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Finite Sets and Natural Numbers in Intuitionistic TT

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Abstract We show how to interpret Heyting's arithmetic in an intuitionistic version of TT, Russell's Simple Theory of Types. We also exhibit properties of finite sets in this theory and compare them with the corresponding properties in classical TT. Finally, we prove that arithmetic can be interpreted in intuitionistic TT_3, the subsystem of intuitionistic TT involving only three types. The definitions of intuitionistic TT and its finite sets and natural numbers are obtained in a straightforward way from the classical definitions. This is very natural and seems to make intuitionistic TT an interesting intuitionistic set theory to study, beside intuitionistic ZF.

1 Introduction In this paper, we want to investigate how natural numbers can be defined in intuitionistic TT, Russell's Simple Theory of Types, in such a way that they satisfy the axioms of HA, Heyting's arithmetic.

We believe it is worth undertaking a study of intuitionistic versions of intuitionistic TT and derived theories such as NF, Quine's New Foundations [10] (see Forster [5]). Indeed, on the one hand, intuitionistic TT can be axiomatized with the same proper axioms as the usual axioms of classical TT (see Dziergowski [4]); this is more natural than for intuitionistic ZF (see for example Myhill [9]), and so, we find intuitionistic TT more elegant, from a philosophical point of view.

And on the other hand, we shall show that Heyting's arithmetic can be interpreted in those intuitionistic theories in a very natural way, which furthermore is very close to the usual interpretation defined in the corresponding classical theories.

2 The axioms of intuitionistic TT We recall (see Boffa [2] or Chapter 3 of Fraenkel, Bar-Hillel, and Levy [6]) that the language $\mathcal{L}_{\mathrm{TT}}$ contains countably many variables of type i, for each $i \in \omega$. So each variable is indexed with a superscript indicating its type: for example, x^0, y^0, z'^0, t^0_{24} are type 0 variables; variables of distinct types are

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distinct (for example, x^0 and x^1 are distinct variables). The atomic formulas are of the form

$$x^i \in y^{i+1} \qquad \text{or} \qquad x^i = y^i,$$

for each $i \in \omega$. In particular, $x^i \in x^i$ is never a well-formed formula. So Russell's paradox involving the set $\{x \,:\, x \notin x\}$ disappears because it simply cannot be expressed in the language! The proper axioms of intuitionistic and classical TT are:

1. extensionality: $(\forall x^{i+1}, y^{i+1})((\forall z^i)(z^i \in x^{i+1} \leftrightarrow z^i \in y^{i+1}) \rightarrow x^{i+1} = y^{i+1})$, for each $i \in \omega$;
2. comprehension schema: $(\exists x^{i+1})(\forall z^i)(z^i \in x^{i+1} \leftrightarrow \varphi(z^i))$, for each formula φ where x^{i+1} does not occur free.

For example, we can easily derive from these axioms the existence, *for each type i*, of an empty set $\varnothing^{i+1} = \{x^i \,:\, x^i \neq x^i\}$, of a universal set $V^{i+1} = \{x^i \,:\, x^i = x^i\}$, of the power set $\mathcal{P}x^{i+1}$ of any set x^{i+1}, of $\mathrm{USC}(X^{i+1}) = \{\{x^i\} \,:\, x^i \in X^{i+1}\}$, etc. Notice, for example, that $\mathcal{P}x^{i+1}$ is a set of type $i+2$. In particular, the paradox inferred by Cantor from $V = \mathcal{P}V$ cannot be reproduced in TT because $V^{i+1} = \mathcal{P}V^{i+1}$ is not a wff ($V^{i+2} = \mathcal{P}V^{i+1}$ is a wff, but it is harmless).

Almost everywhere in this paper we shall not indicate the type of variables in formulas. This will be more readable. Of course, the formulas will be ambiguous, because they can be typed in many different ways: there are infinitely many sets of natural numbers, in the same way that there are infinitely many empty sets. Most often, this ambiguity will not matter: finite sets and natural numbers are defined in the same way at each type, and most properties we shall derive from the definitions do not depend on the types chosen to state them. The only sections where the choice of types matters are

1. Section 8, where we shall study the relations between finite sets of any type $i \geq 1$ and finite sets of type $i+1$;
2. Sections 9 and 10 where we shall see how natural numbers can be defined with only three types.

Finally, let us make precise some notations. A Kripke model of intuitionistic TT will be of the form

$$\mathcal{M} = \Big\langle (\mathcal{M}_k)_{k\in K}\,,\ \langle K, \leq, \mathbf{0}\rangle \Big\rangle$$

where $\langle K, \leq, \mathbf{0}\rangle$ is a partial order with $\mathbf{0}$ as its minimal element, and each $\mathcal{M}_k$ is a structure of the form $\langle M_k^0, M_k^1, \ldots; \in_k^{\mathcal{M}}, =_k^{\mathcal{M}}\rangle$, where each M_k^i is the domain of type i variables (so $\in_k^{\mathcal{M}} \subseteq \bigcup_{i\in\omega} M_k^i \times M_k^{i+1}$).

3 Three notions of finiteness Classically, the set of finite sets is the smallest inductive set. In an intuitionistic framework, we introduce three notions of inductive set, which are all classically equivalent. A set E is

K-inductive	iff	$\varnothing \in E$	$\wedge$	$(\forall x)(x \in E$	$\rightarrow$	$(\forall z)(x \cup \{z\} \in E))$;
S-inductive	iff	$\varnothing \in E$	$\wedge$	$(\forall x)(x \in E$	$\rightarrow$	$(\forall z)(\forall y \subseteq \{z\})(x \cup y \in E))$;
N-inductive	iff	$\varnothing \in E$	$\wedge$	$(\forall x)(x \in E$	$\rightarrow$	$(\forall z \notin x)(x \cup \{z\} \in E))$.

This gives us three notions of finiteness; in each case, the set of finite sets is the smallest inductive set.

$$\begin{aligned}
\mathrm{KFin} &= \bigcap\{E \;:\; E \text{ is K-inductive}\}.\\
\mathrm{SFin} &= \bigcap\{E \;:\; E \text{ is S-inductive}\}.\\
\mathrm{NFin} &= \bigcap\{E \;:\; E \text{ is N-inductive}\}.
\end{aligned}$$

As expected, there is an induction principle corresponding to each notion of finiteness.

Proposition 3.1 *Let $\varphi(x)$ be a formula of $\mathcal{L}_{\mathrm{TT}}$. Then*

1. $[\varphi(\varnothing) \;\wedge\; (\forall x \in \mathrm{KFin})(\varphi(x) \to (\forall z)\varphi(x \cup \{z\}))] \;\;\to\;\; (\forall x \in \mathrm{KFin})\varphi(x)$;
2. $[\varphi(\varnothing) \;\wedge\; (\forall x \in \mathrm{SFin})(\varphi(x) \to (\forall z)(\forall y \subseteq \{z\})\varphi(x \cup y))]$
   $\to \;\; (\forall x \in \mathrm{SFin})\varphi(x)$;
3. $[\varphi(\varnothing) \;\wedge\; (\forall x \in \mathrm{NFin})(\varphi(x) \to (\forall z \notin x)\varphi(x \cup \{z\}))] \;\;\to\;\; (\forall x \in \mathrm{NFin})\varphi(x)$.

Proof: Consider $E = \{x \;:\; x \in \mathrm{KFin} \wedge \varphi(x)\}$. It is easily seen that $\varnothing \in \mathrm{KFin}$ and that $x \cup \{z\} \in \mathrm{KFin}$ whenever $x \in \mathrm{KFin}$. Then it is clear that E is K-inductive. So $\mathrm{KFin} \subseteq E$. In other words, $(\forall x \in \mathrm{KFin})\varphi(x)$. The proofs of (2) and (3) are analogous. □

In classical TT, one proves by induction that NFin = KFin = SFin. In intuitionistic TT, one can also prove by induction that $\mathrm{NFin} \subset \mathrm{KFin} \subset \mathrm{SFin}$. Anyway, one cannot prove that those sets are equal. Indeed, using the technique described in [4], it is easy to find a Kripke model of intuitionistic TT where each singleton $\{a\}$ has a nontrivial subset (i.e., a subset y which is not equal to $\varnothing$ nor to $\{a\}$). Any such subset is in SFin but not in KFin. Also, we can find a Kripke model containing two elements a and b such that neither $a = b$ nor $a \neq b$ is true. Then $\{a, b\}$ is K-finite, but it is not N-finite.

