Notre Dame Journal of Formal Logic
Volume 37, Number 4, Fall 1996

Infinite Versions of Some Problems From Finite Complexity Theory

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Abstract Recently, several authors have explored the connections between NP-complete problems for finite objects and the complexity of their analogs for infinite objects. In this paper, we will categorize infinite versions of several problems arising from finite complexity theory in terms of their recursion theoretic complexity and proof theoretic strength. These infinite analogs can behave in a variety of unexpected ways.

1 Introduction Startling parallels exist between the computational complexity of certain graph theoretic problems and the recursion theoretic complexity and proof theoretic strength of their infinite analogs. For example, the problem of deciding which finite graphs have an Euler path is known to be P-time computable [7], and Beigel and Gasarch have shown in an unpublished work that the problem of deciding which infinite recursive graphs have an Euler path is arithmetical. By contrast, the problem of deciding which finite graphs have Hamilton paths is NP-complete [6], and Harel [4] has shown that the problem of deciding which infinite recursive graphs have a Hamilton graph is Σ^1_1 complete. Thus, the possibly greater computational complexity is paralleled by a demonstrable increase in recursion theoretic complexity. This pattern can also be seen through an application of the techniques of reverse mathematics. The existence of a function that decides which graphs have Euler paths is provably equivalent to $\mathbf{ACA}_0$, while the existence of a similar function for Hamilton paths is equivalent to the much stronger axiom system $\Pi^1_1{-}\mathbf{CA}_0$.

Unfortunately, other graph theoretic problems do not demonstrate this parallelism. We have selected some examples to illustrate two general themes. First, *different infinite statements related to a fixed finite problem can have different recursion theoretic complexities.* This would seem to indicate that the use of a preferred infinite formulation might lead to natural parallels between finite complexity and recursion theoretic complexity. However, the behavior of infinite analogs is not so easily tamed. Indeed, *similar formulations of infinite versions of problems with different finite complexities may have the same recursion theoretic complexity.*

Received September 11, 1995; revised March 22, 1997

2 Variability among graph coloring problems This section contains examples illustrating our first theme. The problem of determining which finite graphs are 3-chromatic is NP-complete [6]. Extrapolating from the problem of finding Hamilton paths, we would expect infinite analogs of the 3-coloring problem to be Σ^1_1 complete. However, the actual recursion theoretic complexity depends on the formulation of the infinite analog, as demonstrated by the following three theorems. Our notation is patterned after that of Soare [10].

The following theorem shows that the set of indices of 3-chromatic recursive graphs is arithmetical, and so is much simpler than the Σ^1_1 complete set we are seeking. This result is implicit in the work of Beigel and Gasarch [1]. By formulating their results in terms of partial recursive functions rather than index sets, Beigel and Gasarch isolate the recursion theoretic complexity contributed by questions of chromaticity from that contributed by the coding of the graphs. To maintain uniformity with later results, we have chosen to use index sets here.

Theorem 2.1 *The set of indices of 3-chromatic recursive graphs is Π^0_2 definable.*

Proof: Let $\mathcal{G}_1$ denote the set of indices of 3-chromatic recursive graphs. Note that $x \in \mathcal{G}_1$ if and only if every finite subgraph of the graph with index x is 3-chromatic. Thus, $\mathcal{G}_1$ is Π^0_1 definable, using the set of indices of all recursive graphs as a parameter. Since the set of indices of recursive graphs is Π^0_2 definable, $\mathcal{G}_1$ is also Π^0_2 definable. $\square$

In order to find an infinite analog of the 3-coloring problem with a complicated associated index set, we examine natural supersets of the 3-chromatic graphs. One candidate is the collection of finitely colorable graphs. The set of indices of the finitely colorable graphs is definable by the conjunction of a Π^0_2 and a Σ^0_2 formula, and so is Δ^0_3 definable (see [1]). By expanding our superset again to the collection of graphs with finitely colorable connected components, we gain some complexity in the index set.

