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A Decidable Temporal Logic of Parallelism

MARK REYNOLDS

Abstract In this paper we shall introduce a simple temporal logic suitable for reasoning about the temporal aspects of parallel universes, parallel processes, distributed systems, or multiple agents. We will use a variant of the mosaic method to prove decidability of this logic. We also show that the logic does not have the finite model property. This shows that the mosaic method is sometimes a stronger way of establishing decidability.

1 Introduction The logic, **FP**$\Diamond$, investigated here is not new. It has been mentioned in Thomason [21], for example, and is one of the simplest ways of combining temporal and modal operators. It is a propositional logic with formulas built using the two Priorean temporal connectives F and P along with a modal operator $\Diamond$. The semantics are evaluated on a rectangular frame consisting of the cross product of a linear order and a nonempty set. The temporal and modal operators act in a perpendicular fashion: thus we have a simple combination Priorean temporal logic of linear time and an **S5** modal logic.

In computer science applications, ordinary one dimensional temporal logic is itself widely recognized as a convenient formalism for representing and reasoning about the behavior of complex and reactive systems. However, adding the modal dimension really does add another dimension to the expressibility. The modality can be used to represent reference to parallel universes or to other processes or agents in a complex system. The increased expressiveness is apparent when we realize that we can describe systems of unbounded or infinite size.

Many similar logics have been studied. They include branching time logics, logics of knowledge and belief, other logics of complex systems, and modal versions of cylindric algebras. We survey some in Section 3. **FP**$\Diamond$ turns out to be very closely related to the simplest versions of many of these logics. However, the combination of a past and future temporal logic over general linear time with an **S5** modal logic does

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not seem to have been studied. Most published work here concerns logics with languages restricted to the future temporal modalities and semantics restricted to natural numbers time. Decidability results for some of these logics exist.

One of the interesting contributions of this paper is the method for the decidability result. From its beginnings in algebraic logic in Németi [15], this mosaic method has grown to be increasingly used in proving decidability and completeness for various multimodal logics. It is well explained in Venema and Marx [23] where it is used to prove completeness and decidability of the logic LC_n—a generalization of a modal version of first-order logic. This paper (along with "Temporal mosaics," forthcoming by Marx, Mikulas, and Reynolds) pioneers the use of the method in the context of temporal logic.

Often, completeness and decidability proofs proceed in a step-by-step manner adding one point at a time to eventually build a model of a satisfiable formula. In the mosaic method we instead try to find a set of small pieces of a model which satisfies a certain closure property. This will be enough to guarantee that the small pieces can be put together to form a model. The actual putting together can either be done by a very simple step-by-step operation (as in [23]) or (as shown recently in Hodkinson et al. [11]) we might be able to use new techniques (of Herwig [10] and Hrushovski [12]) to immediately find the model. In using the mosaic method to give a decidability proof we need to define mosaics appropriate for the logic and define closure properties (dependent on a given formula) for a finite set of mosaics so that the existence of such a set of a certain size will be equivalent to the existence of a model for the formula.

In order to briefly describe the method used in this paper we will picture our rectangular frames with the linear order arranged vertically—greater corresponding to above—and the nonempty set arranged horizontally. Worlds in models correspond to ordered pairs. Since we are interested in the satisfiability of a particular formula we will equate each of these worlds with a set of subformulas of this formula—namely, those true in that world. The mosaics we use here correspond to slices of a rectangular model made along a pair of time points—that is, the slices are horizontal. The slices produce pairs of worlds: one on the bottom slice, the other above it on the top slice. The possibly infinite number of pairs of worlds inhabiting each mosaic are factored out by an equivalence based on the local truth of the subformulas of the original formula. Thus there are only a finite number of equivalence classes of pairs. A particular mosaic should thus be expected to exhibit various sensible "coherency" conditions. For example, if $G\alpha$, "always in the future α", is in the bottom world of a pair then we expect α to be in the top world of the pair. There are corresponding coherency conditions for the past modality and the perpendicular $\Diamond$ modality.

Notice that it may be the case that $F\alpha$ is in the top world of a pair in the mosaic. In this case we say that the mosaic has a defect. There are four types of defect altogether. If we wanted to build a model for our formula then we would hope that we could use other mosaics to put on top of ones with such a defect to cure the defect—that is, provide a witness for α. The decidability method will thus concentrate on finding what we will call a saturated set of mosaics. Such a set should contain all that is needed to cure any defects of mosaics within itself. The proof consists of showing that the existence of such a saturated set is equivalent to the satisfiability of the original formula. We also must show that there is a finite bound on the number of mosaics needed to make a saturated set.

In the long version of this paper [18], we use an IRR style rule (cf. Gabbay, Hodkinson, and Reynolds [8]) to give a straight forward axiomatization of the logic. With such an axiomatization available, it is relevant to ask whether a simple decision procedure can be presented for the logic based on the finite model property. In Section 9, we show that the logic **FP**$\Diamond$ is an interesting candidate for a decidability proof via the mosaic method because, the logic does not have the finite model property. This shows that the finite set of mosaics with the closure property is not just a finite model in disguise. Finally we conclude with some related open problems.

***2 The logic* FP$\Diamond$** Formulas are constructed from propositional atoms (from $\mathcal{L}$ say) and $\top$ using $\neg$, $\wedge$, and the three modalities $\Diamond$, F, and P. F and P will act like the usual mutually dual Priorean temporal connectives. The other modality $\Diamond$ will act in a perpendicular fashion as we will see. We use the usual abbreviations: $\bot$, $\vee$, $\longrightarrow$, G, H, $\Box$. Note that $\Box\alpha$ is thus $\neg\Diamond\neg\alpha$.

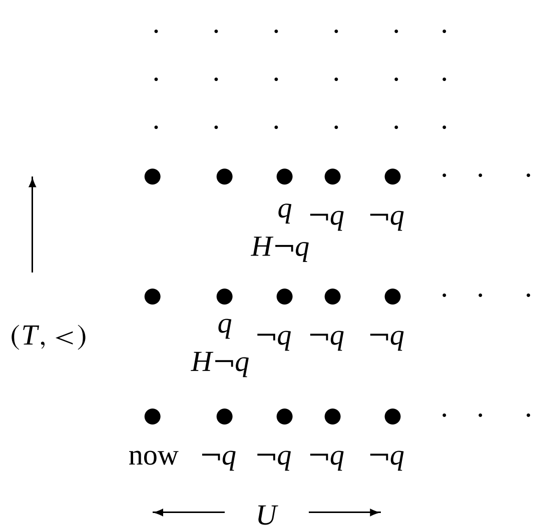
Semantics is over rectangular structures, that is, the frame is $(U, T, <)$ for some nonempty set U and some nonempty irreflexive linear order $(T, <)$. Valuations of the atoms are made at pairs (u, t). Truth in a structure $\mathcal{M} = (U, T, <, g)$ under valuation $g : U \times T \longrightarrow 2^{\mathcal{L}}$ is defined inductively as follows:

$$\begin{array}{lcl}
\mathcal{M}, u, t & \models & p \text{ iff } p \in g(u, t); \\
\mathcal{M}, u, t & \models & \top; \\
\mathcal{M}, u, t & \models & \alpha \wedge \beta \text{ iff } \mathcal{M}, u, t \models \alpha \text{ and } \mathcal{M}, u, t \models \beta; \\
\mathcal{M}, u, t & \models & \neg\alpha \text{ iff } \mathcal{M}, u, t \not\models \alpha; \\
\mathcal{M}, u, t & \models & F\alpha \text{ iff there is } s \in T \text{ such that } t < s \text{ and } \mathcal{M}, u, s \models \alpha; \\
\mathcal{M}, u, t & \models & P\alpha \text{ iff there is } s \in T \text{ such that } s < t \text{ and } \mathcal{M}, u, s \models \alpha; \\
\mathcal{M}, u, t & \models & \Diamond\alpha \text{ iff there is } v \in U \text{ such that } \mathcal{M}, v, t \models \alpha.
\end{array}$$

We say that $(U, T, <)$ is a *brief* structure if and only if T is a singleton (so $<$ is empty).