KFin is the original (classical) definition of the set of finite sets given in *Principia Mathematica*. SFin is the closure of KFin under subsets: it can be proved that $(\forall x)(x \in \mathrm{SFin} \leftrightarrow (\exists k \in \mathrm{KFin})(x \subseteq k))$. And we shall define Nn, the set of natural numbers, as the set of cardinals of N-finite sets. KFin would not have been adequate to define Nn: indeed, the cardinal of a nontrivial subset of a singleton is in some sense strictly between 0 and 1.

It should be worth studying the precise relationship between our three notions of finiteness and notions of finiteness of topos theory (e.g., Kuratowski finiteness or finite cardinals; see Johnstone [8]).

4 Some properties of N-finite sets Recall that HA proves that $(\forall n, m)(n = m \vee n \neq m)$. So if Nn models HA, we can expect NFin to satisfy some "classical" properties. Such properties will be given below. Their proofs, which are easy but lengthy inductions, will not appear here; they can be found in an internal report of the author.

These proofs make clear a nontrivial fact: the "$z \notin x$" making the difference between the definitions of N-inductive and K-inductive is sufficient to obtain all the properties we need.

Proposition 4.1 *Let $x \in \mathrm{NFin}$ and $x' \subseteq x$. Then*

(P1) $(\forall z, z' \in x)(z = z' \vee z \neq z')$;

(P2) $\neg\neg x' \in \mathrm{NFin}$;

(P3) $x' \in \mathrm{NFin} \leftrightarrow (\forall z \in x)(z \in x' \vee z \notin x')$;

(P4) $x' \in \mathrm{NFin} \to (\forall z \in x)(\neg\neg z \in x' \to z \in x')$;

(P5) $(\forall z \in x)(x \setminus \{z\} \in \mathrm{NFin})$;

(P6) if $x' \in \mathrm{NFin}$, then $x = x' \vee x \neq x'$, and $x = x'$ iff $x \setminus x' = \varnothing$;

(P7) $x = \varnothing \vee x \neq \varnothing$.

Remark 4.2 N-finite sets *cannot* be defined as the S-finite sets x such that $(\forall z, z' \in x)(z = z' \vee z \neq z')$ (for example, take x to be a nontrivial subset of a singleton). But on the other hand, it is an open question to know whether N-finite sets can be defined as the K-finite sets x such that $(\forall z, z' \in x)(z = z' \vee z \neq z')$.

Remark 4.3 (P2) implies that $(\forall x \in \mathrm{SFin})(\neg\neg x \in \mathrm{NFin})$. So $\mathrm{SFin} \setminus \mathrm{NFin} = \varnothing$ (but $\neg\neg(\mathrm{SFin} = \mathrm{NFin})$ cannot be proved).

Remark 4.4 The converse of (P4) is not true. Indeed, using the technique of [4], it is possible to find a Kripke model $\mathcal{M}$ of intuitionistic TT consisting in three nodes $\mathcal{M}_{\mathbf{0}}$, $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ ($\mathbf{0} \leq \alpha, \beta$; $\alpha \not\leq \beta \not\leq \alpha$), and containing some x' such that $\mathcal{M} \Vdash_\alpha x' = \{a\}$ and $\mathcal{M} \Vdash_\beta x' = \varnothing$. Clearly, $\mathcal{M} \nVdash_{\mathbf{0}} (a \in x' \vee a \notin x')$, so, by (P3), x' is not N-finite in $\mathcal{M}$. Nevertheless, $\mathcal{M} \Vdash_{\mathbf{0}} (\neg\neg a \in x' \to a \in x')$.

Remark 4.5 (P5) is not true if z is not assumed to be a member of x. Indeed, consider the model $\mathcal{M}$ of the preceding remark. $\mathcal{M}$ can be defined so that it contains some a and b such that $\mathcal{M} \Vdash_\alpha a = b$ and $\mathcal{M} \Vdash_\beta a \neq b$. Let $x = \{a\}$. It is easy to see that $\mathcal{M} \nVdash (x \setminus \{b\} \in \mathrm{NFin})$, using (P3).

4.1 Mutually detachable N-finite sets We define two sets x and y to be *mutually detachable* iff

$$(\forall z \in x)(z \in y \vee z \notin y) \quad \wedge \quad (\forall z \in y)(z \in x \vee z \notin x).$$

So (P3) could be restated: If $x \in \mathrm{NFin}$ and $x' \subseteq x$, then $x' \in \mathrm{NFin}$ iff x and x' are mutually detachable.

The following proposition gives some equivalent definitions. Its proof uses mainly (P3) and induction.

Proposition 4.6 *Let* $x, y \in \mathrm{NFin}$. *The following formulas are equivalent.*

1. *x and y are mutually detachable.*
2. $x \cap y \in \mathrm{NFin}$.
3. $x \cup y \in \mathrm{NFin}$.
4. $(\exists u \in \mathrm{NFin})(x \subseteq u \;\wedge\; y \subseteq u)$.

The second definition and (P2) imply that

(P8) if $x, y \in \mathrm{NFin}$, then $\neg\neg$(x and y are mutually detachable).

Alternatively, (P8) can be proved using the fact that, for any formula φ, and any $x \in \mathrm{NFin}$, $(\forall z \in x)\neg\neg\varphi \leftrightarrow \neg\neg(\forall z \in x)\varphi$, which can be proved by induction on φ.[1] If $x, y \in \mathrm{NFin}$ are mutually detachable, then

(P9) $x \setminus y \in \mathrm{NFin}$,

and

(P10) $\quad x \not\subseteq y$ iff $(\exists t \in x)(t \notin y)$.

To prove (P9), first prove, using mainly (P3) and (P4), that if $x, x' \in \mathrm{NFin}$ and $x' \subseteq x$, then $x \setminus x' \in \mathrm{NFin}$. Then, if $x, y \in \mathrm{NFin}$ are mutually detachable, Proposition 4.6 allows us to conclude because $x \setminus y = x \setminus (x \cap y)$. On the other hand, the nontrivial direction of (P10) is proved by induction on x.

5 Relations on cardinality of N-finite sets We first define the relations $\simeq$, $\preccurlyeq$ and $\prec$:

$$\begin{array}{lll} x \simeq y & \text{iff} & \text{there is a 1–1 function mapping } x \text{ onto } y; \\ x \preccurlyeq y & \text{iff} & \text{there is a 1–1 function mapping } x \text{ into } y; \\ x \prec y & \text{iff} & x \preccurlyeq y \text{ and } x \not\simeq y. \end{array}$$

It is then routine to prove by induction on x that

(P11) $\quad$ if $x \in \mathrm{NFin}$ and $y \simeq x$, then $y \in \mathrm{NFin}$.

The following lemma, which is also needed below, proves that N-finiteness implies Dedekind-finiteness. The problem of comparing Dedekind-finiteness with the three notions of finiteness introduced in this paper remains to be studied.

Lemma 5.1 *Suppose that $x \subseteq x'$ and, for some $y \in \mathrm{NFin}$, $x \simeq y$ and $x' \simeq y$. Then $x = x'$. In particular, if $x \subseteq x' \in \mathrm{NFin}$ and $x \simeq x'$, then $x = x'$.*

Proof: By induction on x (using mainly (P1), in the same style as Lemma 5.2 below). □

The following lemma is the hard part when proving that $\preccurlyeq$ is an order relation.

Lemma 5.2 *Let $x, y \in \mathrm{NFin}$. Then $(x \preccurlyeq y \wedge y \preccurlyeq x) \rightarrow x \simeq y$. So $x \prec y$ iff $x \preccurlyeq y$ and $x \not\simeq y$.*

Proof: We prove that $(x \preccurlyeq y \wedge y \preccurlyeq x) \rightarrow x \simeq y$ by induction on x. This time, as an example, we give the details of the proof. If $x = \varnothing$, then $y = \varnothing$ because $y \preccurlyeq x$, and thus $x \simeq y$.