Theorem 2.2 *The set of indices of recursive graphs with finitely colorable connected components is Π^0_3 complete.*

Proof: Let $\mathcal{G}_1$ denote the set of indices of recursive graphs with finitely colorable connected components. Suppose that x is the index of a graph G. Then $x \in \mathcal{G}_1$ if and only if for every vertex v of G, there is an integer k such that every finite connected subgraph of G containing v is k-chromatic. Thus, $\mathcal{G}_1$ is a Π^0_3 definable subset of the set of indices of recursive graphs.

To show that $\mathcal{G}_1$ is Π^0_3 complete, let $\mathcal{G}_0$ denote the set of indices of those recursive graphs which have connected components that are not finitely colorable. It suffices to show that $(Cof, \overline{Cof}) \le_1 (\mathcal{G}_0, \mathcal{G}_1)$. Here, $Cof = \{e : W_e \text{ is cofinite}\}$.

For each $e \in \omega$, define the graph G_e as follows. G_e will contain vertices labeled $v_{m,n}$ for each m and n in ω, and some additional unlabeled vertices. For each m, the vertex $v_{m,0}$ will be included in a complete graph on $m+1$ vertices. For every m and j all edges of the form $(v_{m,j}, v_{m,j+1})$ will be included in G_e. Finally, the edge $(v_{m,j}, v_{m+1,j})$ will be included in G_e if and only if $\{e\}(m)$ halts by stage j. For every e, G_e is recursive. By the *s-m-n* Theorem, there is a 1-1 recursive function f such that for every e, $f(e)$ is an index for G_e.

Note that if $e \in Cof$, then there is a j such that the vertices $\{v_{m,n} : m > j\}$ are all in the same connected component. Consequently, arbitrarily large complete finite subgraphs are contained in this component, and it is not finitely colorable. Thus, if $e \in Cof$, $f(e) \in \mathcal{G}_0$. Now suppose that $e \in \overline{Cof}$ and C is a connected component of G_e. C must contain a vertex of the form $v_{m,0}$. Since $e \in \overline{Cof}$, there is a least j greater than m such that $\{e\}(j)$ never halts. Consequently, C cannot contain any vertex $v_{n,k}$ such that $n > j$. This ensures that C is $j+1$-chromatic, so $f(e) \in \mathcal{G}_1$. Thus, f witnesses that $(Cof, \overline{Cof}) \leq_1 (\mathcal{G}_0, \mathcal{G}_1)$, as desired. □

For the next proof, we will need the following notation for finite sequences of natural numbers. Assuming a recursive bijection between ω and $\omega^{<\omega}$, we will use a Greek letter (usually σ or τ) to denote both a sequence and its integer code. The formula $\sigma \subseteq \tau$ means that σ is an initial (not necessarily proper) segment of τ. Thus, T is a *tree* if whenever $\tau \in T$ and $\sigma \subseteq \tau$, then $\sigma \in T$.

Given an arbitrary index e, $\{e\}$ may or may not be the characteristic function for a recursive tree. To streamline our discussion, consider the following auxiliary function.

Definition 2.3 For $e \in \omega$, the partial recursive function η_e is defined by:

$$\eta_e(\tau) = \begin{cases} 1 & \text{if} \quad \forall \sigma \subseteq \tau\, (\{e\}(\sigma) = 1), \\ 0 & \text{if} \quad \{e\}(\tau) = 0 \wedge [\forall \sigma \subseteq \tau\, (\{e\}(\sigma) = 0 \vee \{e\}(\sigma) = 1)] \\ & \qquad \wedge [\forall \sigma \subseteq \tau\, \forall \alpha \subseteq \sigma\, (\{e\}(\alpha) = 0 \to \{e\}(\sigma) = 0)], \\ \uparrow & \text{otherwise.} \end{cases}$$

Naïvely, η_e approximates the characteristic function of a tree. In particular, η_e is total if and only if e is the index of a recursive tree. Note that by the *s*-*m*-*n* Theorem, there is a 1-1 recursive function which maps each e to an index for η_e.

So far, we have examined sets of graphs that can be colored with a set of colors that is “small” in some sense. In finite graphs, coloring with a small number of colors forces repeated use of some color. Thus, it seems reasonable to consider graphs with colorings that use one color infinitely often. Note that this set of graphs is a superset of the graphs with finitely colorable connected components.

