (A2) and (A3), but otherwise parallels the one given above for $\mathbf{HST}^*_\lambda$. It is, however, simpler and is thus left to the reader. Given this outcome, the diagnosis for λ**HST**$*$ is, *mutatis mutandis*, as for $\mathbf{HST}^*_\lambda$, except that, of course, option 1 above is no longer available.

Although these results are specific to λ**HST**$*$ and $\mathbf{HST}^*_\lambda$, there is a general lesson. Just as in formal theories of truth we must pay attention not only to simple liar sentences, but also to contingent liar sentences (cf. Tarski [13] and Kripke [8]), similarly, in formal theories of properties and predication, we must guard against contingent Russellian paradoxes as well as the standard Russell's paradox.[18]

Acknowledgments I wish to thank an anonymous referee and professors Nino Cocchiarella and Gregory Landini for some useful comments.

NOTES

1. A slight variation on the theme of $\mathbf{HST}^*_\lambda$ is used in Orilia [11] to deal with intensional contexts.

2. This runs as follows: $[\lambda x_1, \ldots, x_n\varphi](a_1, \ldots, a_n) \leftrightarrow \varphi(a_1/x_1, \ldots, a_n/x_n)$, where each a_i is free for x_i, for all $i \leq n$. Here and in the rest of this paper I follow the (rather standard) metalinguistic terminology and formal conventions of [5], although I shall informally drop some parentheses in quite obvious ways.

3. A well-formed expression ξ is said to be *homogeneously stratified* iff there is an assignment t of natural numbers to the set of terms occurring in ξ (including ξ itself) such that (1) for all terms a and b, if $(a = b)$ occurs in ξ, then $t(a) = t(b)$; (2) for all $n \geq 1$, all n-place predicate expressions π and all terms $a_1, \ldots, a_n$, if $\pi(a_1, \ldots, a_n)$ is a wff occurring in ξ, then (i) $t(a_i) = t(a_j)$, for $1 \leq i,\ j \leq n$, and $t(\pi) = t(a_i) + 1$; and (3) for all $k \in \omega$, all individual variables $x_1, \ldots, x_k$, and all wffs φ, if $[\lambda x_1, \ldots, x_k\varphi]$ occurs in ξ, then $t(x_i) = t(x_j)$, for $1 \leq i,\ j \leq k$, and $t([\lambda x_1, \ldots, x_k\varphi]) = t(x_1) + 1$ (p. 217).

4. A well-formed expression ξ is *bound to individuals* iff for all $n \in \omega$, all n-place predicate variables F^n, and all wffs φ, if $(\forall F^n)\varphi$ is a wff occurring in ξ, then for some individual variable x, and some wff ψ, φ is the wff $((\exists x)(F^n = x) \rightarrow \psi)$.

5. This is in line with Cocchiarella's [6] proposed treatment of "to seek" in its *de dicto* interpretation (he has not explicitly considered the formal rendering of "to think of ____ at time t," and the like, in his systems).

6. In the following, for simplicity's sake, I may drop from formulas the superscripts indicating adicity.

7. The axioms ($\exists/\lambda$-conv$*$), ($\mathbf{CP}^*_\lambda$) and ($\exists$/HSCP$^*_\lambda$) read, respectively, as follows:

$$[\lambda x_1, \ldots, x_n\varphi](a_1, \ldots, a_n) \leftrightarrow (\exists x_1)\ldots(\exists x_n)(a_1 = x_1\ \&\ \ldots\ \&\ a_n = x_n\ \&\ \varphi),$$

where no x_i is free in any a_j, for $1 \leq i,\ j \leq n$;

$$(\exists F^n)([\lambda x_1, \ldots, x_n\varphi] = F^n),$$

where F^n does not occur free in φ;

$$((\exists y)(a_1 = y)\ \&\ \ldots\ \&\ (\exists y)(a_k = y)) \rightarrow (\exists y)([\lambda x_1, \ldots, x_n\varphi] = y),$$

where $[\lambda x_1, \ldots, x_n\varphi]$ is homogeneously stratified, φ is bound to individuals, y is an individual variable not occurring in φ, and $a_1, \ldots, a_k$ are all the variables or nonlogical constants occurring free in $[\lambda x_1, \ldots, x_n\varphi]$.

8. As the following assignment of integers to the relevant terms shows:

$$z, F, y, Q, R:\ 2;$$
$$x, u, w:\ 1.$$

9. As the occurrence of the sub-wff "$(\exists z)(z = F^1)$" in G grants.

10. Note that ($\mathbf{CP}^*_\lambda$) guarantees that all closed λ-abstracts stand for values of variables occurring in predicate position, and therefore F can be instantiated to G.