Suppose that $x = x_1 \cup \{z_1\}$, where $z_1 \notin x_1$. Let i be a 1–1 function mapping x into y, and j be a 1–1 function mapping y into x.

$$x \setminus \{z_1\} \preccurlyeq y \setminus \{i(z_1)\}, \text{ and } y \setminus \{i(z_1)\} \preccurlyeq x \setminus \{j(i(z_1))\}.$$

If $j(i(z_1)) = z_1$, it is easy. In that case, indeed, (P5) and the induction hypothesis implies that $x \setminus \{z_1\} \simeq y \setminus \{i(z_1)\}$. So $x = (x \setminus \{z_1\}) \cup \{z_1\} \simeq (y \setminus \{i(z_1)\}) \cup \{i(z_1)\} = y$ (both equalities can be proved using (P3)). Furthermore, we can always assume that $j(i(z_1)) = z_1$. More precisely, we are going to prove that j can be transformed into j' such that $j'(i(z_1)) = z_1$. Indeed, let $j``y = \{j(u) : u \in y\}$. Then by (P11) and (P3), there are two cases.

Case 1: $z_1 \in j``y$. So there exists $t \in y$ such that $z_1 = j(t)$ and we can define j' as follows:

$$\begin{array}{rcl} j'(i(z_1)) & = & z_1; \\ j'(t) & = & j(i(z_1)); \\ j'(u) & = & j(u) \quad \text{if } u \neq t \text{ and } u \neq i(z_1). \end{array}$$

Case 2: $z_1 \notin j``y$. This is easier. Just define j' as follows:

$$\begin{aligned} j'(i(z_1)) &= z_1; \\ j'(u) &= j(u) \quad \text{if } u \neq i(z_1). \end{aligned}$$

In both cases, we leave it to the reader to check that $j' : y \to x$ is 1–1. (Notice that (P1) has been implicitly used throughout the definition of j'.) Finally, it is then easy to check that $x \prec y \Leftrightarrow x \preccurlyeq y \wedge x \not\simeq y$. $\square$

Now, by induction on x, one proves the following expected trichotomy.

Proposition 5.3 *If $x, y \in$ NFin, then $x \prec y$ or $x \simeq y$ or $x \succ y$.*

This of course implies the decidability of $\simeq$:

(P12) If $x, y \in$ NFin, then $(x \simeq y \vee x \not\simeq y)$ and $(\neg\neg x \simeq y \to x \simeq y)$.

6 The natural numbers

6.1 Basic definitions Now we are in position to define Nn, the set of natural numbers:

$$\mathrm{Nn} = \{n \ : \ (\exists x \in \mathrm{NFin})(\forall z)(z \in n \leftrightarrow z \simeq x)\}.$$

In particular, the natural number 0 is $\{\varnothing\}$, and 1 is the set of all singletons, that is, $\mathrm{USC}(V)$. The order on Nn, is defined from the order $\prec$ on NFin: if $n, m \in$ Nn, then

$$n < m \quad \text{iff} \quad (\exists x \in n)(\exists x' \in m)(x \prec x')$$

(one or both of the $\exists$s above may be equivalently replaced with a $\forall$). And

$$n \leq m \quad \text{iff} \quad n < m \vee n = m \quad \text{iff} \quad (\exists x \in n)(\exists x' \in m)(x \preccurlyeq x').$$

Let us now define S, the *successor function* on Nn.

$$S = \{\langle n, m\rangle \ : \ (\exists x \in n)(\exists z)(z \notin x \wedge x \cup \{z\} \in m))\}.$$

S satisfies the expected properties, as stated in the following lemma.

Lemma 6.1

1. *S is a 1–1 function whose domain and range are subsets of* Nn*.*
2. *If $m \in \mathrm{dom}S$ and $n \leq m$, then $n \in \mathrm{dom}S$.*
3. $(\forall n \in \mathrm{dom}S)(S(n) \neq 0)$.
4. $(\forall n \in \mathrm{Nn})(n \neq 0 \to (\exists m \in \mathrm{Nn})(m = S(n)))$.

Notice that $(\forall n \in \mathrm{Nn})(n \in \mathrm{dom}S \vee n \notin \mathrm{dom}S)$ cannot be proved in intuitionistic TT. Indeed, by the technique described in [4], it is easy to obtain a Kripke model $\mathcal{M}$ of intuitionistic TT satisfying the following properties. $\mathcal{M}$ consists of three nodes: $\mathcal{M}_0$, $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ $(0 \leq \alpha, \beta; \alpha \not\leq \beta \not\leq \alpha)$. And $M_0^0 = \{a\} = M_\alpha^0$, while $M_\beta^0 = \{a, b\}$. Then $\mathcal{M} \Vdash_\alpha 1 \notin \mathrm{dom}S$, but $\mathcal{M} \Vdash_\beta 1 \in \mathrm{dom}S$.

6.2 Induction To write properties in a simpler form, we introduce some notation. If F denotes a function, then we shall write

$$\begin{aligned} m = F(n) \quad &\text{for} \quad \langle n, m\rangle \in F, \text{ and} \\ F(n)\downarrow \quad &\text{for} \quad n \in \mathrm{dom} F. \end{aligned}$$

Of course, all the formulas appearing in the sequel and using these notations can be translated into wff of $\mathcal{L}_{\mathrm{TT}}$. In particular, an expression such as $(F(n)\downarrow \wedge H(F(n))\downarrow)$ denotes the formula $(\exists k)(\langle n, k\rangle \in F \wedge (\exists k')(\langle k, k'\rangle \in H))$.

Here now is an *induction principle* on Nn.

Proposition 6.2 *Let $\varphi(x)$ be a formula. Then* $[\varphi(0) \wedge (\forall n \in \mathrm{Nn})((S(n)\downarrow \wedge \varphi(n)) \to \varphi(S(n)))] \to (\forall n \in \mathrm{Nn})\varphi(n)$.

Proof: The idea is to replace induction in Nn by induction in NFin. Let $\psi(x) \equiv (\forall n \in \mathrm{Nn})(x \in n \to \varphi(n))$. Then $\varphi(0)$ implies $\psi(\varnothing)$. On the other hand, suppose $(\forall n \in \mathrm{Nn})((S(n)\downarrow \wedge \varphi(n)) \to \varphi(S(n)))$. This translates into $(\forall x \in \mathrm{NFin})(\psi(x) \to (\forall z \notin x)\psi(x \cup \{z\})$. By induction on NFin, we conclude that $(\forall x \in \mathrm{NFin})\psi(x)$, which implies $(\forall n \in \mathrm{Nn})\varphi(n)$. □

The next proposition guarantees the existence of functions $\mathrm{Nn} \to \mathrm{Nn}$ defined inductively.

Proposition 6.3 *Let $a \in \mathrm{Nn}$ and $H : D_H \to \mathrm{Nn}$, where $D_H \subseteq \mathrm{Nn}$. Then there exists a unique function F such that*

(R1) the domain and the range of F are subsets of Nn;

(R2) $F(0)\downarrow$ and $F(0) = a$;

(R3) if $S(n)\downarrow$, then $F(S(n))\downarrow$ iff $F(n)\downarrow$ and $H(F(n))\downarrow$;

(R4) if $S(n)\downarrow$ and $F(S(n))\downarrow$, then $F(S(n)) = H(F(n))$.

Proof (Uniqueness): Suppose that F and F' both satisfy (R1) – (R4). By induction on n, it can be proved that, for all $n \in \mathrm{Nn}$, $F(n)\downarrow$ iff $F'(n)\downarrow$, and that if $F(n)\downarrow$, then $F(n) = F'(n)$.

Proof (Existence): First, we are going to prove that, for all $k \in \mathrm{Nn}$, there exists a unique function F_k such that

($\mathrm{R1}_k$) the domain of F_k is a subset of $\{0, \dots, k\}$ and its range a subset of Nn;

($\mathrm{R2}_k$) $F_k(0)\downarrow$ and $F_k(0) = a$;

($\mathrm{R3}_k$) if $S(n)\downarrow$ and $S(n) \leq k$, then $F_k(S(n))\downarrow$ iff $F_k(n)\downarrow$ and $H(F_k(n))\downarrow$;

($\mathrm{R4}_k$) if $S(n)\downarrow$, $S(n) \leq k$ and $F_k(S(n))\downarrow$, then $F_k(S(n)) = H(F_k(n))$.