Theorem 2.4 ([5]) *The set of indices of recursive graphs with colorings which use one color infinitely often is Σ^1_1 complete.*

Proof: Let $\mathcal{G}$ denote the set of indices of recursive graphs with colorings that use one color infinitely often. Note that $x \in \mathcal{G}$ if and only if there is a function χ mapping the vertices of the graph with index x into ω, such that χ maps neighboring vertices to different values, and 0 appears infinitely often in the range of χ. This statement can be formalized using a single existential set quantifier followed by an arithmetical formula, so $\mathcal{G}$ is Σ^1_1 definable.

To show that $\mathcal{G}$ is Σ^1_1 complete, we will show that $\mathcal{T} \leq_1 \mathcal{G}$, where $\mathcal{T}$ denotes the set of indices of recursive trees which are not well founded. With each $e \in \omega$, we associate a partial recursive graph, G_e. The vertex set for G_e consists of (codes for) elements of $\omega^{<\omega}$. For every $\sigma, \tau \in \omega^{<\omega}$, the characteristic function for the edge set of G_e is defined by

$$E_e(\sigma,\tau) = \begin{cases} 0 & \text{if} \quad \eta_e(\sigma) = 1 \wedge \eta_e(\tau) = 1 \wedge (\sigma \subseteq \tau \vee \tau \subseteq \sigma), \\ 1 & \text{if} \quad \eta_e(\sigma) = 1 \wedge \eta_e(\tau) = 1 \wedge \neg(\sigma \subseteq \tau \vee \tau \subseteq \sigma), \\ 1 & \text{if} \quad \eta_e(\sigma) \downarrow \wedge \eta_e(\tau) \downarrow \wedge (\eta_e(\sigma) = 0 \vee \eta_e(\tau) = 0), \\ \uparrow & \text{otherwise.} \end{cases}$$

Roughly, we connect σ and τ by an edge if they are incomparable nodes on the tree or if one of them is not in the tree, ignoring those nodes whose status is suspect. By the *s*-*m*-*n* Theorem, there is a 1-1 recursive function f such that for every e, $f(e)$ is an index for G_e.

If $e \in \mathcal{T}$, then e is the index of a recursive tree containing an infinite path P. Consequently, $f(e)$ is the index of a recursive graph. We can color this graph by mapping every node of P to 0, and mapping all other nodes to their integer codes. Since 0 is used infinitely often in this coloring, $f(e) \in \mathcal{G}$.

Now suppose $e \notin \mathcal{T}$. If e is not the index of a recursive tree, then $f(e)$ is not the index of a recursive graph, so $f(e) \notin \mathcal{G}$. If we suppose that e is the index of a recursive tree T, then T is well founded. Suppose, by way of contradiction, that there is a coloring of the associated recursive graph G_e that uses 0 infinitely often. All the nodes of G_e that are colored 0 correspond to comparable nodes of T, contradicting the claim that T is well founded. Again, we have $f(e) \notin \mathcal{G}$, completing the proof that $\mathcal{T} \leq_1 \mathcal{G}$. □

The techniques of reverse mathematics can be used to draw a distinction between the first two of our infinite analogs and the third. The following two results make use of the axiom systems $\mathbf{RCA}_0$(**R**ecursive **C**omprehension **A**xiom), $\mathbf{ACA}_0$ (**A**rithmetical **C**omprehension **A**xiom), and Π^1_1–$\mathbf{CA}_0$ (Π^1_1 **C**omprehension **A**xiom). For a brief overview of reverse mathematics, see Simpson [9].

Theorem 2.5 ($\mathbf{RCA}_0$) *The following are equivalent:*

1. $\boldsymbol{ACA}_0$.
2. *For any sequence of graphs* $\langle G_i : i \in \omega\rangle$, *there is a function* $s : \omega \to 2$ *such that* $s(i) = 1$ *if and only if* G_i *is 3-chromatic.*
3. *For any sequence of graphs* $\langle G_i : i \in \omega\rangle$, *there is a function* $s : \omega \to 2$ *such that* $s(i) = 1$ *if and only if every connected component of* G_i *is finitely colorable.*

Proof: To prove (1) → (2) and (1) → (3), it suffices to show that the function s is arithmetically definable in $\langle G_i : i \in \omega\rangle$. For (2), a Π^0_1 defining formula for s can be extracted from the proof of Theorem2.1. Similarly, for (3), imitating the proof of Theorem 2.2 yields a Π^0_3 defining formula.