The simple modal fragment involving $\Diamond$ is made a lot more powerful by its combination with the temporal logic. For example, we can say that there are an infinite number of parallel time-lines:

$$F\top \wedge GF\top \wedge G\Diamond(q \wedge H\neg q).$$



The only rectangular models of this formula are infinite. In Section 9, we will consider whether this implies that the logic lacks the finite model property.

3 Related logics **FP**$\Diamond$ logic is a restriction of the Synchronized Ockhamist branching-time logic of Di Maio and Zanardo [3]. The semantical structures (called $T \times W$ frames in [21]) for this logic involve the cross product of a linear order $(T, <)$ and a set W along with equivalence relations $\sim_t$ on W for each $t \in T$. The equivalence relations must satisfy the property that $w \sim_t w'$ and $t' < t$ implies $w \sim_{t'} w'$. The order $(T, <)$ represents time and the elements of the set W represent alternative histories. The $\sim_t$-class containing w can be used to represent the histories which are possible from the point of view of the world (t, w). Thus the modality $\Diamond_2$ defined by

$$\begin{aligned}&(T, <, W, \{\sim_t\}, g), t, w \models \Diamond_2 \alpha \quad \text{iff} \\ &\quad \exists w' \in W, \quad \text{such that} \quad w \sim_t w' \quad \text{and} \quad (T, <, W, \{\sim_t\}, g), t, w' \models \alpha\end{aligned}$$

represents the idea of "at this time in some history which is currently considered possible". The modality $\Diamond_1$ defined by

$$\begin{aligned}&(T, <, W, \{\sim_t\}, g), t, w \models \Diamond_1 \alpha \quad \text{iff} \\ &\quad \exists w' \in W, \quad \text{such that} \quad (T, <, W, \{\sim_t\}, g), t, w' \models \alpha\end{aligned}$$

represents the idea of "at this time in some history". It is this latter modality which extends the nontemporal modality in the logic **FP**$\Diamond$. Logics very similar to the synchronized logic form bases for logics of agency in Belnap and Perloff [1] and causation in von Kutschera [24]. There are axiomatizations of such logic in von Kutschera [25] and Di Maio and Zanardo [4]. It seems to be an open problem whether this logic is decidable.

Logics of historical necessity or Ockhamist logics are closely related examples of a combined logic. They are not neatly two-dimensional logics but we do have a modal logic of possibility in some sense orthogonal to a linear temporal logic. These logics are obtained by removing the $\Diamond_1$ modality from the synchronized logic above. They are described in Burgess [2] while there are axiomatizations in Zanardo [26], Zanardo [27], and [8]. A special case of this logic is proved decidable in Gurevich and Shelah [9].

Many combinations of time and other modalities arise from formal investigations into how knowledge (or belief) changes over time. These logics are usually designed for reasoning about systems of multiple agents. See Fagin et al. [5] for a comprehensive survey. A temporal-epistemic logic for n agents will use n knowledge modalities. Thus the versions which are of relevance to us here are simple ones, formalizing the changes in knowledge of one lone agent who knows about the world and her or his own knowledge. **S5** is commonly taken to be the nontemporal logic of knowledge appropriate for one agent. So we can formalize the semantics of the temporal-epistemic logic using a two-dimensional frame very similar in general form to those for synchronized historical necessity. However, the accessibility relation for the knowledge modality does not have to be restricted to being between worlds (t, w) and (t', w') with $t = t'$. In the case with time being the natural numbers these logics are well studied. In [5] a logic is introduced which is like **FP**$\Diamond$ but involves tomorrow and until temporal operators, has a natural numbers frame, has an EXPSPACE-complete decision procedure, and is given a complete axiomatization. This is a temporal-epistemic logic of one agent who doesn't forget, doesn't learn and who knows the time.

We have mentioned that**FP**$\Diamond$ logic has applications to systems of parallel processes. There has been some work in developing two-dimensional logic for such applications. In Reif and Sistla [17], for example, we find a logic combining temporal and spatial modalities. Once again the temporal dimension is the natural numbers and we have the other dimension based on a set of processes. However, there is a set of names for links which may or may not connect one process to another. The language uses *until* in the temporal direction but has a spatial modality for each link as well as one for the transitive closure of all links. This leads to a highly undecidable, unaxiomatizable logic. In Sistla and German [20], on the other hand, we have a similar logic but without the linking modalities. There is just the one existential spatial modality as in **FP**$\Diamond$. With *until* as the temporal connective and the natural numbers as time, deciding validity in this logic is stated to be EXPSPACE-complete.

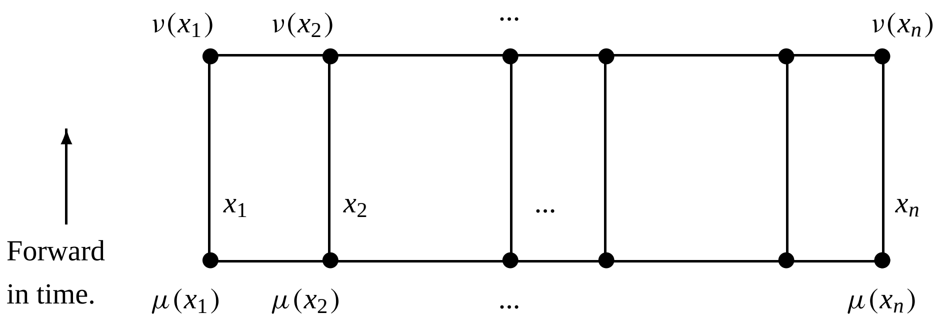
There are other temporal logics in the literature with a two-dimensional flavor. For example, there are the logics arising from general *temporalizing* (Finger and Gabbay [7]) and combining (Finger and Gabbay [6]) techniques. Temporalizing allows the adding of a temporal logic on top of any other logic. Truth is evaluated in two-dimensional structures but only a restricted language is available—formulas with a horizontal modality nested inside a vertical one, say, are outlawed. Combining or Fibring techniques, on the other hand, allow the full two-dimensional language but also allow very complex models without commutativity of the two accessibility relations $<_1$ and $<_2$. Such structures are sometimes known as independent combinations of modal logics (Thomason [22]). Such logics are used to investigate the preservation of various logical properties under combination logics. They can also sometimes be the only way of keeping combinations of logic decidable.

There is a wealth of two-dimensional nontemporal modal logics which have been investigated. One of the most fruitful areas here has been the investigation of modal versions of first-order logic and their cylindric algebra counterparts. If we look at first-order logic with no function symbols, relations of arity only 1 or 2 and only two variable symbols, then we can regard the existential quantifier as a modal operator and come up with a two-dimensional modal logic which is the same as that in Segerberg [19]. In Marx [14], a similar modal logic is studied. This logic is proved decidable and the proof is an example of the mosaic method which we now turn to.

4 Segments Suppose that φ is a formula. Let S be the set of subformulas of φ including the single negations of nonnegated subformulas. We are interested in deciding whether φ is satisfiable, that is, there exists valuated structure $(U, T, <, g)$ and $u \in U$, $t \in T$ such that $(U, T, <, g), u, t \models \varphi$. As described in the introduction, we will be using the mosaic method to provide the decision procedure. First we will define a mosaic—a small piece of model which we will eventually use to build whole models. Because the pieces in this proof are supposed to correspond to whole segments across a rectangular model, we will call the pieces "segments" rather than mosaics.