The uniqueness of each F_k is proved exactly as the uniqueness of F above. Now, let us prove the existence of F_k, by induction on k. If $k = 0$, we define $F_0 = \{\langle 0, a\rangle\}$ (conditions ($\mathrm{R3}_0$) and ($\mathrm{R4}_0$) are empty). On the other hand, suppose that $S(k)\downarrow$. Then, using the induction hypothesis, we define

$$F_{S(k)} = F_k \cup \{\langle S(k), m\rangle \;:\; F_k(k)\downarrow \wedge H(F_k(k))\downarrow \wedge m = H(F_k(k))\}.$$

Clearly, $F_{S(k)}$ satisfies $(\mathrm{R1}_{S(k)})$ – $(\mathrm{R4}_{S(k)})$. Now we can define

$$F = \{\langle n, m\rangle \ : \ F_n(n)\downarrow \wedge m = F_n(n)\}.$$

Clearly, F satisfies (R1) – (R4). □

6.3 Addition and multiplication Now we can use Proposition 6.3 to define addition and multiplication on Nn. First, for each $m \in \mathrm{Nn}$, A_m is the unique function such that

1. the domain and the range of A_m are subsets of Nn;
2. $A_m(0)\downarrow$ and $A_m(0) = m$;
3. if $S(n)\downarrow$, then $A_m(S(n))\downarrow$ iff $A_m(n)\downarrow$ and $S(A_m(n))\downarrow$;
4. if $S(n)\downarrow$ and $A_m(S(n))\downarrow$, then $A_m(S(n)) = S(A_m(n))$.

Then $m + n = k$ denotes the formula $A_m(n)\downarrow \wedge k = A_m(n)$. On the other hand, for each $m \in \mathrm{Nn}$, we define P_m to be the unique function such that

1. the domain and the range of P_m are subsets of Nn;
2. $P_m(0)\downarrow$ and $P_m(0) = 0$;
3. if $S(n)\downarrow$, then $P_m(S(n))\downarrow$ iff $P_m(n)\downarrow$ and $A_m(P_m(n))\downarrow$;
4. if $S(n)\downarrow$ and $P_m(S(n))\downarrow$, then $P_m(S(n)) = A_m(P_m(n))$.

Naturally, $m.n = k$ denotes the formula $P_m(n)\downarrow \wedge k = P_m(n)$.

We leave it to the reader to check all the usual properties such as $n + m = m + n$, $n + (m + k) = (n + m) + k$, $(m + n).k = m.k + n.k$, etc. This task is rather tedious and boring, due to the fact that the A_m's and P_m's are not defined everywhere on Nn. Alternative definitions of addition and multiplication will be given in Section 10.

7 The axiom of infinity

7.1 The axiom Our goal is to prove in intuitionistic TT that Nn is a model of HA. To achieve this goal, we need $A_m(n)\downarrow$ and $P_m(n)\downarrow$ to be true for all $m, n \in \mathrm{Nn}$. And this is true if $S(n)\downarrow$ for all $n \in \mathrm{Nn}$, which in turn is equivalent to our *axiom of infinity*:

(AxInf) $\quad (\forall x \in \mathrm{NFin})(\exists y \in \mathrm{NFin})(y \succ x).$

7.2 Intuitionistic and classical equivalent forms Here are four equivalent ways to state this axiom of infinity. The equivalences which are not trivial can be proved by induction.

Proposition 7.1 *The following four formulas are equivalent to* (AxInf).

(AI_1) $\quad (\forall n \in \mathrm{Nn})(\exists m \in \mathrm{Nn})(m > n).$

(AI_2) $\quad (\forall x, y \in \mathrm{NFin})(\exists x', y' \in \mathrm{NFin})(x \simeq x' \ \wedge \ y \simeq y' \ \wedge \ x' \cap y' = \varnothing).$

(AI_3) $\quad (\forall x \in \mathrm{NFin})(\exists x' \in \mathrm{NFin})(x \simeq x' \wedge (\exists z)(z \notin x')).$

(AI_4) $\quad (\forall n \in \mathrm{Nn})(S(n)\downarrow).$

Classically, (AxInf) is equivalent to $(\forall x \in \mathrm{NFin})(\exists y)(y \notin x)$ and also to $(V \notin \mathrm{NFin})$. We are going to prove that, in an intuitionistic framework, only one direction of each equivalence still holds.

Lemma 7.2 *$(\forall x \in \mathrm{NFin})(\exists y)(y \notin x)$ implies* (AxInf)*, but the converse does not hold. Notice that $(\forall x \in \mathrm{NFin})(\exists y)(y \notin x)$ is equivalent to*

$$(\forall x, y \in \mathrm{NFin})(\exists x', y' \in \mathrm{NFin})(x \cap x' = \varnothing \ \wedge\ y \cap y' = \varnothing \ \wedge \\ x \simeq x' \ \wedge\ y \simeq y' \ \wedge\ x' \cap y' = \varnothing). \quad (1)$$

Compare with (AI_2) and (AI_3).

Proof: It trivial to check that $(\forall x \in \mathrm{NFin})(\exists y)(y \notin x)$ implies (AxInf). But it is more tricky to see why the converse does not hold. Here is a counterexample. Using the technique of [4], as in all counterexamples, we can prove the existence of a Kripke model $\mathcal{M}$ of intuitionistic TT satisfying the following conditions. $\mathcal{M} = \langle(\mathcal{M}_i)_{i \in K}, \langle K, \leq, k_0\rangle\rangle$, where $K = \{k_0, k_1, k_2, \ldots\}$ and for all i, $k_0 \leq k_i$, while $k_i \not\leq k_j$ if $0 \neq i \neq j$. For each $k_i \in K$, the domain of type 0 objects of $\mathcal{M}_{k_i}$ is $\{x_0, x_1, x_2, \ldots\}$. And equality in $\mathcal{M}$ is defined as follows.

$$\mathcal{M} \Vdash_{k_i} x_0 = x_i, \ \text{for all } i \in \omega$$

and

$$\mathcal{M} \Vdash_{k_i} x_j \neq x_l, \ \ \text{if } \{j, l\} \neq \{0, i\} \text{ and } j \neq l.$$

Clearly, $\mathcal{M} \Vdash (\{x_0\} \in \mathrm{NFin})$ but $\mathcal{M} \nVdash (\exists y)(y \notin \{x_0\})$. But one can prove that $\mathcal{M} \Vdash$ AxInf (Hint: if $\mathcal{M} \Vdash x \in \mathrm{NFin}$, then there exists x' such that $\mathcal{M} \Vdash x' \simeq x$ and $\mathcal{M} \nVdash x_0 \in x'$).

Also, $(\forall x \in \mathrm{NFin})(\exists y)(y \notin x)$ is equivalent to (1); one direction is trivial, the other one can be proved by induction on x. We remark that (1) is not satisfied in $\mathcal{M}$ when $x = \{x_0\}$. □

Lemma 7.3 (AxInf) *implies* $(V \notin \mathrm{NFin})$*, but the converse does not hold.*

Proof: Suppose (AxInf) and $V \in \mathrm{NFin}$. Then there exists some $v \in \mathrm{NFin}$ such that $v \succ V$. This is absurd because $v \preccurlyeq V$ because $v \subseteq V$. So $V \notin \mathrm{NFin}$. See [4] for a counterexample showing that the converse does not hold. □

7.3 Infinity and arithmetic Now, it is routine to define an interpretation of $\mathcal{L}_{\mathrm{HA}}$ in $\mathcal{L}_{\mathrm{TT}}$ and to prove the following theorem, using the results presented above, on Nn and (AxInf).

Theorem 7.4 (AxInf) *implies that* $\langle \mathrm{Nn}, S, +, \cdot, 0\rangle$ *satisfies* HA.

According to (AI_4), this implication is in fact an equivalence. So (AxInf) is exactly strong enough for arithmetic to be interpreted.