By Lemma 2.7 of [8], to prove that (2) → (1) and (3) → (1), it suffices to show that $\mathbf{RCA}_0$ can prove that for any injection $g : \omega \to \omega$, there is a sequence of graphs $\langle G_i : i \in \omega\rangle$ such that the range of g is Δ^0_1 definable in the associated function s. Fix g and assume $\mathbf{RCA}_0$. We will define a sequence of graphs that works for both (2) and (3). Let G_n have ω as its vertex set. For every $j \in \omega$, include the edge $(j, j+1)$ in G_n. For $j < k$, add the edge (j,k) to G_n if and only if $\exists t \leq j\,(g(t) = n)$. The sequence $\langle G_i : i \in \omega\rangle$ is Δ^0_1 definable in g, so $\mathbf{RCA}_0$ proves it exists. Let s be as in (2) or (3). Then $s(n) = 1$ if and only if n is not in the range of g. Thus, the range of g is Δ^0_1 definable in s, as desired. □

The proceeding proof still holds if 3-chromatic is replaced by 2-chromatic in the statement of (2). Thus, these infinite analogs of the 2-coloring and 3-coloring problems are provably equivalent.

Theorem 2.6 (**RCA**$_0$) *The following are equivalent:*

1. Π^1_1–$\mathbf{CA}_0$.
2. *For any sequence of graphs* $\langle G_i : i \in \omega\rangle$*, there is a function* $s : \omega \to 2$ *such that* $s(i) = 1$ *if and only if* G_i *has a coloring in which one color is used infinitely often.*

Proof: To prove that (1) → (2), it suffices to note that the function s is Σ^1_1 definable in $\langle G_i : i \in \omega\rangle$, and so exists by Π^1_1–$\mathbf{CA}_0$. To prove the converse, we will use the fact that Π^1_1–$\mathbf{CA}_0$ is equivalent to the existence of a function that decides which members of a sequence of trees are well founded. (This is an easy consequence of Lemma 6.1 in [3].) Assume **RCA**$_0$, and suppose that $\langle T_i : i \in \omega\rangle$ is a sequence of trees. With each tree T_n, we associate a graph G_n as follows. The vertices of G_n are the nodes of T_n, and two vertices of G_n are connected if and only if the associated nodes are incomparable in the tree ordering. The sequence $\langle G_i : i \in \omega\rangle$ is Δ^0_1 definable in $\langle T_i : i \in \omega\rangle$, and so exists by **RCA**$_0$. Let s be as in (2). Then $s(i) = 1$ if and only if G_i contains an infinite collection of pairwise disconnected vertices, which occurs if and only if T_i is not well founded. Thus (2) implies Π^1_1–$\mathbf{CA}_0$, completing the proof. □

3 Variability among graph isomorphism problems From the results in the preceding section, it is clear that the recursion theoretic strength of infinite analogs depends in part on their formulation. As shown by Harel and Tirza Hirst in [5], adoption of a standardized translation yields interesting parallels between finite complexity and recursion theoretic complexity for restricted classes of problems. However, for broader classes of problems, the parallels break down. In this section, we will consider three problems of diverse finite complexity that all have Σ^1_1 complete infinite analogs, thus illustrating our second theme. Consider the following three variants of the subgraph isomorphism problem:

- **P1** *Given a pair of finite graphs, H and G, determine if H is isomorphic to a subgraph of G.*
- **P2** *For a fixed finite graph H, given a finite graph G, determine if H is isomorphic to a subgraph of G.*
- **P3** *For a fixed finite graph G, given a finite graph H, determine if H is isomorphic to a subgraph of G.*

