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A Conjecture on Numeral Systems

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Abstract A numeral system is an infinite sequence of different closed normal λ-terms intended to code the integers in λ-calculus. Barendregt has shown that if we can represent, for a numeral system, the functions Successor, Predecessor, and Zero Test, then all total recursive functions can be represented. In this paper we prove the independancy of these three particular functions. We give at the end a conjecture on the number of unary functions necessary to represent all total recursive functions.

1 Introduction A numeral system is an infinite sequence of different closed $\beta\eta$-normal λ-terms $\mathbf{d} = d_0, d_1, \dots, d_n, \dots$ intended to code the integers in λ-calculus. For each numeral system $\mathbf{d}$, we can represent total numeric functions as follows. A total numeric function $\varphi : \mathbb{N}^p \longrightarrow \mathbb{N}$ is λ-definable with respect to $\mathbf{d}$ if and only if

$$\exists F_\varphi \forall n_1, \dots, n_p \in \mathbb{N}(F_\varphi\, d_{n_1}, \dots, d_{n_p}) \simeq_\beta d_{\varphi(n_1,\dots,n_p)}$$

One of the differences between our numeral system definition and the Barendregt's definition given in [1] is the fact that the λ-terms d_i are normal and different. The last conditions allow with some fixed reduction strategies (for example, the left reduction strategy) to find the exact value of a function computed on arguments. Barendregt has shown that if we can represent, for a numeral system, the functions successor, predecessor, and zero test, then all total recursive functions can be represented. We prove in this paper that these three particular functions are independent. We think it is, at least, necessary to have three unary functions to represent all total recursive functions.

This paper is organized in the following way. Section 2 is devoted to preliminaries. In Section 3, we define the numeral systems and we present the result of Barendregt. In Section 4, we prove the independency of the functions successor, predecessor, and zero test. We give at the end a conjecture on the number of unary functions necessary to represent all total recursive functions.

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2 Notations and definitions The notations are standard (see [1] and [2]).

1. We denote by I (for identity) the λ-term λxx, T (for true) the λ-term $\lambda x\lambda yx$, and by F (for false) the λ-term $\lambda x\lambda yy$.
2. The pair $\langle M, N\rangle$ denotes the λ-term $\lambda x(x\, M\, N)$.
3. The *β-equivalence* relation is denoted by $M \simeq_\beta N$.
4. The notation $\sigma(M)$ represents the result of the simultaneous substitution σ to the free variables of M after a suitable renaming of the bound variables of M.
5. A *$\beta\eta$-normal* λ-term is a λ-term which does not contain either a β-redex (i.e., a λ-term of the form $(\lambda x\, M\, N)$) or an η-redex (i.e., a λ-term of the form $\lambda x(M\, x)$ where x does not appear in M).

The following result is the well-known Böhm Theorem.

Theorem 2.1 *If U, V are two distinct closed $\beta\eta$-normal λ-terms then there is a closed λ-term W such that $(W\, U) \simeq_\beta T$ and $(W\, V) \simeq_\beta F$.*

1. A λ-term M either has a *head redex* (i.e., $M = \lambda x_1, \ldots, \lambda x_n((\lambda xU\, V)\, V_1, \ldots, V_m)$, the head redex being $(\lambda xU\, V)$), or is in *head normal form* (i.e., $M = \lambda x_1, \ldots, \lambda x_n\ (x\, V_1, \ldots, V_m)$).
2. The notation $U \succ V$ means that V is obtained from U by some head reductions and we denote by $h(U, V)$ the length of the head reduction between U and V.
3. A λ-term is said to be *solvable* if and only if its head reduction terminates.

The following results are well known.

1. If M is β-equivalent to a head normal form then M is solvable.
2. If $U \succ V$, then for any substitution σ, $\sigma(U) \succ \sigma(V)$, and $h(\sigma(U), \sigma(V)) = h(U, V)$.

In particular, if for some substitution σ, $\sigma(M)$ is solvable, then M is solvable.

3 Numeral systems

1. A *numeral system* is an infinite sequence of different closed $\beta\eta$-normal λ-terms $\mathbf{d} = d_0, d_1, \ldots, d_n, \ldots$.
2. Let $\mathbf{d}$ be a numeral system.
   (a) A closed λ-term S_d is called *successor* for $\mathbf{d}$ if and only if
   $$(S_d\, d_n) \simeq_\beta d_{n+1} \text{ for all } n \in \mathbb{N}.$$
   (b) A closed λ-term P_d is called *predecessor* for $\mathbf{d}$ if and only if
   $$(P_d\, d_{n+1}) \simeq_\beta d_n \text{ for all } n \in \mathbb{N}.$$
   (c) A closed λ-term Z_d is called *zero test* for $\mathbf{d}$ if and only if
   $$(Z_d\, d_0) \simeq_\beta T \quad \text{and} \quad (Z_d\, d_{n+1}) \simeq_\beta F \quad \text{for all } n \in \mathbb{N}.$$
3. A numeral system is called *adequate* if and only if it possesses closed λ-terms for successor, predecessor, and zero test.