7.4 A relation between N-finiteness and K-finiteness

Proposition 7.5 *Assuming* (AxInf)*, the following characterization holds:*

$$(\forall x)((x \in \mathrm{KFin}) \leftrightarrow \\ (\exists t \in \mathrm{NFin})(\exists f)(f \textit{ is a function} \wedge t \subseteq \mathrm{dom}\, f \wedge f``t = x)).^2 \qquad (2)$$

Proof: First, assuming (AxInf), we prove that $(\forall x \in \mathrm{KFin})(\exists t \in \mathrm{NFin})(\exists f)(f$ is a function $\wedge f``t = x)$, by induction on $x$ (using mainly $(\mathrm{AI}_3)$). As $\mathrm{NFin} \subset \mathrm{KFin}$, the other direction is a particular case of a more general fact: if $x \in \mathrm{KFin}$ and $f$ is a function whose domain contains $x$, then $f``x \in \mathrm{KFin}$. □

Notice that (2) does not imply (AxInf). In fact, (2) does not even imply (AxInf) in classical TT. Indeed (2) is a theorem of classical TT because KFin = NFin, but it is well-known that (AxInf) is not a theorem of classical TT.

8 Shifting the axiom of infinity When we defined NFin, we did not write the type of this set. In fact, there exists a set NFin at each type $i \geq 2$. In this section, we want to study how each NFin^i is related to NFin^{i+1}. This relation does not depend on i. So in what follows, we shall write NFin instead of NFin^i, and NFin^+ instead NFin^{i+1}, for some fixed i. And in the same way, we shall call (AxInf) the axiom of infinity about NFin^i, and (AxInf^+) the axiom of infinity about NFin^{i+1}. As in the previous sections, we shall not give more indication about the types of variables; they should be clear from the context.

***8.1 Infinity,* USC *and* T** The usual way to compare objects of a given type with objects of the next type is to use USC (defined in Section 2). So the following lemma should not be surprising. At this point, it is worth remembering that $\bigcup$ is in some sense the *inverse* of USC: $\bigcup \mathrm{USC}(x) = x$.

Lemma 8.1

1. *For all* x, $\mathrm{USC}(x) \in \mathrm{NFin}^+$ *iff* $x \in \mathrm{NFin}$.
2. *For all* $x, y \in \mathrm{NFin}$, $\mathrm{USC}(x) \simeq \mathrm{USC}(y)$ *iff* $x \simeq y$, $\mathrm{USC}(x) \preccurlyeq \mathrm{USC}(y)$ *iff* $x \preccurlyeq y$ *and* $\mathrm{USC}(x) \prec \mathrm{USC}(y)$ *iff* $x \prec y$.

Proof: The first part is proved by induction. Then, if $R \subseteq x \times y$, define $R^+ = \{\langle\{t\}, \{u\}\rangle : \langle t, u\rangle \in R\}$. It is easy to prove that R is a function (resp. 1–1, resp. onto) iff R^+ is a function (resp. 1–1, resp. onto). Conversely, if $R \subseteq \mathrm{USC}(x) \times \mathrm{USC}(y)$, define $R^- = \{\langle \bigcup t, \bigcup u\rangle : \langle t, u\rangle \in R\}$. And we can also prove that R is a function (resp. 1–1, resp. onto) iff R^- is a function (resp. 1–1, resp. onto). Knowing this, it is routine to prove the second part of the lemma. □

So if $x \in \mathrm{NFin}$, then $\mathrm{USC}(x) \in \mathrm{NFin}^+$. But $\varnothing \notin \mathrm{USC}(x)$. Thus $\mathrm{USC}(x) \cup \{\varnothing\} \in \mathrm{NFin}^+$. Furthermore, $\mathrm{USC}(x) \prec \mathrm{USC}(x) \cup \{\varnothing\}$. This entails two consequences. On the one hand, we obtain, for all $x \in \mathrm{NFin}$, an easy proof of $\mathrm{USC}(x) \prec \mathcal{P}x$, because $\mathrm{USC}(x) \cup \{\varnothing\} \subseteq \mathcal{P}x$.

On the other hand, if we could find some $y \in \mathrm{NFin}$ such that $\mathrm{USC}(y) \simeq \mathrm{USC}(x) \cup \{\varnothing\}$, then we would have a $y \succ x$. With this remark it is easy to prove the following proposition.

Proposition 8.2 (AxInf) *is equivalent to* $(\forall x \in \mathrm{NFin}^+)(\exists y \in \mathrm{NFin})(x \simeq \mathrm{USC}(y))$.

Classically, the *operation* USC, mapping NFin into NFin$^+$, can be transformed into another *operation*, T, mapping Nn into Nn$^+$: if $n \in \mathrm{Nn}$ and $m \in \mathrm{Nn}^+$, then

$$Tn = m \quad \text{iff} \quad (\exists x \in n)(\mathrm{USC}(x) \in m).$$

(Since Nn and Nn$^+$ do not have the same type, the axiom of comprehension cannot be used to define T as a real function; the same remark applies to USC.)

Corollary 8.3 *The operation T is a 1–1 monomorphism mapping* Nn *into* Nn$^+$. *Furthermore,* (AxInf) *is equivalent to "T is onto."*

Proof: Using Lemma 8.1, one can prove that, if $n, m \in \mathrm{Nn}$, then $Tn = Tm \leftrightarrow n = m$, $Tn < Tm \leftrightarrow n < m$. It is also easy to prove that $T(n+m) = Tn + Tm$, $T(n \cdot m) = Tn \cdot Tm$ and $T0 = 0$. Finally, Proposition 8.2 implies that (AxInf) is equivalent to "T is onto," that is, $(\forall m \in \mathrm{Nn}^+)(\exists n \in \mathrm{Nn})(m = Tn)$. □

Also, notice that if $x \in \mathrm{NFin}$, then $\mathcal{P}x \cap \mathrm{NFin} \in \mathrm{NFin}^+$ (this can be proved by induction). But $(\exists a)(\mathcal{P}\{a\} \in \mathrm{NFin}^+)$ implies the excluded middle. Indeed, consider $z \subseteq \{a\}$. As $\{a\}, z \in \mathcal{P}\{a\}$, then $z = \{a\}$ or $z \neq \{a\}$, by (P1). In particular, if φ is any formula, consider $z = \{t \ : \ t = a \wedge \varphi\}$ (we assume that a and t do not occur free in φ). Then it is easy to check that $(z = \{a\} \vee z \neq \{a\})$ implies $(\varphi \vee \neg\varphi)$.

8.2 N-Infinity shifts up but does not shift down Classically, (AxInf) is ambiguous, i.e., (AxInf) is equivalent to (AxInf$^+$). In an intuitionistic framework, this is no longer valid, as proved by the following proposition.

Proposition 8.4 (AxInf) *implies* (AxInf$^+$)*, but the converse does not hold.*

Proof: Consider $x \in \mathrm{NFin}^+$. We want to find $y \in \mathrm{NFin}^+$ such that $y \succ x$. By Proposition 8.2, we can find $y' \in \mathrm{NFin}$ such that $x \simeq \mathrm{USC}(y')$. But then (AxInf) allows us to find $y'' \in \mathrm{NFin}$ such that $y'' \succ y'$. We can let $y = \mathrm{USC}(y'')$.

To prove that (AxInf$^+$) does not imply (AxInf), we consider the following counterexample. We can construct (using, as always, the technique described in [4]) the following Kripke model $\mathcal{M} = \langle (\mathcal{M}_k)_{k \in K}, \langle K, \leq, 0\rangle\rangle$, where $K = \{0\} \cup \{\{i, j\} \ : \ i, j \in \omega \wedge i \neq j\}$ and $\leq\, = \{\langle 0, 0\rangle\} \cup \{\langle 0, \{i, j\}\rangle \ : \ \{i, j\} \in K\}$.

Assume the domain of objects of type 0 of each $\mathcal{M}_k$ is equal to $\{x_0, x_1, x_2, \ldots\}$. The equality relation on these objects is defined as follows.

$$\begin{aligned}
&\mathcal{M} \Vdash_{\{i,j\}} x_i = x_j. \\
&\mathcal{M} \Vdash_{\{i,j\}} x_k = x_k, \quad \text{for all } k. \\
&\mathcal{M} \Vdash_{\{i,j\}} x_k \neq x_l, \quad \text{if } \{k, l\} \neq \{i, j\} \text{ and } k \neq l.
\end{aligned}$$

So, if $i \neq j$, $\mathcal{M} \nVdash (x_i = x_j) \vee (x_i \neq x_j)$. This implies that the only N-finite sets of type 1 are $\varnothing$ and the singletons.

$$\mathcal{M} \Vdash \mathrm{NFin}^2 = \{\varnothing\} \cup \mathrm{USC}(V).$$

This clearly implies that $\mathcal{M} \nVdash \mathrm{AxInf}^2$.