P1 is the familiar form of the subgraph isomorphism problem, and is known to be NP-complete [2]. One algorithm for solving **P2** and **P3** consists of enumerating all functions from H into G, and checking each one to see if it is the desired isomorphism. The number of functions to check is bounded by $\mid G \mid^{\mid H \mid}$, where $\mid G \mid$ denotes the number of vertices of G. Since H is fixed in **P2**, the number of functions to check is a constant power of $\mid G \mid$. Furthermore, the number of steps required to check each function is bounded by a constant based on the fixed value $\mid H \mid$. Thus, **P2** can be solved in a number of steps which is bounded by a polynomial in $\mid G \mid$. In **P3**, G is fixed, and we can discard any graphs H such that $\mid H \mid > \mid G \mid$, so the number of steps required to solve an instance of **P3** is bounded by a constant based on the fixed value $|G|$. Summarizing, the complexity of three problems ranges from NP-complete to constant time computable.

Compared to the coloring problem in §2, these subgraph isomorphism problems have very straightforward infinite analogs. Despite the variation in the computational complexity of the finite problems, their infinite analogs are all Σ^1_1 complete, as is shown in the following three theorems.

Theorem 3.1 ([5]) *The set of indices of ordered pairs of recursive graphs, (H, G), such that H is isomorphic to a subgraph of G is Σ^1_1 complete.*

Proof: Let $\mathcal{G}$ be the set of indices of ordered pairs of recursive graphs such that the first graph is isomorphic to a subgraph of the second. Since $x \in \mathcal{G}$ if and only if an appropriate isomorphism exists, it is easy to see that $\mathcal{G}$ is Σ^1_1 definable.

To prove that $\mathcal{G}$ is Σ^1_1 complete, we will show that $\mathcal{T} \leq_1 \mathcal{G}$, where $\mathcal{T}$ denotes the set of indices of recursive trees which are not well founded. With each $e \in \omega$, we associate a pair of partial recursive graphs, H_e and G_e. H_e is a countably infinite linear graph with a triangle attached at one end. To be precise, the vertex set of H_e is $\{v_n : n \in \omega\}$ and the edge set is $\{(v_0, v_2)\} \cup \{(v_n, v_{n+1}) : n \in \omega\}$. If e is the index of a recursive tree T, then G_e consists of a copy of T with a triangle attached to the root, and a collection of disconnected vertices. In general, the vertex set for G_e consists of $\{v_0, v_1, v_2\}$ and (codes for) the elements of $\omega^{<\omega}$. Let σ_0 denote the code for the empty sequence. The edge (v_0, σ_0) and the three edges of the form (v_i, v_j) where $i \neq j$ are included in G_e. For every σ and τ in $\omega^{<\omega}$, the edge (σ, τ) is included in G_e if and only if

$$\eta_e(\sigma) = \eta_e(\tau) = 1 \wedge \sigma \subseteq \tau \wedge \neg\exists\alpha(\sigma \subsetneq \alpha \subsetneq \tau),$$

where η_e is the function defined in §2. By the *s-m-n* Theorem, there is a recursive 1-1 function f such that for every e, $f(e)$ is an index for the pair (H_e, G_e).

If $e \in \mathcal{T}$, then e is the index of a recursive tree containing an infinite path P. In this case, H_e is isomorphic to the subgraph of G_e consisting of the base triangle and a copy of P. Thus $f(e) \in \mathcal{G}$.

Now suppose that $e \notin \mathcal{T}$. If e is not the index of a recursive tree, then G_e is not a recursive graph, so $f_e \notin \mathcal{G}$. If e is the index of a recursive tree $\mathcal{T}$, then $\mathcal{T}$ is well founded. The graph G_e is a copy of T with a triangle attached to its base. Any isomorphism mapping H_e into G_e must map the triangle in H_e into the triangle in G_e, and the linear portion of H_e to an infinite path in the copy of T. Since T is well founded, no such isomorphism exists. Thus $f(e) \notin \mathcal{G}$, completing the proof that $\mathcal{T} \leq_1 \mathcal{G}$. □

Theorem 3.2 *There is a recursive graph H, such that the set of indices of recursive graphs containing a subgraph isomorphic to H is Σ^1_1 complete.*

Proof: In the proof of Theorem 3.1, H_e is a fixed recursive graph defined without reference to e. Any recursive 1-1 function mapping e to an index for the graph G_e (defined as in the proof of Theorem 3.1) witnesses the desired 1-reduction. □

Theorem 3.3 *There is a recursive graph G, such that the set of indices of recursive graphs that are isomorphic to a subgraph of G is Σ^1_1 complete.*

Proof: We begin the proof by constructing the recursive graph G. This graph will consist of a countable collection of subgraphs $\langle G_e : e \in \omega \rangle$, where each G_e consists of a treelike substructure together with some spurious disconnected subgraphs.