The idea of a segment comes from considering a pair of slices cut horizontally across the whole of a rectangular model. Such a pair of slices gives rise to a set of pairs of worlds: a world on the bottom slice paired with the world directly above it on the top slice. We shall equate worlds with the set of formulas from S locally true in that world in the sliced model and thus we have just a finite set X of pairs. We will let $\mu(x)$ be the lower set of formulas in a pair x and $\nu(x)$ be the upper set.

We can picture a segment as follows:



where $\{x_1, \ldots, x_2\}$ are the parallel time lines. This leads us to the following definition.

Definition 4.1 A φ-segment is a triple (X, μ, ν) such that

1. X is a finite set;
2. $\mu, \nu : X \longrightarrow 2^S$;
3. if $\mu(x) = \mu(y)$ and $\nu(x) = \nu(y)$ then $x = y$;
4. each $\mu(x)$ and $\nu(x)$ is a maximally Boolean consistent set;
5. if $G\alpha \in \mu(x)$ then $\alpha \in \nu(x)$ and $G\alpha \in \nu(x)$;
6. if $H\alpha \in \nu(x)$ then $\alpha \in \mu(x)$ and $H\alpha \in \mu(x)$;
7. if $\Box\alpha \in \mu(x)$ then for all $y \in X$, $\alpha \in \mu(y)$;
8. if $\Box\alpha \in \nu(x)$ then for all $y \in X$, $\alpha \in \nu(y)$;
9. if $\Diamond\alpha \in \mu(x)$ then there is $y \in X$ such that $\alpha \in \mu(y)$;
10. if $\Diamond\alpha \in \nu(x)$ then there is $y \in X$ such that $\alpha \in \nu(y)$.

Such conditions are often called coherency conditions for mosaics.

The decision procedure for satisfiability of φ amounts to looking for a set of segments which are sufficient so that they can be put together to build a rectangular model of φ. As we have seen, the models of φ may all be infinite. Nevertheless, we will show that a finite set of segments will suffice to build a model. Of course, we may need many copies of the same segment at various stages of the construction.

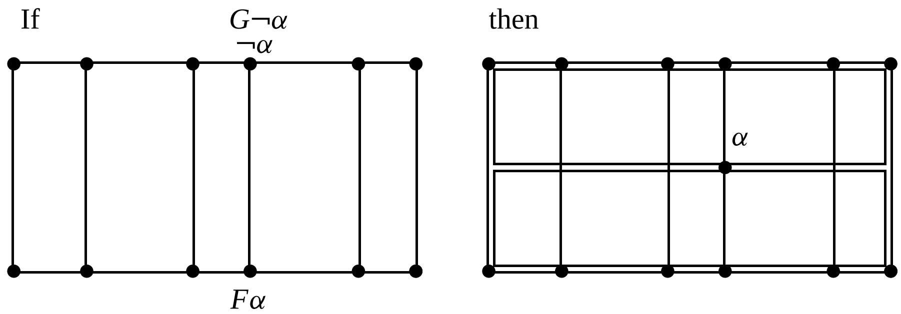
The test for sufficiency of a given finite set L of segments is surprisingly simple. In actual fact, we need only consider one segment at a time from L and make sure that certain closely related segments are also in L. Consider a segment $A = (X, \mu, \nu)$. Suppose that $F\alpha \in \nu(x)$ for some $x \in X$. This means that if we ever use A in building our model then there will be a point on top of the model (if A has just been placed on top) at which $F\alpha$ should be true. It is clear that at some later stage we are going to have to place a segment on top of the partially completed model which makes α true at the point above $\nu(x)$. The $F\alpha$ in $\nu(x)$ is called a defect in A and the process of later placing a witness for α at the right point is called curing the defect. We will require that a segment for curing the defect in A is also in L available for our use.

There are other sorts of defects as described in B3, B4, and B5 below. Returning to consider our set L of segments, it turns out that we need only make sure that we can cure defects in each of the segments in L using only segments from L. Such considerations give rise to the following definition.

Definition 4.2 A *saturated set of segments* (SSS) for φ is a set L of φ-segments such that

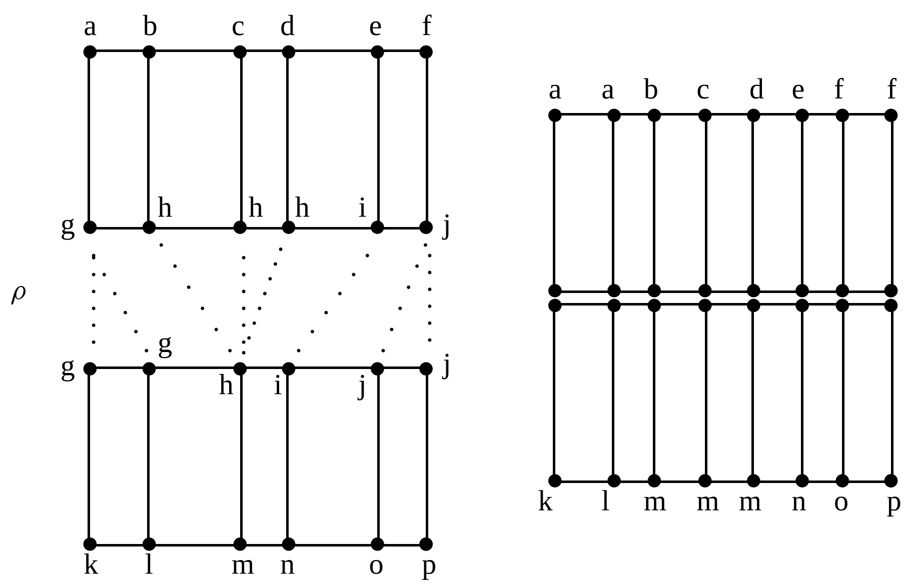
- B1 there is $(X, \mu, \nu) \in L$, and $x \in X$ such that either $\varphi \in \mu(x)$ or $\varphi \in \nu(x)$;
- B2 if $(X, \mu, \nu) \in L$, $x \in X$ and $F\alpha \in \nu(x)$ then there is $(X', \mu', \nu') \in L$, $\rho \subseteq X \times X'$ and $x' \in X'$ such that
  - B2.1 for all $y \in X$ there is $y' \in X'$ such that $\rho(y, y')$,
  - B2.2 for all $y' \in X'$ there is $y \in X$ such that $\rho(y, y')$,
  - B2.3 if $\rho(y, y')$ then $\nu(y) = \mu'(y')$,
  - B2.4 $\rho(x, x')$,
  - B2.5 $\alpha \in \nu(x')$;
- B3 similarly for $P\alpha$;
- B4 if $(X, \mu, \nu) \in L$, $x \in X$, $F\alpha \in \mu(x)$, $F\alpha \notin \nu(x)$ and $\alpha \notin \nu(x)$ then there is $(X', \mu', \nu') \in L$, $(X'', \mu'', \nu'') \in L$, $\rho \subseteq X \times X'$, $\sigma \subseteq X' \times X''$ and $\tau \subseteq X'' \times X$, $x'' \in X''$ and $x' \in X'$ such that
  - B4.1 for all $y \in X$ there is $y' \in X'$ and $y'' \in X''$ such that $\rho(y, y')$, $\sigma(y', y'')$ and $\tau(y'', y)$,
  - B4.2 for all $y' \in X'$ there is $y \in X$ and $y'' \in X''$ such that $\rho(y, y')$, $\sigma(y', y'')$ and $\tau(y'', y)$,
  - B4.3 for all $y'' \in X''$ there is there is $y' \in X'$ and $y \in X$ such that $\rho(y, y')$, $\sigma(y', y'')$ and $\tau(y'', y)$,
  - B4.4 if $\rho(y, y')$ then $\mu(y) = \mu'(y')$,
  - B4.5 if $\sigma(y', y'')$ then $\nu'(y') = \mu''(y'')$,
  - B4.6 if $\tau(y'', y)$ then $\nu''(y'') = \nu(y)$,
  - B4.7 $\rho(x, x')$,
  - B4.8 $\sigma(x', x'')$,
  - B4.9 $\tau(x'', x)$,
  - B4.10 $\alpha \in \nu(x')$;
- B5 similarly for $P\alpha$.