But $\mathcal{M} \Vdash \mathrm{AxInf}^3$. Indeed, consider a set of the form $\{x_0, x_1, \ldots, x_k\}$. In each $\mathcal{M}_{\{i,j\}}$, this set has exactly k or $k+1$ distinct members. So, in $\mathcal{M}$, $\{x_0, \ldots, x_k\}$ is always distinct from $\{x_0, \ldots, x_{k+2}\}$. Thus, in $\mathcal{M}$, the following sets of type 2 are N-finite:

$$\begin{aligned}
&\{\varnothing\},\\
&\{\varnothing, \{x_0, x_1\}\},\\
&\{\varnothing, \{x_0, x_1\}, \{x_0, \ldots, x_3\}\},\\
&\{\varnothing, \{x_0, x_1\}, \{x_0, \ldots, x_3\}, \{x_0, \ldots, x_5\}\},\\
&\text{etc.}
\end{aligned}$$

But, *outside* $\mathcal{M}$ (i.e., in the model of set theory within which $\mathcal{M}$ has been defined), one can prove by induction that if $\mathcal{M} \Vdash (\varnothing \neq x \in \mathrm{NFin}^3)$, then $\mathcal{M} \Vdash (x \simeq \{\varnothing, \{x_0, x_1\}, \{x_0, \ldots, x_{2k+1}\}\})$, for some k. So $\mathcal{M} \Vdash (x \prec \{\varnothing, \{x_0, x_1\}, \{x_0, \ldots, x_{2(k+1)+1}\}\})$. In other words $\mathcal{M} \Vdash (\forall x \in \mathrm{NFin}^3)(\exists y \in \mathrm{NFin}^3)(y \succ x)$. $\square$

9 *Defining N-finite cardinals with three types*

9.1 The need for a new definition of equinumerosity Consider a set x^1 of type 1. Then $|x^1|$, the cardinal of x, is defined by: $|x^1| = \{y^1 : y^1 \simeq x^1\}$. So $|x^1|$ is a set of type 2.

Nevertheless, if $x^0 \in x^1$ and $y^0 \in y^1$, then $\langle x^0, y^0\rangle = \{\{x^0\}, \{x^0, y^0\}\}$ is a set of type 2. So a function $f : x^1 \to y^1$, being a set of ordered pairs, is a set of type 3. Thus $y^1 \simeq x^1$ denotes a formula where some variables of type 3 occur. This entails that $|x^1|$ is a type 2 object, whose definition requires type 3. In other words, the definition we gave of $|x^1|$ cannot be written in TT_3, the fragment of TT, whose langage is restricted to types 0, 1 and 2.

This is not a minor detail. Indeed, intuitionistic NF is finitely axiomatizable with 4-stratified sentences (see Dzierzgowski [3], an intuitionistic adaptation of Hailperin [7]). In other words, NF is identical with NF_4, where the comprehension axioms are 4-stratified. The consistency of intuitionistic NF is still an open problem. Nevertheless, being identical with NF_4, NF is *close* to NF_3, which is known to be consistent, even if a suitable axiom of infinity is added. So it would be nice to define cardinals of N-finite sets without using type 3, in order to be able to interpret arithmetic into a consistent subtheory of intuitionistic NF.

To achieve this, notice that if $x^1 \cap y^1 = \varnothing$, then a function $f : x^1 \to y^1$ can be coded by a set of *pairs* $\{x^0, y^0\}$, instead of a set of *ordered pairs* $\langle x^0, y^0\rangle$. In such a way, a function becomes a set of type 2.

For N-finite sets, the trick is to replace $x \simeq y$ with $x \simeq_\Delta y$, where $x \simeq_\Delta y$ iff there exists a set of pairs coding a 1–1 function mapping $x \setminus y$ onto $y \setminus x$ (it is clear that $(x \setminus y) \cap (y \setminus x) = \varnothing$).

Classically, if x and y are finite, then $x \simeq y$ is equivalent to $x \simeq_\Delta y$ (see Boffa [1], where this was used in order to interpret second order arithmetic in classical TT with only three types). Intuitionistically, we are going to prove that, if x and y are N-finite, then $x \simeq y$ is equivalent to $\neg\neg x \simeq_\Delta y$. The double negation cannot be removed. Indeed, by the technique presented in [4], we can devise the following Kripke model $\mathcal{M}$ of intuitionistic TT. $\mathcal{M} = \langle(\mathcal{M}_k)_{k\in K}, \langle K, \leq 0\rangle\rangle$, where $K = \{0, k, l\}$, with $0 \leq k, l$ and $k \not\leq l \not\leq k$, and such that the domains of type 0 objects contain four elements a, b, α, β satisfying

$$\mathcal{M} \Vdash_k a \neq b \wedge \alpha \neq \beta \wedge \alpha \neq a \wedge \alpha = b, \text{ and}$$
$$\mathcal{M} \Vdash_l a \neq b \wedge \alpha \neq \beta \wedge \alpha = a \wedge \alpha \neq b.$$

Let $\mathcal{M} \Vdash x = \{a, b\}$ and $\mathcal{M} \Vdash y = \{\alpha, \beta\}$. So $\mathcal{M} \Vdash_k x \setminus y = \{a\}$, $\mathcal{M} \Vdash_k x \setminus y = \{b\}$ and $\mathcal{M} \Vdash y \setminus x = \{\beta\}$. Thus $\mathcal{M} \Vdash \neg\neg x \simeq_\Delta y$, while $\mathcal{M} \nVdash x \simeq_\Delta y$.

9.2 Equivalence with the previous definition in intuitionistic TT

Proposition 9.1 *Suppose $x, y \in$ NFin are mutually detachable. Then $x \simeq y$ iff $x \simeq_\Delta y$.*

Proof: As x and y are mutually detachable, $x \cap y \in$ NFin by Proposition 4.6. Then, using (P3), we can see that $x = (x \setminus y) \cup (x \cap y)$ and $y = (y \setminus x) \cup (x \cap y)$. From this, it is easy to infer that $x \simeq_\Delta y$ implies $x \simeq y$.

The other direction is more tedious. We are going to prove it by induction on x. As it is trivial if $x = \varnothing$, suppose that $x = x_1 \cup \{z_1\}$, with $z_1 \notin x_1$. Let f be a 1–1 function mapping x onto y.

By (P9) and (P3) then, there are two cases.

Case 1: $z_1 \in x \setminus y$. If $y \subseteq x$, then, by Lemma 5.1, $y = x$, which is absurd because $z_1 \notin y$. So by (P10), there exists some $t \in y \setminus x$. Now it is easy to transform f so that $f(z_1) = t$ and the conclusion easily follows from the induction hypothesis.

Case 2: $z_1 \notin x \setminus y$, i.e., $z_1 \in x \cap y$. We can transform f so that $f(z_1) = z_1$, and then it is easy to conclude. □

Corollary 9.2 *If* $x, y \in \mathrm{NFin}$, *then* $x \simeq y$ *iff* $\neg\neg x \simeq_\Delta y$.

Proof: If $x, y \in \mathrm{NFin}$, then (P8) implies that $\neg\neg$ (x and y are mutually detachable). So, by Proposition 9.1, $\neg\neg\,(x \simeq y)$ iff $\neg\neg\,(x \simeq_\Delta y)$. We can then conclude by (P12). □

Notice that the above corollary is not true if x and y are not assumed to be in NFin. Indeed, using as usual the technique of [4], we find a model $\mathcal{M}$ of intuitionistic TT where x is some singleton, and y a nontrivial part of x such that $\mathcal{M} \nVdash (y = \varnothing \vee y = x)$ and $\mathcal{M} \Vdash \neg\neg x = y$. As $\mathcal{M} \Vdash \neg\neg x = y$ then, a fortiori, $\mathcal{M} \Vdash \neg\neg x \simeq_\Delta y$. But $\mathcal{M} \nVdash x \simeq y$ (otherwise, $x = y$). Furthermore, this counterexample proves that $x \in \mathrm{NFin}$ and $\neg\neg(y \simeq_\Delta x)$ does not imply that $y \in \mathrm{NFin}$. So (P11) does *not* generalize to $\neg\neg(\cdot \simeq_\Delta \cdot)$.