For each $e \in \omega$, G_e will be constructed from cycles labeled $C(e, \sigma, k)$ for each nonempty $\sigma \in \omega^{<\omega}$ and each $k \in \omega$. The cycle $C(e, \sigma, k)$ consists of $2(e+1)+2$ vertices joined to make a circular graph. We designate two vertices of $C(e, \sigma, k)$ as $v^0_{e,\sigma,k}$ and $v^1_{e,\sigma,k}$, and require that the paths joining them contain $e+2$ edges. To give a concrete example, $C(1, \sigma, k)$ looks like a hexagon, with the bottom vertex labeled $v^0_{1,\sigma,k}$ and the top vertex labeled $v^1_{1,\sigma,k}$.

The treelike substructure of G_e consists of a triangular base with a vertex labeled t_0, and branches consisting of linked cycles. We say that a cycle $C(e, \sigma, k)$ is *exact* if k is the least integer such that (1) $\eta_e(\tau)\downarrow$ by stage k for every τ which is an initial subsequence of σ or has a code less than σ, and (2) $\eta_e(\sigma) = 1$. (Here η_e is the function defined in §2.) Edges are added to G_e by the following two rules. Connect $v^0_{e,\sigma,k}$ to t_0 if and only if $C(e, \sigma, k)$ is an exact cycle and σ is a sequence of length 1. Connect $v^1_{e,\sigma,k}$ to $v^0_{e,\tau,j}$ if and only if $C(e, \sigma, k)$ and $C(e, \tau, j)$ are exact cycles and $\tau = \sigma * \langle m \rangle$ for some $m \in \omega$. Cycles which are not exact are *spurious*; they are included in G_e, but are never connected to the treelike substructure.

Let G be the union of all the G_e's. G is recursive, since the rules for adding edges involve only bounded computations. Furthermore, if e is the code of a recursive tree T, then the treelike substructure of G_e can be mapped homomorphically onto T by identifying exact cycles with corresponding nodes. Viewing the cycles as nodes, the substructure is well founded if and only if T is a well-founded tree. If e is not the code of a recursive tree, η_e is not total, and the treelike substructure of G_e is finite.

Let $\mathcal{G}$ be the set of indices of recursive graphs that are isomorphic to a subgraph of G. Since $x \in \mathcal{G}$ if and only if an isomorphism exists, it is easy to see that $\mathcal{G}$ is Σ^1_1 definable. To prove that $\mathcal{G}$ is Σ^1_1 complete, we will show that $\mathcal{T} \leq_1 \mathcal{G}$, where $\mathcal{T}$ denotes the set of indices of recursive trees which are not well founded. With each $e \in \omega$, we associate a recursive graph H_e consisting of a countable linear graph with each node replaced by a $2(e+1)+2$ cycle and with a triangle attached at one end. More precisely, H_e contains a triangle with one vertex labeled t_0, and (copies of) the cycles $C(e, \langle 0 \rangle, k)$ for each $k \in \omega$. To the edges already specified, we add the edge $(t_0, v^0_{e,\langle 0 \rangle,k})$ and the edges $(v^1_{e,\langle 0 \rangle,k}, v^0_{e,\langle 0 \rangle,k+1})$ for each $k \in \omega$. By the *s-m-n* Theorem, there is a recursive 1-1 function f such that for every e, $f(e)$ is an index for H_e.

If $e \in \mathcal{T}$, then e is the index of a recursive tree containing an infinite path P. In this case, H_e is isomorphic to the subgraph of G_e consisting of the base triangle and a copy of P with nodes replaced by cycles. Thus $f(e) \in \mathcal{G}$.