There are several aspects of this definition that need to be explained. First, the defect and cure considered in B4. Here we have supposed we have a segment $(X, \mu, \nu) \in L$ with $F\alpha \in \mu(x)$ but neither $F\alpha$ nor $\alpha \in \nu(x)$. When we are building a model and we use this segment then we will have to, at a later stage, replace it by two segments which fitted together one immediately above the other match the original. This is because there must be a witness to α in between the two slices which formed the original segment.



The other important aspect of curing defects is the seemingly very complicated way in which segments fit together, one above the other. For example, in B2, we introduce a relation $\rho \subseteq X \times X'$ to do the fitting or gluing together. Of course, ρ must be a total relation (condition B2.1) and ρ must be onto (condition B2.2) so that time lines do not end or begin at joins in our segmented model. But why do we not require that ρ is one-to-one? This is because, it may be recalled, segments do not correspond exactly to pairs of slices in a model but to sets of pairs of worlds at slices factored out by equivalence in truth of formulas from S. It so happens that sometimes during construction the set of time-lines constructed by some stage needs to suffer a splitting of two (or more) equivalent lines in order to continue each of them in a different way. This is why our gluing allows forking in one direction or another.

Here is an example of the result of gluing segments together where the lower case letters refer to sets of formulas which label the points of the segments:



5 ***Soundness of the saturation condition*** The main result we prove in the next few sections is that the existence of a rectangular model for φ is almost equivalent to the existence of a saturated set of segments for φ. In this section we will show that if a formula φ is satisfiable in a rectangular model then there exists an SSS for φ or φ has a brief model. If φ does not have a brief model then the SSS we find will be simply constructed by taking all pairs of slices from a model of φ.

Lemma 5.1 *If φ is satisfiable then there is an SSS L for φ or a brief model of φ.*

Proof: Let $\mathcal{T} = (U, T, <, g)$ be a model of φ. Say $u_0 \in U$, $t_0 \in T$ and $\mathcal{T}, u_0, t_0 \models \varphi$. If $T = \{t_0\}$ then we have a brief model. So suppose there is another element of T. For each $u \in U$, $t \in T$ define $S(u, t) = \{\beta \in S \mid \mathcal{T}, u, t \models \beta\}$. For each $s, t \in T$ such that $s < t$ define $\text{seg}(s, t) = (X, \mu, \nu)$ by putting $X = \{(S(u, s), S(u, t)) \mid u \in U\}$, $\mu(B, C) = B$ and $\nu(B, C) = C$. This is a φ-segment.

Let $L = \{\text{seg}(s, t) \mid s, t \in T, \text{ and } s < t\}$. It is not hard to show that L is an SSS: L is finite as there are at most $2^{2|\varphi|}$ pairs of possible segments. For example, let us check B4. Suppose $(X, \mu, \nu) = \text{seg}(s, t) \in L$ for some $s < t$ in T, $x = (S(u, s), S(u, t)) \in X$ for some $u \in U$, $F\alpha \in \mu(x)$, $F\alpha \notin \nu(x)$ and $\alpha \notin \nu(x)$. Thus $\mathcal{T}, u, s \models F\alpha$ and $\mathcal{T}, u, t \models \neg F\alpha$ and $\mathcal{T}, u, t \models \neg\alpha$. So there is $r \in T$ such that $s < r < t$ and $\mathcal{T}, u, r \models \alpha$. Let

1. $(X', \mu', \nu') = \text{seg}(s, r) \in L$,
2. $(X'', \mu'', \nu'') = \text{seg}(r, t) \in L$,
3. $\rho \subseteq X \times X'$ contain just $((S(v, s), S(v, t)), (S(v, s), S(v, r)))$ for each $v \in U$,
4. $\sigma \subseteq X' \times X''$ contain just $((S(v, s), S(v, r)), (S(v, r), S(v, t)))$ for each $v \in U$,
5. $\tau \subseteq X'' \times X$ contain just $((S(v, r), S(v, t)), (S(v, s), S(v, t)))$ for each $v \in U$,
6. $x'' = (S(u, r), S(u, t)) \in X''$,
7. $x' = (S(u, s), S(u, r)) \in X'$.

Checking all the conditions is straightforward. For example let us check B4.1. Suppose $y \in X$. Say $y = (S(v, s), S(v, t))$ for some $v \in U$. Simply let $y' = (S(v, s), S(v, r))$ and $y'' = (S(v, r), S(v, t))$. It is clear that this will do. $\square$

6 Curing defects Suppose that L is an SSS for φ. We will show that φ is satisfiable in a rectangular model by gradually building a model by gluing together the segments in L. In this section we will see how to extend a model constructed from segments so that one defect in it is cured. In the next section we will see how continual curing of defects eventually ends in a model for φ.

We will build a model for φ by concentrating on labeled structures. A (2^S-) *labeled structure* is a tuple $(U, T, <, \lambda)$ where U is a set, $(T, <)$ a linear irreflexive ordered set, and $\lambda : (U \times T) \longrightarrow 2^S$. Traditionally such a map λ has been called a *chronicle*. Our construction will only involve labeled structures which are built from segments. We introduce the following definition.

Definition 6.1 The labeled structure $\mathcal{T} = (U, T, <, \lambda)$ is an (L-) *segmented* structure if and only if

- C1 $(T, <)$ is a finite suborder of the rationals of size at least two and U is a finite set;
- C2 for each $(u, t) \in (U \times T)$, $\lambda(u, t)$ is a maximally Boolean consistent set;
- C3 for each $(u, t) \in (U \times T)$, if $G\alpha \in \lambda(u, t)$ then for all $s > t$ in T, $G\alpha \in \lambda(u, s)$ and $\alpha \in \lambda(u, s)$;
- C4 for each $(u, t) \in (U \times T)$, if $H\alpha \in \lambda(u, t)$ then for all $s < t$ in T, $H\alpha \in \lambda(u, s)$ and $\alpha \in \lambda(u, s)$;

C5 for each $(u, t) \in (U \times T)$, if $\Box\alpha \in \lambda(u, t)$ then for all $v \in U$, $\alpha \in \lambda(v, t)$;
C6 for each $(u, t) \in (U \times T)$, if $\Diamond\alpha \in \lambda(u, t)$ then there is $v \in U$ such that $\alpha \in \lambda(v, t)$;
C7 for each $s < t$ in T that are immediate neighbors, there is $(X, \mu, \nu) \in L$ and a map $q : U \longrightarrow X$ such that

C7.1 q is onto,
C7.2 for all $u \in U$, $\mu(q(u)) = \lambda(u, s)$ and $\nu(q(u)) = \lambda(u, t)$.

Definition 6.2 For a segmented structure $\mathcal{T} = (U, T, <, \lambda)$ and $(u, t) \in U \times T$, we say that $(u, t, F\alpha)$ is a *defect* in $\mathcal{T}$ if and only if $F\alpha \in \lambda(u, t)$ but there is no $s > t$ in T such that $\alpha \in \lambda(u, s)$. Similarly we have defects $(u, t, P\alpha)$.