11. This could perhaps be avoided by means of some other policy, instead of the one assumed here, for the translation of English sentences involving intensional verbs into the language of $\mathbf{HST}^*_\lambda$. Such a policy should give us, instead of G, a λ-abstract somehow corresponding to it, which however fails to denote an abstract individual, e.g., because it is not homogeneously stratified. I do not know what kind of policy would do this job.

12. This was suggested in correspondence by Cocchiarella.

13. Where G' is obtained from G by replacing R with a predicate standing for "is intrigued by," and G'' is obtained from G by replacing R with a predicate standing for "is puzzled by."

14. As the following assignment of integers to the relevant terms shows: F, y, Q, R : 2; x, w, u : 1.

15. The system λ**HST**∗ includes (p. 227) the subsystem $\lambda M**$, and λ-conversion is provable therein (p. 222).

16. This results from the fact that λ**HST**∗ includes $\lambda M**$ (p. 222).

17. A similar proof (where Plato is assumed to think of a certain set rather than a certain intensional object) could be carried out in Quine's set theories NF and ML (cf. [12]), once they are appropriately embedded in a theory of intensional contexts, for these systems also rely on stratification to avoid Russell's paradox. But it is not obvious that this is of any intrinsic interest, for Quine's set theories are primarily meant to be tools to be used in the foundations of mathematics, and how they should relate to intensional contexts has not been explored.

18. In particular, there seems to be an analogy worth noting between the systems discussed here and the classical language with its own truth predicate, but with otherwise impoverished expressive resources, considered by Gupta [7]. Gupta provides a consistency proof showing that the standard Liar cannot arise, given such a language. But this does not rule out a contingent Liar depending, e.g., on the fact that certain contingent, appropriately vicious, "sentence naming ceremonies" are assumed to take place.

REFERENCES

[1] Chierchia, G., *Topics in the Syntax and Semantics of Infinitives and Gerunds*, Ph.D. dissertation, University of Massachusetts, Amherst, 1984. 1

[2] Cocchiarella, N., "The theory of homogeneous simple types as a second order logic," *Notre Dame Journal of Formal Logic*, vol. 20 (1979), pp. 505–524. Zbl 0314.02027 MR 80k:03058 1

[3] Cocchiarella, N., "Two λ-extensions of the theory of homogeneous simple types," *Notre Dame Journal of Formal Logic*, vol. 26 (1985), pp. 377–407. Zbl 0588.03006 MR 87c:03014 1

[4] Cocchiarella, N., "Frege, Russell and logicism: a logical reconstruction," pp. 197–252 in *Frege Synthesized*, edited by L. Haaparanta and J. Hintikka, Reidel, Dordrecht, 1986. 1

[5] Cocchiarella, N., *Logical Investigations of Predication Theory and the Problem of Universals*, Bibliopolis Press, Naples, 1986. Zbl 0614.03003 MR 88k:03010 1, 1, 1, 1

[6] Cocchiarella, N., "Conceptualism, realism, and intensional logic," *Topoi*, vol. 5 (1989), pp. 75–87. MR 91a:03029 1, 1, 1

[7] Gupta, A., "Truth and paradox," *Journal of Philosophical Logic*, vol. 11 (1982), pp. 1–60. Zbl 0512.03002 MR 84i:03020a 1

[8] Kripke, S., "Outline of a theory of truth," *Journal of Philosophy*, vol. 72 (1975), pp. 690–716. Zbl 0952.03513 1

[9] Landini, G., *Meinong Reconstructed Versus Early Russell Reconstructed: A Study in the Formal Ontology of Fiction*, Ph.D. dissertation, Indiana University, Bloomington, 1986. 1

[10] Landini G., "How to Russell another Meinongian: a Russellian theory of fictional objects versus Zalta's theory of abstract objects," *Grazer Philosophische Studien*, vol. 37 (1990), pp. 93–122. 1

[11] Orilia, F., "Belief representation in a deductivist type-free doxastic logic," *Minds and Machines*, vol. 4 (1994), pp. 163–203. 1

[12] Quine, W., *Set Theory and Its Logic*, Harvard University Press, Cambridge, 1963. Zbl 0122.24601 1

[13] Tarski, A., "The concept of truth in formalized languages," pp. 152–278 in *Logic, Semantics, Metamathematics*, Clarendon, Oxford, 1956. 1

Università di Cagliari
Facoltà di Lettere e Filosofia
Dipartimento di Filosofia e Teoria delle Scienze Umane
09123 Cagliari
Italy
email: orilia@vaxca1.unica.it