Now suppose that $e \notin \mathcal{T}$. Note that because the size of the cycles varies with e, if H_e is isomorphic to a subgraph of G, then H_e is isomorphic to a subgraph of G_e. Since $e \notin \mathcal{T}$, G_e consists of disconnected cycles and a well-founded treelike substructure. If H_e is isomorphic to a subgraph of G_e, then the treelike substructure of G_e contains an infinite path, yielding a contradiction. Thus $f(e) \notin \mathcal{G}$ completing the proof that $\mathcal{T} \leq_1 \mathcal{G}$. □

Using the reverse mathematics framework, the preceding three theorems can be lumped together into a single equivalence result.

Theorem 3.4 ($\mathbf{RCA}_0$) *The following are equivalent:*

1. $\Pi^1_1\text{–}\mathbf{CA}_0$.
2. *For any sequence of ordered pairs of graphs,* $\langle (H_i, G_i) : i \in \omega \rangle$*, there is a function* $s : \omega \to 2$ *such that* $s(i) = 1$ *if and only if* H_i *is isomorphic to a subgraph of* G_i*.*
3. *For any graph* H*, and any sequence of graphs* $\langle G_i : i \in \omega \rangle$*, there is a function* $s : \omega \to 2$ *such that* $s(i) = 1$ *if and only if* H *is isomorphic to a subgraph of* G_i*.*
4. *For any graph* G*, and any sequence of graphs* $\langle H_i : i \in \omega \rangle$*, there is a function* $s : \omega \to 2$ *such that* $s(i) = 1$ *if and only if* H_i *is isomorphic to a subgraph of* G*.*

Proof: To prove that (1) implies (2), (3), or (4), it suffices to note that the function s is Σ^1_1 definable in the appropriate sequence of graphs. Since (3) is a special case of (2), we need only show that (3) $\to$ (1) and (4) $\to$ (1) to complete the proof. As in the proof of Theorem 2.6, we will determine which members of a sequence of trees are well founded. For the remainder of the proof, assume $\mathbf{RCA}_0$ and let $\langle T_i : i \in \omega \rangle$ be a sequence of trees.

To prove that (3) $\to$ (1), we use a simplified version of the construction in the proof of Theorem 3.2. As in that proof, let H be a countable linear graph with a triangle attached to one end. For each $n \in \omega$, let G_n be a copy of T_n, with a triangle attached to the root. The graph H and the sequence $\langle G_i : i \in \omega \rangle$ are Δ^0_1 definable in $\langle T_i : i \in \omega \rangle$, so $\mathbf{RCA}_0$ proves that they exist. Let s be as in (3). Then $s(i) = 1$ if and only if H is isomorphic to a subgraph of G_i, which occurs if and only if T_i has an infinite path. Thus (3) implies $\Pi^1_1\text{–}\mathbf{CA}_0$.

To prove that (4) $\to$ (1), we use a simplified version of the proof of Theorem 3.3. As in that proof, let H_n consist of a linear graph with each node replaced by a $2(n+1)+2$ cycle, and with a triangle attached to one end. The graph G consists of subgraphs G_n for each $n \in \omega$, where each G_n is a copy of T_n with nonbase nodes replaced by $2(n+1)+2$ cycles, and a triangle attached to the base node. The graph G and the sequence $\langle H_i : i \in \omega \rangle$ are Δ^0_1 definable in $\langle T_i : i \in \omega \rangle$, so $\mathbf{RCA}_0$ proves that they exist. If s is as in (4), then $s(i) = 1$ if and only if H_i is isomorphic to a subgraph of G, which occurs if and only if H_i is isomorphic to a subgraph of G_i. Finally, H_i is isomorphic to a subgraph of G_i if and only if T_i is not well founded, so (4) implies $\Pi^1_1\text{–}\mathbf{CA}_0$, completing the proof. □

Although infinite analogs are useful for studying restricted classes of problems, the preceding examples indicate that, in a general setting, their behavior does not necessarily parallel that of the associated finite problems. However, examination of results in finite complexity can provide motivation for appealing results in recursion theory and reverse mathematics.

Acknowledgments Lempp's research was partially supported by NSF grant DMS-9100114.

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