9.3 Correctness of the definition in intuitionistic TT_3 In the previous section, we have proved, in intuitionistic TT, that $\neg\neg(\cdot \simeq_\Delta \cdot)$ is the same relation on Nn as $(\cdot \simeq \cdot)$. But this does not prove, *in intuitionistic* TT_3, that $\neg\neg(\cdot \simeq_\Delta \cdot)$ is an equivalence relation on Nn. The proof in intuitionistic TT_3 will be given in Proposition 9.4, which needs the following lemma, whose proof is a very long and tedious induction on x, in the same style as the second part of the proof of Proposition 9.1, but with more than two cases.

Lemma 9.3 *Let* $x, y, z \in \mathrm{NFin}$. *Assume that* $x \cap y, x \cap z, y \cap z, x \cap y \cap z \in \mathrm{NFin}$. *Then, in intuitionistic* TT_3, $(x \simeq_\Delta y \ \wedge\ y \simeq_\Delta z) \rightarrow x \simeq_\Delta z$.

Proposition 9.4 *In intuitionistic* TT_3, $\neg\neg(\cdot \simeq_\Delta \cdot)$ *is an equivalence relation.*

Proof: The only nontrivial part consists in proving that $\neg\neg(\cdot \simeq_\Delta \cdot)$ is transitive. Consider $x, y, z \in \mathrm{NFin}$. By (P2), $\neg\neg\,(x \cap y, x \cap z, y \cap z, x \cap y \cap z \in \mathrm{NFin})$. Then Lemma 9.3 can be used to prove that $(\neg\neg x \simeq_\Delta y \ \wedge\ \neg\neg y \simeq_\Delta z) \rightarrow \neg\neg x \simeq_\Delta z$. □

9.4 Trichotomy If $x, y \in \mathrm{NFin}$, we define

$$\begin{array}{lll} x \simeq_3 y & \text{iff} & \neg\neg(x \simeq_\Delta y), \\ x \preccurlyeq_3 y & \text{iff} & (\exists y' \in \mathrm{NFin})(y' \subseteq y \wedge x \simeq_3 y'), \\ x \prec_3 y & \text{iff} & (\exists y' \in \mathrm{NFin})(y' \subsetneq y \wedge x \simeq_3 y'). \end{array}$$

With these definitions, we shall be able to reproduce Lemma 5.2 and Proposition 5.3.

Lemma 9.5 *Let $x, y \in$ NFin. Then, in intuitionistic* TT_3, $(x \preccurlyeq_3 y \ \wedge \ y \preccurlyeq_3 x) \ \rightarrow$ $x \simeq_3 y$. *So $x \prec_3 y$ iff $x \preccurlyeq_3 y$ and $x \not\simeq_3 y$.*

Proof: Let us prove that $(x \preccurlyeq_3 y \ \wedge \ y \preccurlyeq_3 x) \ \rightarrow \ x \simeq_3 y$. If $x \preccurlyeq_3 y$, then there exists $y' \subseteq y$ such that $x \simeq_3 y'$. Also, if $y \preccurlyeq_3 x$, then there exists $x' \subseteq x$ such that $y \simeq_3 x'$. So $\neg\neg(x \simeq_\Delta y' \wedge y \simeq_\Delta x')$. We want to prove that this implies $x \simeq_3 y$, i.e., $\neg\neg(x \simeq_\Delta y)$.

On the one hand, suppose that $y \simeq_\Delta x'$ and consider a 1–1 function f, coded as a set of pairs, mapping $y \setminus x'$ onto $x' \setminus y$. Let $x'' = f``(y' \setminus x') \cup (x' \cap y')$. Clearly, $x'' \setminus y' = f``(y' \setminus x')$, and $y' \setminus x'' = y' \setminus x'$. So $x'' \simeq_\Delta y'$, which implies $\neg\neg(x'' \simeq_\Delta y')$. On the other hand, suppose that $x \simeq_\Delta y'$. This implies $\neg\neg(x \simeq_\Delta y')$. As $\neg\neg(x'' \in$ NFin), by (P2), then $\neg\neg(x \simeq_\Delta x'')$, by Proposition 9.4. But as $x'' \setminus x = \varnothing$ because $x'' \subseteq x$, this implies that $\neg\neg(x \setminus x'' = \varnothing)$. So $\neg\neg(x = x'')$ by (P6). But $x'' \subseteq x' \subseteq x$. Thus $\neg\neg(x' = x)$. As we supposed that $y \simeq_\Delta x'$, we get $\neg\neg(x \simeq_\Delta y)$.

Let us summarize. We want to prove $x \simeq_3 y$, i.e., $\neg\neg(x \simeq_\Delta y)$. To that aim, we first proved that $\neg\neg(x \simeq_\Delta y' \wedge x' \simeq_\Delta y)$. Then, we proved that $((x' \simeq_\Delta y \wedge x \simeq_\Delta y') \rightarrow \neg\neg(x \simeq_\Delta y))$, which is equivalent to $(\neg\neg(x' \simeq_\Delta y \wedge x \simeq_\Delta y') \rightarrow \neg\neg(x \simeq_\Delta y))$. We then conclude by modus ponens.

Finally, by the definitions of $\prec_3$ and $\preccurlyeq_3$, it is trivial to prove that $x \prec_3 y \Leftrightarrow (x \preccurlyeq_3 y \wedge x \not\simeq_3 y)$. $\square$

The following lemma is used in the proof of Proposition 9.7. It will also play an important role in the definition of addition in Section 10.

Lemma 9.6 *Let $x, x', y, y' \in$ NFin such that $x \cap y = \varnothing$, $x \simeq_3 x'$, $y \simeq_3 y'$, $x' \cap y' = \varnothing$. Then $x \cup y \simeq_3 x' \cup y'$.*

Proof: First prove by induction on x that if $x, x', y, y' \in$ NFin are such that $x \cap y = \varnothing$, $x \simeq_\Delta x'$, $y \simeq_\Delta y'$, $x' \cap y' = \varnothing$ and $x \cup y$ and $x' \cup y'$ are mutually detachable, then $x \cup y \simeq_\Delta x' \cup y'$. By taking the double negation of this, we infer that if $x \cap y = \varnothing$, $x \simeq_3 x'$, $y \simeq_3 y'$, $x' \cap y' = \varnothing$ and $\neg\neg$ ($x \cup y$ and $x' \cup y'$ are mutually detachable), then $x \cup y \simeq_3 x' \cup y'$. We may then conclude by using (P8), because $x \cup y, x' \cup y' \in$ NFin by Proposition 4.6, which is provable in intuitionistic TT_3. $\square$

Proposition 9.7 *If $x, y \in$ NFin, then, in intuitionistic* TT_3, *$x \prec_3 y$ or $x \simeq_3 y$ or $x \succ_3 y$.*

Proof: The proof is by induction on x, using the previous lemma (in a simplified form: if $x, y \in$ NFin, $z \notin x$ and $z' \notin y$, then $x \simeq_3 y$ implies $x \cup \{z\} \simeq_3 y \cup \{z'\}$). $\square$

10 Defining natural numbers with three types Finite sets of type 0 objects are of type 1. So a natural number is of type 2, and we proved in the preceding section that it can be defined in intuitionistic TT_3, that is, without sets of type 3. Now Nn, the set of natural numbers does not exist in TT_3, because it should be of type 3.

Anyway, we shall use the notation Nn below (in the same way that **ON** is used in ZF). More precisely, a formula such as $n \in$ Nn should be considered as an abbreviation for $(\exists x \in \mathrm{NFin})(\forall y)(y \in n \leftrightarrow (y \in \mathrm{NFin} \wedge y \simeq_3 x))$.

10.1 Successor, addition and multiplication The main point of this section is to define addition and multiplication. As we have insufficient types to define functions from Nn to Nn, we cannot use definitions by induction (Proposition 6.3). So we shall use the alternative definitions of [1], which of course remain valid in intuitionistic TT.