Example 3.1 (The Barendregt numeral system) For each $n \in \mathbb{N}$, we define the Barendregt integer $\overline{n}$ by : $\overline{0} = I$ and $\overline{n+1} = \langle F, \overline{n}\rangle$. It is easy to check that

$$\begin{array}{rcl}\overline{S} & = & \lambda x\langle F, x\rangle,\\ \overline{P} & = & \lambda x(x\,F),\\ \overline{Z} & = & \lambda x(x\,T).\end{array}$$

are respectively λ-terms for successor, predecessor, and zero test.

Example 3.2 (The Church numeral system) For each $n \in \mathbb{N}$, we define the Church integer $\underline{n} = \lambda f\lambda x(f(f, \ldots, (f\,x), \ldots))$ (f occurs n times). It is easy to check that

$$\begin{array}{rcl}\underline{S} & = & \lambda n\lambda f\lambda x(f\,(n\,f\,x)),\\ \underline{P} & = & \lambda n(n\,U\,\langle\underline{0}, \underline{0}\rangle\,T) \text{ where } U = \lambda a\langle(\underline{s}\,(a\,T)), (a\,F)\rangle,\\ \underline{Z} & = & \lambda n(n\,\lambda x F\,T).\end{array}$$

are respectively λ-terms for successor, predecessor, and zero test.

Each numeral system can be naturally considered as a coding of integers into λ-calculus and then we can represent total numeric functions as follows.

> A total numeric function $\varphi : \mathbb{N}^p \longrightarrow \mathbb{N}$ is λ-*definable* with respect to a numeral system **d** if and only if
>
> $$\exists\, F_\varphi\, \forall\, n_1, \ldots, n_p \in \mathbb{N}\ (F_\varphi\, d_{n_1}, \ldots, d_{n_p}) \simeq_\beta d_{\varphi(n_1,\ldots,n_p)}.$$

The zero test can be considered as a function on integers.

Lemma 3.3 *A numeral system* **d** *has a* λ*-term for zero test if and only if the function* φ *defined by :* $\varphi(0) = 0$ *and* $\varphi(n) = 1$ *for every* $n \geq 1$ *is* λ*-definable with respect to* **d**.

Proof: It suffices to see that d_0 and d_1 are distinct $\beta\eta$-normal λ-terms. □

Barendregt has shown in [1] that

Theorem 3.4 *A numeral system* **d** *is adequate if and only if all total recursive functions are* λ*-definable with respect to* **d**.

4 Some results on numeral systems

Theorem 4.1 *There is a numeral system with successor and predecessor but without zero test.*[1]

Proof: For every $n \in \mathbb{N}$, let $a_n = \lambda x_1, \ldots, \lambda x_n I$. It is easy to check that the λ-terms $S_a = \lambda n\lambda x n$ and $P_a = \lambda n(n\,I)$ are λ-terms for successor and predecessor for **a**. Let ν, x, y be different variables. If **a** possesses a closed λ-term Z_a for zero test, then

$$(Z_a\, a_n\, x\, y) \simeq_\beta \begin{cases} x & \text{if } n = 0\\ y & \text{if } n \geq 1\end{cases}$$

and

$$(Z_a\, a_n\, x\, y) \succ \begin{cases} x & \text{if } n = 0\\ y & \text{if } n \geq 1\end{cases}$$

Therefore $(Z_a\, \nu\, x\, y)$ is solvable and its head normal form does not begin with λ. We must look at three cases.

1. $(Z_a\, \nu\, x\, y) \succ (x\, u_1, \ldots, u_k)$, then $(Z_a\, a_1\, x\, y) \not\succ y$.
2. $(Z_a\, \nu\, x\, y) \succ (y\, u_1, \ldots, u_k)$, then $(Z_a\, a_0\, x\, y) \not\succ x$.
3. $(Z_a\, \nu\, x\, y) \succ (\nu\, u_1, \ldots, u_k)$, then $(Z_a\, a_{k+2}\, x\, y) \not\succ y$.

Each case is impossible. □

Theorem 4.2 *There is a numeral system with successor and zero test but without predecessor.*

Proof: Let $b_0 = \langle T, I\rangle$ and for every $n \geq 1$, $b_n = \langle F, a_{n-1}\rangle$. It is easy to check that the λ-terms $S_b = \lambda n\langle F, ((n\,T)\,a_0\,\lambda x(n\,F))\rangle$ and $Z_b = \lambda n(n\,T)$ are λ-terms for successor and zero test for **b**. If **b** possesses a closed λ-term P_b for predecessor, then the λ-term $P'_b = \lambda n(P_b\,\langle F, n\rangle\, T)$ is a λ-term for zero test for **a**. This is a contradiction. □

4.1 Remarks

Remark 4.3 Let $b'_0 = b_1$, $b'_1 = b_0$, and for every $n \geq 2$, $b'_n = b_n$. It is easy to check that the numeral system $\mathbf{b}'$ does not have λ-terms for successor, predecessor, and zero test.