There are only finitely many defects in any segmented structure. In this section we show how to cure defects. First we give the following definition:

Definition 6.3 A defect (u, t, χ) in a segmented structure $\mathcal{T} = (U, T, <, \lambda)$ is said to be *cured* in segmented structure $\mathcal{T}' = (U', T', <', \lambda')$ if and only if $T \subseteq T'$ and there is an injection $i : U \longrightarrow U'$ such that for all $s \in T$, $v \in U$, $\lambda'(i(v), s)) = \lambda(v, s)$, and $(i(u), t, \chi)$ is not a defect in $\mathcal{T}'$.

Claim 6.4 *Under the assumption that L is an SSS for φ, we can cure any defect in any L-segmented structure.*

Suppose that the defect is $(u, t_1, F\alpha)$. Dual defects are dealt with similarly. A simple induction establishes that there is $t \in T$ such that $t_1 \leq t$ and $F\alpha \in \lambda(u, t)$ but either

1. t is the $<$-maximal element in $(T, <)$ or
2. s is the immediate successor of t in $(T, <)$ and $F\alpha \notin \lambda(u, s)$ and $\alpha \notin \lambda(u, s)$.

It is clear that there is a defect $(u, t, F\alpha)$ in $\mathcal{T}$ and that curing this defect will also cure the defect $(u, t_1, F\alpha)$. The two cases are set out in the following two subsections.

6.1 The case of a maximal defect Suppose that s is the immediate predecessor of t in $(T, <)$. There is such an s by C1. By condition C7, there is $(X, \mu, \nu) \in L$ and a map $q : U \longrightarrow X$ such that q is onto and for all $v \in U$, $\mu(q(v)) = \lambda(v, s)$ and $\nu(q(v)) = \lambda(v, t)$. Thus $F\alpha \in \mu(q(u))$. By condition B2 on L there is $(X', \mu', \nu') \in L$, $\rho \subseteq X \times X'$ and $x' \in X'$ such that:

D1 for all $y \in X$ there is $y' \in X'$ such that $\rho(y, y')$;
D2 for all $y' \in X'$ there is $y \in X$ such that $\rho(y, y')$;
D3 if $\rho(y, y')$ then $\nu(y) = \mu'(y')$;
D4 $\rho(q(u), x')$;
D5 $\alpha \in \nu'(x')$.

Choose an element t' in the rationals greater than t. We define a new structure $\mathcal{T}' = (U', T', <', \lambda')$ by

1. $U' = \{(v, y') \in U \times X' \mid (q(v), y') \in \rho\}$;
2. $T' = T \cup \{t'\}$;
3. $(T', <')$ is a suborder of the rationals;
4. $\lambda'((v, y'), r) = \lambda(v, r)$ for $v \in U$, $y' \in X'$ and $r \in T$;

5. $\lambda'((v, y'), t') = \nu'(y')$ for $v \in U$ and $y' \in X'$.

We will now check that $\mathcal{T}'$ is segmented. C1 and C2 are clear.

For C3 suppose that $(v, y') \in U'$, $s_1 <' s_2$ in T', and $G\alpha \in \lambda'((v, y'), s_1)$. Clearly $s_1 \in T$ so that $G\alpha \in \lambda(v, s_1)$. Either $s_2 \in T$ when the result follows immediately from C3 in $\mathcal{T}$ or $s_2 = t'$. In this latter case, $\lambda'((v, y'), s_2) = \nu'(y')$. But either $s_1 = t$ when $G\alpha \in \lambda(v, t)$ or $s_1 < t$ when condition C3 on $\mathcal{T}$ also gives us $G\alpha \in \lambda(v, t)$. By condition C7 applied to (X, μ, ν) and q as mentioned above, $G\alpha \in \nu(q(v))$. Since $(v, y') \in U'$, $(q(v), y') \in \rho$ and so, by D3, $G\alpha \in \mu'(y')$. Finally coherency of the segment $(X'\mu'\nu')$ gives us $\alpha \in \nu'(y') = \lambda'((v, y'), s_2)$ and $G\alpha \in \nu'(y') = \lambda'((v, y'), s_2)$ as required. Condition C4 is dual while conditions C5 and C6 are similarly straightforward.

For C7 there are two cases. If $s_1 <' s_2$ are both in T then we find (any of) the same (X_1, μ_1, ν_1) and $q_1 : U \longrightarrow X_1$ as we would use to show C7 in $\mathcal{T}$. However we use $q_2 : U' \longrightarrow X_1$ given by $q_2((v, y')) = q_1(v)$. For the specific immediate neighbors t and t' from T' we use (X', μ', ν') and the map $q' : U' \longrightarrow X'$ given by $q'((v, y')) = y'$. It is easy to check that this will do.

We also define a map $i : U \longrightarrow U'$. Let $i(u) = (u, x')$. This is in U' by D4. For each other $v \in U$ just choose any $y' \in X'$ such that $(q(v), y') \in \rho$ and put $i(v) = (v, y') \in U'$. There is such a y' by D1. It is clear that i is an injection. From the fact that $\lambda'((v, y'), s_1) = \lambda(v, s_1)$ it is also clear that i will do as the required injection for extending our defective structure $\mathcal{T}$: for any $v \in U$, for any $s_1 \in T$, $\lambda(v, s_1) = \lambda'(i(v), s_1)$.

Finally we check $(i(u), t, F\alpha)$ is not a defect of $\mathcal{T}'$. But this is clear as $t < t'$ and $\alpha \in \lambda'(i(u), t') = \lambda'((u, x'), t') = \nu'(x')$ by D5.

6.2 The case of a nonmaximal defect We have $u \in U$ and $t \in T$ such that $F\alpha \in \lambda(u, t)$ but s is the immediate successor of t in $(T, <)$ and $F\alpha \notin \lambda(u, s)$ and $\alpha \notin \lambda(u, s)$. By condition C7 there is $(X, \mu, \nu) \in L$ and a map $q : U \longrightarrow X$ such that

1. q is onto;
2. for all $u \in U$, $\mu(q(u)) = \lambda(u, t)$ and $\nu(q(u)) = \lambda(u, s)$.

Thus $F\alpha \in \mu(q(u))$ but $F\alpha \notin \nu(q(u))$ and $\alpha \notin \nu(q(u))$. By condition B4 on L, there is $(X', \mu', \nu') \in L$, $(X'', \mu'', \nu'') \in L$, $\rho \subseteq X \times X'$, $\sigma \subseteq X' \times X''$ and $\tau \subseteq X'' \times X$, $x'' \in X''$ and $x' \in X'$ such that conditions B4.1 to B4.10 hold for $x = q(u)$.

Choose an element t' in the rationals between t and s. We define a new structure $\mathcal{T}' = (U', T', <', \lambda')$ by

1. $U' = \{(v, y', y'') \in U \times X' \times X'' \mid (q(v), y') \in \rho, (y', y'') \in \sigma$ and $(y'', q(v)) \in \tau\}$;
2. $T' = T \cup \{t'\}$;
3. $(T', <')$ is a suborder of the rationals;
4. $\lambda'((v, y', y''), r) = \lambda(v, r)$ for $v \in U$, $y' \in X'$, $y'' \in X''$ and $r \in T$;
5. $\lambda'((v, y', y''), t') = \nu'(y')$ for $v \in U$, $y' \in X'$ and $y'' \in X''$.

We can now check that $\mathcal{T}'$ is segmented. This is quite straightforward in parts and similar to the proof in the last subsection in other parts.