If $s, m, n \in \mathrm{Nn}$, we define $s = m + n$ as an abbreviation for

$$(\exists x \in m)(\exists y \in n)(x \cap y = \varnothing \wedge x \cup y \in s).$$

By Lemma 9.6, the above definition makes sense.

Defining multiplication on Nn is less trivial. If $m, n, p \in \mathrm{Nn}$, we would like to define $p = m \cdot n$ as $(\exists x \in m)(\exists y \in n)(\exists z \in p)(z \simeq_3 x \times y)$. The problem is that the type of p would then be higher than the type of m and n. If it existed, a function mapping z onto $x \times y$ would be a set of pairs of the form $\langle c, \langle a, b\rangle\rangle$, with $a \in x$, $b \in y$ and $c \in z$. Nevertheless, if $x \cap y = \varnothing$, $x \cap z = \varnothing$ and $y \cap z = \varnothing$, then such a function can be coded as a set of triples of the form $\{c, a, b\}$, with $a \in x$, $b \in y$ and $c \in z$. Let us write $z \approx x \times y$ to denote the existence of such a coded bijection mapping z onto $x \times y$. Then if $m, n, p \in \mathrm{Nn}$, we define $p = m \cdot n$ as an abbreviation for

$$(\exists x \in m)(\exists y \in n)(\exists z \in p)(x \cap y = \varnothing \wedge x \cap z = \varnothing \wedge y \cap z = \varnothing \wedge z \approx x \times y).$$

The following lemma ensures that $p = m \cdot n$ is independent of the choice of x, y, z. To prove it, one should first prove that if $x, y, z \in \mathrm{NFin}$, $a \notin y$ and $z \approx x \times (y \cup \{a\})$, then there exist $z_1, z_2 \in \mathrm{NFin}$ such that $z_1 \approx x \times y$, $z_2 \approx x \times \{a\}$, $z_1 \cap z_2 = \varnothing$ and $z_1 \cup z_2 = z$.

Lemma 10.1 *Let $x, x', y, y', z, z' \in \mathrm{NFin}$ such that $x \cap y = \varnothing$, $x \cap z = \varnothing$, $y \cap z = \varnothing$, $z \approx x \times y$, $x \simeq_3 x'$, $y \simeq_3 y'$, $x' \cap y' = \varnothing$, $x' \cap z' = \varnothing$, $y' \cap z' = \varnothing$, $z' \approx x' \times y'$. Then $z \simeq_3 z'$.*

Now, in intuitionistic TT, the definitions given in this section can be proved to be equivalent to those given in Section 6. Also, the induction principle given by Proposition 6.2 remains valid in intuitionistic TT_3 if we consider $(m = S(n))$ as an abbreviation for $(m = n + 1)$, where $1 = \mathrm{USC}(V)$ (the sets of all singletons), and $S(n)\downarrow$ as an abbreviation for $(\exists m \in \mathrm{Nn})(m = S(n))$.

10.2 Infinity in intuitionistic TT_3 As in intuitionistic TT, Nn will be a model of HA if $(\forall n \in \mathrm{Nn})(S(n)\downarrow)$. This will be the case if we suppose the axiom of infinity. Precisely, in intuitionistic TT_3, we define the axiom of infinity to be the formula

$$(\mathrm{AxInf}_3) \qquad (\forall x \in \mathrm{NFin})(\exists y \in \mathrm{NFin})(y \succ_3 x)$$

In intuitionistic TT, (AxInf_3) is of course equivalent to (AxInf).

Proposition 10.3 is an adaptation of Proposition 7.1. In its proof, we shall need the following lemma.

Lemma 10.2 *Let $x, y, u \in \mathrm{NFin}$ such that $x \cap y = \varnothing$ and $x \cup y \simeq_3 u$. Then there exist $x', y' \in \mathrm{NFin}$ such that $x' \cap y' = \varnothing$, $x' \cup y' = u$, $x \simeq_3 x'$ and $y \simeq_3 y'$.*

Proof: The proof is by induction on x. It is of course trivial if $x = \varnothing$. So let $x = x_1 \cup \{z_1\}$, where $z_1 \notin x_1$. As $x \cup y \neq \varnothing$, then $u \neq \varnothing$, and we can find $u_1 \in \mathrm{NFin}$ and $t_1 \notin u_1$ such that $u = u_1 \cup \{t_1\}$. $x_1 \cup y$ and u_1 are in NFin. If $x_1 \cup y \prec_3 u_1$, then it is easy to prove that $x \cup y \prec_3 u$, which contradicts $x \cup y \simeq_3 u$. In the same way, $x_1 \cup y \not\prec_3 u_1$. So, by Proposition 9.7, $x_1 \cup y \simeq_3 u_1$. By the induction hypothesis, we can find x_1' and y' in NFin such that $x_1' \cap y' = \varnothing$, $x_1' \cup y' = u_1$, $x_1 \simeq_3 x_1'$ and $y \simeq_3 y'$. But then, we can conclude because $x \simeq_3 x_1' \cup \{t_1\}$, by Lemma 9.6. □

Proposition 10.3 *In intuitionistic* TT_3, *the following four formulas are equivalent to* (AxInf_3).

1. $(\forall n \in \mathrm{Nn})(\exists m \in \mathrm{Nn})(m > n)$.
2. $(\forall x, y \in \mathrm{NFin})(\exists x', y' \in \mathrm{NFin})(x \simeq_3 x' \ \wedge\ y \simeq_3 y' \ \wedge\ x' \cap y' = \varnothing)$.
3. $(\forall x \in \mathrm{NFin})(\exists x' \in \mathrm{NFin})(x \simeq_3 x' \wedge (\exists z)(z \notin x'))$.
4. $(\forall n \in \mathrm{Nn})(S(n)\downarrow)$.

Proof: It is easy to prove that (AxInf_3) is equivalent to (1), that (2) implies (3), that (3) implies (AxInf_3), and that (4) is equivalent to (3). Now we prove that (AxInf_3) implies (2), by induction on x. If $x = \varnothing$, then, clearly, we can take $x' = x$ and $y' = y$.

Suppose that $x = x_1 \cup \{z_1\}$, where $z_1 \notin x_1$. By the induction hypothesis, there exist x_1'' and y'' such that $x_1'' \simeq_3 x_1$, $y'' \simeq_3 y$ and $x_1'' \cap y'' = \varnothing$. As $x_1'' \cap y'' \in \mathrm{NFin}$, we know by Proposition 4.6 that $x_1'' \cup y'' \in \mathrm{NFin}$. Using (AxInf_3), we can find some $u \in \mathrm{NFin}$ such that $x_1'' \cup y'' \prec_3 u$. So there is some $u' \subsetneq u$ such that $x_1'' \cup y'' \simeq_3 u'$. By Lemma 10.2, we can find x_1' and y' such that $x_1' \cap y' = \varnothing$, $x_1' \cup y' = u'$, $x_1 \simeq_3 x_1'$ and $y' \simeq_3 y''$.

Finally, as $u' \neq u$, we can find some $z_1' \in u \setminus u'$ (by (P6)). We can now conclude by letting $x' = x_1' \cup \{z_1'\}$. □

10.3 Infinity and arithmetic in intuitionistic TT_3 If (AxInf_3) is assumed, then HA can be interpreted in intuitionistic TT_3. As in Section 7, we leave it to the reader to state the following theorem formally (recalling that Nn does not exist).

Theorem 10.4 *In intuitionistic* TT_3, (AxInf_3) *is equivalent to* "$\langle \mathrm{Nn}, S, +, \cdot, 0\rangle$ *satisfies* HA."

The proof of this theorem easily follows from the properties stated in this section.

We have now completed the presentation of definitions and basic properties related to finite sets and natural numbers in intuitionistic TT and TT_3. This is the starting point of a study of these notions. Interesting problems include, among others, finding a model of intuitionistic TT showing that the excluded middle cannot be derived for arithmetic formulas, and studying the axiom of infinity in the intuitionistic version of Quine's New Foundations and related systems.

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NOTES

1. This remark was inspired by Holmes.

2. This proposition arose from a discussion with T. Forster.

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