Remark 4.4 The proofs of Theorems 4.1 and 4.2 rest on the fact that we are considering sequences of λ-terms with a strictly increasing order (number of abstractions). Considering sequences of λ-terms with a strictly increasing degree (number of arguments) does not work as well.

See the following example. We define $\tilde{0} = I$ and for each $n \geq 1$, $\tilde{n} = \lambda x(x\,x, \ldots, x)$ (x occurs $n+1$ times). Let

$$\begin{aligned}
\tilde{S} &= \lambda n\lambda x(n\,x\,x),\\
\tilde{Z} &= \lambda n(n\,A\,I\,I\,T) \ \text{ where } \ A = \lambda x\lambda y(y\,x),\\
\tilde{P} &= \lambda n\lambda x(n\,U\,F) \ \text{ where } \ U = \lambda y(y\,V\,I) \ \text{ and } \ V = \lambda a\lambda b\lambda c\lambda d(d\,a\,(c\,x))
\end{aligned}$$

It is easy to check that $\tilde{S}$, $\tilde{Z}$, and $\tilde{P}$ are, respectively, λ-terms for successor, zero test, and predecessor.

4.2 Definitions

1. We denote by Λ^0 the set of closed λ-terms and by Λ^1 the set of the infinite sequences of closed normal λ-terms. It is easy to see that Λ^0 is countable but Λ^1 is not countable.
2. For every finite sequence of λ-terms $U_1, U_2, \ldots, U_n$ we denote by $\langle U_1, U_2, \ldots, U_n\rangle$ the λ-term $\langle \ldots, \langle\langle I, U_1\rangle, U_2\rangle, \ldots, U_n\rangle$.
3. Let $\mathbf{U} = U_1, U_2, \ldots$ be a sequence of normal closed λ-terms. A closed λ-term A is called *generator* for $\mathbf{U}$ if and only if :

$$(A\,I) \simeq_\beta U_1$$

and

$$(A\,\langle U_1, U_2, \ldots, U_n\rangle) \simeq_\beta U_{n+1} \text{ for every } n \geq 1$$

Lemma 4.5 *There is a sequence of normal closed λ-terms without generator.*

Proof: If not, let φ be a bijection between Λ^0 and $\mathbb{N}$ and Φ the function from Λ^1 into Λ^0 defined by the following: $\Phi(\mathbf{U})$ is the generator $G_{\mathbf{U}}$ such that $\varphi(G_{\mathbf{U}})$ is minimum. It is easy to check that Φ is a one to one mapping. This is a contradiction. □

Theorem 4.6 *There is a numeral system with predecessor and zero test but without successor.*

Proof: Let **e** be a sequence of normal closed λ-terms without generator. Let $c_0 = I$ and for every $n \geq 1$, $c_n = \langle c_{n-1}, e_n\rangle$. It is easy to check that the λ-terms $P_c = \lambda n(n\,T)$ and $Z_c = \lambda n(n\,\lambda x\lambda y I\;T\;F\;T)$ are λ-terms for predecessor and zero test for **c**. If **c** possesses a closed λ-term S_c for successor, then the λ-term $S'_c = \lambda n(S_c\,n\,F)$ is a generator for **e**. This is a contradiction. □

The result of Barendregt (Theorem 3.4) means that, for a numeral system, it suffices to represent three particular functions in order to represent all total recursive functions. We have proved that these three particular functions are independent. We think it is, at least, necessary to have three functions as is mentioned below.

Conjecture 4.7 *There are no total recursive functions* $f, g : \mathbb{N} \longrightarrow \mathbb{N}$ *such that for all numeral systems* **d**, f, g *are λ-definable if and only if all total recursive functions are λ-definable with respect to* **d**.

If we authorize the binary functions we obtain the following result.

Theorem 4.8 *There is a binary total function k such that for all numeral systems* **d**, *k is λ-definable if and only if all total recursive functions are λ-definable with respect to* **d**.

Proof: Let k be the total binary function defined by

$$k(n,m) = \begin{cases} n+1 & \text{if } m = 0 \\ |n-m| & \text{if } m \neq 0 \end{cases}$$

It suffices to see that

$$k(n,n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0, \end{cases}$$

$$k(n,0) = n+1,$$

$$k(n,1) = n-1 \quad \text{if } n \neq 0.$$

□

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NOTE

1. This theorem is the exercise 6.8.21 of Barendregt's book (see [1]). We give here a proof based on the techniques developed by J.-L. Krivine in [3].

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