We establish C6. Suppose $(v', r) \in U' \times T'$ and $\Diamond\alpha \in \lambda'(v', r)$. Say $v' = (v, y, y'') \in U \times X' \times X''$ for $(q(v), y') \in \rho$, $(y', y'') \in \sigma$ and $(y'', q(v)) \in \tau$. There are two cases. First suppose that $r \in t$. Thus $\Diamond\alpha \in \lambda'(v', r) = \lambda'((v, y', y''), r) = \lambda(v, r)$. But $\mathcal{T}$ satisfies C6 and $(v, r) \in U \times T$ so there is $w \in U$ such that $\alpha \in \lambda(w, r)$. Now $q(w) \in X$ so we can use B4.1 to find $z' \in X'$ and $z'' \in X''$ such that $(q(w), z') \in \rho$, $(z', z'') \in \sigma$ and $(z'', q(w)) \in \tau$. Then $w' = (w, z', z'') \in U'$ and $\alpha \in \lambda(w, r) = \lambda'((w, z', z''), r) = \lambda'(w', r)$. The other case is that $r = t'$. Thus $\Diamond\alpha \in \lambda'(v', t') = \lambda'((v, y', y''), t') = \nu'(y')$. By the coherency of (X', μ', ν'), we have $z' \in X'$ such that $\alpha \in \nu'(z')$. By B4.2 there is $z \in X$ and $z'' \in X''$ such that $(z, z') \in \rho$, $(z', z'') \in \sigma$ and $(z'', z) \in \tau$. As q is onto X, there is $w \in U$ such that $q(w) = z$. Let $w' = (w, z, z'')$ then $\alpha \in \nu'(z') = \lambda'((w, z', z''), t') = \lambda'(w', t')$ as required. This establishes C6.

We will also define a map $i : U \longrightarrow U'$. Let $i(u) = (u, x', x'')$. This is in U' by B4.7, B4.8, and B4.9. For each other $v \in U$ just choose any $y' \in X'$ and $y'' \in X''$ such that $(q(v), y') \in \rho$, $(y', y'') \in \sigma$ and $(y'', q(v)) \in \tau$ and put $i(v) = (v, y', y'') \in U'$. There are such y', y'' by B4.1. It is clear that i is an injection. From the fact that $\lambda'((v, y', y''), s_1) = \lambda(v, s_1)$ it is also clear that i will do as the required injection for extending our defective structure: for any $v \in U$, for any $s_1 \in T$, $\lambda(v, s_1) = \lambda'(i(v), s_1)$.

Finally we check that we have cured the defect $(u, t, F\alpha)$: we show that $(i(u), t, F\alpha)$ is not a defect of $\mathcal{T}'$. But this is clear as $t < t'$ and $\alpha \in \lambda'(i(u), t') = \lambda'((u, x', x''), t') = \nu'(x')$ by B4.10.

7 Building a model from an SSS In this section we will show that any formula with an SSS also has a rectangular model. It is possible that Fraïssé techniques can be used in combination with the defect-curing results of the previous sections. However, we will use a straightforward step-by-step approach. We start with a small segmented structure and build it up slowly, curing the defects successively. Finally, we extract a model for the formula from the limit of the process. Suppose that L is an SSS for φ and we wish to build a model for φ.

7.1 The induction By B1 there is $(X, \mu, \nu) \in L$ and $x \in X$ such that either $\varphi \in \mu(x)$ or $\varphi \in \nu(x)$. Define $U_0 = X$, $T_0 = \{0, 1\}$ and $<_0 = (0, 1)$ and $\lambda_0 : (U_0 \times T_0) \longrightarrow 2^S$ by $\lambda_0(x, 0) = \mu(x)$ and $\lambda_0(x, 1) = \nu(x)$. Let $\mathcal{T}_0 = (U_0, T_0, <_0, \lambda_0)$. Then $\mathcal{T}_0$ is a segmented structure. We will need to keep an account of the defects in our succession of structures. Number the defects in $\mathcal{T}_0$ $1, \dots, k$. It is clear that we have the condition I^1 as defined below.

Our induction hypothesis I^δ, for an ordinal δ, is that the following conditions hold for each n with $0 < n < \delta$.

1. $\mathcal{T}_n = (U_n, T_n, <_, \lambda_n)$ is an L-segmented structure.
2. We have an injection $i_{n-1} : U_{n-1} \longrightarrow U_n$.
3. $T_{n-1} \subseteq T_n$.
4. For each $u \in U_{n-1}$, for each $t \in T_{n-1}$, $\lambda_n(i_{n-1}(u), t) = \lambda_{n-1}(u, t)$.
5. The defects (if any) in $\mathcal{T}_n$ are numbered with distinct numbers greater than n.

6. For each $u \in U_{n-1}$, for each $t \in T_{n-1}$, if (u, t, χ) is a defect numbered d in $\mathcal{T}_{n-1}$, then either $(i_{n-1}(u), t, \chi)$ is not a defect in $\mathcal{T}_n$, or it is a defect numbered d in $\mathcal{T}_n$.
7. $\mathcal{T}_0$ is a segmented structure, its defects are numbered > 0, and there is $t_0 \in T_0$ and $u_0 \in U_0$ such that $\varphi \in \lambda_0(u_0, t_0)$.

We have already seen that we have I^1 holding.

Suppose that we have constructed the $\mathcal{T}_n$, i_n and the numbering of defects so that I^δ holds. Now check whether there is any defect numbered δ in $\mathcal{T}_{\delta-1}$. If there is not then we just let $\mathcal{T}_\delta = \mathcal{T}_{\delta-1}$, $i_{\delta-1}$ be the identity and we leave the numbering the same in $\mathcal{T}_\delta$ as it is in $\mathcal{T}_{\delta-1}$.

If there is a defect (u, t, χ) say, numbered δ in $\mathcal{T}_{\delta-1}$, then we use the defect-curing technique described above to construct $\mathcal{T}_\delta$ and the injection $i_{\delta-1} : U_{\delta-1} \longrightarrow U_\delta$ so that $(i_{\delta-1}(u), t, \chi)$ is not a defect in $\mathcal{T}_\delta$. To number each defect in $\mathcal{T}_\delta$ we simply check whether it is $(i_{\delta-1}(v), s, \psi)$ for some defect (v, s, ψ) in $\mathcal{T}_{\delta-1}$. If it is, then the defect inherits the number from the defect in $\mathcal{T}_{\delta-1}$. Note that as we have cured the defect numbered δ, then this number will not be inherited. After we have numbered all the inherited defects in this way we give any new defects distinct unused numbers greater than δ. After this it is clear that we have $I^{\delta+1}$ holding. Thus, by induction, we have I^ω holding.

7.2 The limit We may assume that all the U_ns are distinct. Define $\sim$ on $\bigcup U_n$ as follows. Suppose $u, v \in \bigcup U_n$, say that $u \in U_n$ and $v \in U_m$. Put $u \sim v$ if $n < m$ and $v = i_{m-1}(i_{m-2}(\ldots, i_n(u), \ldots))$ and close $\sim$ under reflexivity and symmetry to make it an equivalence relation. Note that we have the following.

- H1 For all n, for all $u \in \bigcup U_n$, for all $t \in T$, if U_n contains some v such that $u \sim v$ then for all $m \geq n$, U_m contains some w such that $u \sim w$.
- H2 If $u \sim v$ and $u \in U_n$, $v \in U_m$ then for any $t \in T_n \cap T_m$, $\lambda_n(u, t) = \lambda_m(v, t)$.
- H3 If (u, t, χ) is a defect in $\mathcal{T}_m$ then there is $n > m$ and $v \in U_n$ such that $u \sim v$ and (v, t, χ) is not a defect in $\mathcal{T}_n$.

These properties are simple consequences of I^ω. To prove H3, suppose that the defect (u, t, χ) is numbered k in $\mathcal{T}_m$. So $k > m$. By I^ω_6 and a simple induction we can find a smallest $n > m$ such that $(i_{n-1}(i_{n-2}(\ldots, i_m(u), \ldots)), t, \chi)$ is not a defect in $\mathcal{T}_n$. There must be such an n for otherwise we would end up with a defect numbered k in $\mathcal{T}_k$ contradicting I^ω property 5). Now simply put $v = i_{n-1}(i_{n-2}(\ldots, i_m(u), \ldots))$.

We define the structure $(E, T, <)$ by $E = (\bigcup U_n)/\sim$, $T = \bigcup T_n \subseteq \mathbb{Q}$ and $<$ being inherited from $(\mathbb{Q}, <)$. We also label this structure by λ. For $\lambda(e, t)$ for any $e \in E$ and $t \in T$, choose n large enough so that U_n contains some $v \in e$ and $t \in T_n$. This can be done by conditions H1 and I^ω property 3. Now simply put $\lambda(e, t) = \lambda_n(v, t)$. This is well defined by H2.

$(E, T, <, \lambda)$ has the following properties.

- G1 $(T, <)$ is a linear order.
- G2 Each $\lambda(e, t)$ is a maximally Boolean consistent subset of S. This follows from the definition of λ and C2.
- G3 If $\Box\alpha \in \lambda(e, t)$ then for all $e' \in E$, $\alpha \in \lambda(e', t)$. Suppose $e, e' \in E$, $t \in T$ and $\Box\alpha \in \lambda(e, t)$. Using H1 and I^ω, choose m large enough so that

$t \in T_m$ and there is $u, u' \in U_m$ with $u \in e$ and $u' \in e'$. By the definition of λ, $\Box\alpha \in \lambda_m(u, t) = \lambda(e, t)$. By I^ω, $\mathcal{T}_m = (U_m, T_m, <_m, \lambda_m)$ satisfies C1–7. By C5, $\alpha \in \lambda_m(u', t)$. By the definition of λ, $\alpha \in \lambda(e', t)$ as required.

G4 If $\Diamond\alpha \in \lambda(e, t)$ then there is $e' \in E$ such that $\alpha \in \lambda(e', t)$. This is proved similarly to G3.

G5 If $G\alpha \in \lambda(e, t)$ then for all $s \in T$, if $t < s$, $\alpha \in \lambda(e, s)$. Suppose that $G\alpha \in \lambda(e, t)$ and $t < s$. Choose m large enough that $s, t \in T_m$ and there is $u \in U_m$ such that $u \in e$. By the definition of λ, $G\alpha \in \lambda_m(u, t)$. By C1 and the definition of $<$, $t <_m s$ in $(T_m, <_m)$. By C3, $\alpha \in \lambda_m(u, s)$. By the definition of λ, $\alpha \in \lambda(e, s)$ as required.

G6 We have the dual of G5 for $H\alpha$.

G7 If $F\alpha \in \lambda(e, t)$ then there is $s \in T$, such that $t < s$ and $\alpha \in \lambda(e, s)$.

Choose m large enough so that $t \in T_m$ and there is $u \in U_m$ with $u \in e$. By the definition of λ, $F\alpha \in \lambda_m(u, t)$. Possibly, there is $s \in T_n$ such that $t <_m s$ and $\alpha \in \lambda_m(u, s)$. Then by C1 and the definition of $<$, $t < s$ in $(T, <)$ and by the definition of λ, $\alpha \in \lambda(e, s)$ as required.

Otherwise, $(u, t, F\alpha)$ is a defect in $\mathcal{T}_m$. Thus by H3 there is $n > m$ and $v \in U_n$ such that $v \sim u$ and $(v, t, F\alpha)$ is not a defect in $\mathcal{T}_n$. So there is $s \in T_n$ such that $t <_n s$ and $\alpha \in \lambda_n(v, s)$. Then by C1 and the definition of $<$, $t < s$ in $(T, <)$ and by the definition of λ, $\alpha \in \lambda(e, s)$ as required.

G8 We have the dual of G7 for $P\alpha$.

G9 $\varphi \in \lambda([x_0], t_0)$.

Our model of φ will be $\mathcal{T} = (E, T, <, g)$ where $g : (E \times T) \longrightarrow 2^{\mathcal{L}}$ is defined by $g(e, t) = \{p \in \mathcal{L} \mid p \in \lambda(e, t)\}$. By G9, there is $([x_0], t_0) \in E \times T$ such that φ is in $\lambda([x_0], t_0)$. The next result will show that $(E, T, <, g), [x_0], t_0 \models \varphi$.

Lemma 7.1 *For all $\psi \in S$, for all $e \in E$, for all $t \in T$, $\psi \in \lambda(e, t)$ if and only if $\mathcal{T}, e, t \models \psi$.*

Proof: We proceed by induction on the construction of ψ. The cases of atoms $\top$, $\neg\psi$, and $\psi \wedge \chi$ are trivial by the definition of g and G2. For $F\psi$, suppose $F\psi \in \lambda(e, t)$. By G7 there is $s \in t$ such that $t < s$ and $\psi \in \lambda(e, s)$. By the inductive hypothesis, $\mathcal{T}, e, s \models \psi$ so $\mathcal{T}, e, t \models F\psi$. Conversely, suppose $\mathcal{T}, e, t \models F\psi$ so there is $s \in T$ with $t < s$ and $\mathcal{T}, e, s \models \psi$. By the inductive hypothesis, $\psi \in \lambda(e, s)$. By G2 and G5, $F\psi \in \lambda(e, t)$. The case of $P\psi$ is analogous.

For $\Diamond\psi$, suppose $\Diamond\psi \in \lambda(e, t)$. By G4 there is $e' \in E$ such that $\psi \in \lambda(e', t)$. By the inductive hypothesis, $\mathcal{T}, e', t \models \psi$ so $\mathcal{T}, e, t \models \Diamond\psi$. Conversely, suppose $\mathcal{T}, e, t \models \Diamond\psi$ so there is $e' \in E$ with $\mathcal{T}, e', t \models \psi$. By the inductive hypothesis, $\psi \in \lambda(e', t)$. By G2 and G3, $\Diamond\psi \in \lambda(e, t)$. □

8 Decidability Before summarizing our decision procedure, we need to check a few facts about brief models.

Lemma 8.1 *φ has a brief model if and only if there is a set $A \subseteq 2^S$ such that*

1. *each $a \in A$ is a maximally Boolean consistent set;*
2. *no $F\alpha$ or $P\alpha$ is in any $a \in A$;*

3. *if* $\Box\alpha \in a$ *then for all* $b \in A$ *we have* $\alpha \in b$*;*
4. *if* $a \in A$ *and* $\Diamond\alpha \in a$ *then there is* $b \in A$ *such that* $\alpha \in b$*;*
5. *there is* $a \in A$ *with* $\varphi \in a$*.*

Proof: For the forward direction, say $(U, \{t_0\}, \emptyset, g), u_0, t_0 \models \varphi$. Let $A = \{\|u\| \mid u \in U\}$ where for each $u \in U$, $\|u\| = \{\alpha \in S \mid (U, \{t_0\}, \emptyset, g), u, t_0 \models \alpha\}$. Conversely, let $U = A$, $T = \{0\}$, $<$ be empty and define g to take $(a, 0)$ to $\{p \in \mathcal{L} \mid p \in a\}$. $\Box$

Since there are only a finite number of subsets of 2^S, it is straightforward to describe an algorithm which decides whether φ has a brief model. We are now able to present a deciding algorithm for the satisfiability of formulas in the logic **FP**$\Diamond$.

Theorem 8.2 *Satisfiability of* **FP**$\Diamond$ *formulas is decidable.*

Proof: Suppose we are to check whether φ has a model. First use the algorithm above to see whether φ has a brief model. If not, then proceed as follows.

Suppose that the complexity (length) of φ is n. Thus the size of S is at most $2n$. There are at most 2^{2n} maximally Boolean consistent subsets of S. There are at most $2^{2^{4n}}$ different φ-segments (X, μ, ν)—up to difference in X. Using an idea of Pratt from [16], we can start with this set of all different φ-segments and gradually whittle it down to an SSS for φ by throwing away incoherent or incurable segments. Eventually we lose any witness for φ in the set of segments (in which case there is no SSS for φ) or the process reaches a fixed set which will be an SSS for φ.

If there is an SSS for φ then φ has a model. If there is no SSS for φ nor a brief model of φ then φ is not satisfiable. $\Box$

Of course, this may not be the most efficient decision procedure possible. In [5], a related logic (the temporal-knowledge logic of one agent who knows the time but neither learns nor forgets) over natural numbers time, is shown to have an EXPSPACE-hard decision problem. It might be possible to use the techniques of these authors to show that our decision problem is also EXPSPACE-hard, but finding matching lower and upper bounds on its complexity will have to remain as an open problem.

9 No finite model property In this section we will show that the logic **FP**$\Diamond$ does not have the finite model property. This will show that the mosaic method for proving decidability is stronger than the traditional method of using the finite model property. Having to use other decidability techniques is not entirely new to temporal logic—for example, many temporal logics over the integers or natural numbers have no finite models and we usually resort to automata techniques to show their decidability. However, mosaics look even more like finite models than automata do and so it is worth presenting the result of this section to demonstrate a clear distinction between the techniques.

First we define our terms. We have already met (valued) rectangular structures: that is, $(U, T, <, g)$ in which U is a nonempty set, $(T, <)$ is a nonempty linear irreflexive order and $g : U \times T \longrightarrow 2^{\mathcal{L}}$. Recall that a formula φ of **FP**$\Diamond$ is a *validity* if and only if for every rectangular structure $(U, T, <, g)$, for every $u \in U$, for every $t \in T$, we have $(U, T, <, g), u, t \models \varphi$.

Now we introduce more general structures. Say that $\mathcal{M} = (W, \pi, \rho, \sigma, g)$ is a (valued) Kripke structure (for **FP**$\Diamond$) if and only if W is a nonempty set, $\pi, \rho, \sigma \subseteq W \times W$ and $g : W \longrightarrow 2^{\mathcal{L}}$. We define truth of **FP**$\Diamond$-formulas at worlds in Kripke structures by using π, ρ, and σ as the accessibility relations for F, P, and $\Diamond$ in the usual way. For example, we have clauses

1. $\mathcal{M}, w \models p$ iff $p \in g(w)$;
2. $\mathcal{M}, w \models F\alpha$ iff there is $w' \in W$ such that $(w, w') \in \pi$ and $\mathcal{M}, w' \models \alpha$;
3. $\mathcal{M}, w \models \Diamond\alpha$ iff there is $w' \in W$ such that $(w, w') \in \sigma$ and $\mathcal{M}, w' \models \alpha$.

We say that $\mathcal{M}$ is finite if and only if W is. We say that the Kripke structure $\mathcal{M}$ is a *model* of **FP**$\Diamond$ if and only if for every validity φ of **FP**$\Diamond$ and for every $w \in W$ we have $\mathcal{M}, w \models \varphi$. We say that the logic **FP**$\Diamond$ has the finite model property if and only if for every formula φ of **FP**$\Diamond$ which is not a validity there is a finite Kripke structure $\mathcal{M}$ which is a model of **FP**$\Diamond$ but for which there is $w \in W$ with $\mathcal{M}, w \models \neg\varphi$. Note that since we can easily axiomatize **FP**$\Diamond$ using an IRR-style rule (see the long version [18]) then it would follow that **FP**$\Diamond$ is decidable if we could show that **FP**$\Diamond$ has the finite model property. Now we show the following theorem.

Theorem 9.1 **FP**$\Diamond$ *does not have the finite model property.*

Proof: Let $\psi_0 = q \wedge H\neg q$ and for each $i \geq 0$, let $\psi_{i+1} = P\psi_i \wedge HH\neg\psi_i$. In fact we will show that the formula

$$\varphi = F\top \wedge G\Diamond(\psi_0 \wedge F\psi_1)$$

does not have a finite Kripke model in which all the validities of **FP**$\Diamond$ are valid. Note that φ is satisfiable in a rectangular structure so $\neg\varphi$ is not a validity.

Suppose for contradiction that $\mathcal{M} = (W, \pi, \rho, \sigma, g)$ is a finite Kripke model of **FP**$\Diamond$ such that there is $w \in W$ with $\mathcal{M}, w \models \varphi$. First, it is clear that there is $w_0 \in W$ with $\mathcal{M}, w_0 \models \psi_0$. We also have $(w, w_0) \in \pi \circ \sigma$. We will show by induction that for each $i \geq 0$ there is some $w_i \in W$ with

1. $\mathcal{M}, w_i \models \psi_i$ and
2. $(w, w_i) \in (\pi \circ \sigma \circ \pi^i)$.

To help we will define $\chi_0 = \Diamond(\psi_0 \wedge F\psi_1)$ and for each $i \geq 0$ define $\chi_{i+1} = G\chi_i$. Note that $G\chi_0 \longrightarrow G\Box\chi_i$ is a validity of**FP**$\Diamond$ for any $i \geq 0$.

We have already established the inductive property for $i = 0$. Now suppose that it is true for some $i \geq 0$. Since $\mathcal{M}$ is a model of **FP**$\Diamond$ we have $\mathcal{M}, w \models G\Box\chi_i$. Since $(w, w_i) \in (\pi \circ \sigma \circ \pi^i)$ we thus have $\mathcal{M}, w_i \models \Diamond(\psi_0 \wedge F\psi_1)$ as well as the assumed $\mathcal{M}, w_i \models \psi_i$. Another validity of **FP**$\Diamond$ is

$$(\Diamond(\psi_0 \wedge F\psi_1) \wedge \psi_i) \longrightarrow F\psi_{i+1}.$$

Thus $\mathcal{M}, w_i \models F\psi_{i+1}$. This gives us the required $w_{i+1} \in W$.

Now that we have our set $\{w_i | i \geq 0\}$ of elements of W we are finished when we show that they are all distinct for $i > 0$. Assume for contradiction that $0 < i < j$ but $w_i = w_j$. Thus $\mathcal{M}, w_i \models P\psi_{j-1} \wedge HH\neg\psi_{i-1}$. Thus there is $v \in W$ with $\mathcal{M}, v \models \psi_{j-1} \wedge H\neg\psi_{i-1}$. However, it is a validity of **FP**$\Diamond$ that $\psi_{j-1} \longrightarrow P\psi_{i-1}$ so we have our contradiction. $\Box$

10 Conclusion Although the logic **FP**$\Diamond$ is not new, we have been able, using the new mosaic techniques, to provide the first decidability result for it. As we have also shown that the logic does not have the finite model property we can conclude that the mosaic method provides useful new tools for investigating modal and temporal logics.

As we have seen there are many related logics so it is very possible that applying the mosaic method in similar ways will provide a very fruitful research opportunity. One open problem which is very close to the problem considered in this paper concerns the use of the horizontal difference operator D where the semantic clause for D is

$$\mathcal{M}, u, t \models D\alpha \text{ iff there is } v \in U \text{ such that } v \neq u \text{ and } \mathcal{M}, v, t \models \alpha.$$

It seems that the addition of the D operator to our rectangular logic results in a subtly more complicated logic for which no decision procedure is known.

Another open problem closely connected with the results here concerns the complexity of the decision procedure. The algorithm presented in Section 8 is highly complex but it is very probable that much less complex procedures exist. We have briefly described a double exponentially complex procedure and mentioned some possible techniques for finding an EXPSPACE lower bound. An existing lower bound on the complexity is given by the recent result in Marx [13] in which it is shown (also using mosaic techniques) that the product logic of **S5** and **S5** is nexptime complete.

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Department of Computer Science
King’s College
Strand
London, WC2R 2LS
UNITED KINGDOM
email: *markr@dcs.kcl.ac.uk*