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Strong Normalization Theorem for a Constructive Arithmetic with Definition by Transfinite Recursion and Bar Induction

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Abstract We prove the strong normalization theorem for the natural deduction system for the constructive arithmetic **TRDB** (the system with **D**efinition by **T**ransfinite **R**ecursion and **B**ar induction), which was introduced by Yasugi and Hayashi. We also establish the consistency of this system, applying the strong normalization theorem.

1 Introduction The main result of this paper is the strong normalization theorem for the natural deduction system for the constructive arithmetic **TRDB**. This system is a renewal version of the system **ASOD** (**A**nalytic **S**ystem especially designed for **O**rdinal **D**iagrams) which was introduced by Yasugi in [3]. In **ASOD**, Yasugi succeeded in constructing an accessibility proof of ordinal diagrams (see [3] and [4]). Yasugi and Hayashi [5] also have studied functional interpretations of proofs formalized in **TRDB** (see [3], [5], and [6]). For such studies, the normalizability of a proof formalized in **TRDB** is important.

The proof of the main result is based on [5] and on the proof of the strong normalization theorem for **HA** by Troelstra in [2]. For example, using *degrees* defined similarly to those in [5], we define *reducibility sets* similarly to the *strong validity predicate* in [2]. However, since a reducibility set in this paper consists of deductions whose consequences are *closed* formulas, there arises new difficulty in dealing with the reducibility of a deduction. The difficulty arises essentially from the inference rule: definition by transfinite recursion. We think that, in order to settle the difficulty, it is necessary to study a relation between the reducibility of a deduction Π (whose consequence is a closed formula) and that of a deduction Π' obtained from Π by substituting closed terms for free variables which are not eigenvariables. (See Lemma 3.19, Lemma 3.22 and Remark 3.21.) These lemmas are most crucial in our

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proof of the main result. Our proof would be applied to prove the strong normalization theorem for other systems with definition by transfinite recursion (and/or bar induction).

This paper consists of three sections: in Section 2, we define **TRDB** and degrees which give an order on **TRDB**-formulas; in Section 3, we prove the strong normalization theorem of reductions (of **TRDB**-deductions) defined in the same section; in Section 4, applying the strong normalization theorem, we establish the consistency of **TRDB** and prove the existence property and the disjunction property of **TRDB**.

Notation 1.1 In Section 2, we define **TRDB** as the system formalized by the natural deduction system.

1. Lower case alphabets $x, \ldots, a, \ldots, t, \ldots$ denote terms. In particular, a, b, c, x, y and z denote variables. Upper case alphabets $A, \ldots, A[t], \ldots$ denote formulas. Greek alphabets $\Pi, \Sigma, \Delta, \ldots$ denote (natural) deductions. $\vec{t}, \ldots \vec{x}, \ldots$ denote finite sets of terms.
2. Given a deduction Π, $Cnsq(\Pi)$ denotes the consequence of Π. $[A]$ denotes a live assumption of a deduction. We do not write explicitly the label of an assumption.
3. We define *subdeductions* of a deduction as follows: if Π is a deduction of the form

$$\dfrac{\begin{array}{c}\vdots\ \Pi_1\\ A_1\end{array}\quad\cdots\quad\begin{array}{c}\vdots\ \Pi_n\\ A_n\end{array}}{A}\ R\quad,$$

then the set consisting of Π and all subdeductions of Π_i $(i = 1, \ldots, n)$ forms the set of all subdeductions of Π.
4. Let Π be a deduction with $Cnsq(\Pi) = A$, and let Σ be a deduction having a live assumption of the form $[A]$. Then $\Sigma[\Pi/A]$ denotes the deduction obtained from Σ by substituting Π for the live assumption $[A]$.
5. We assume that variables in a deduction are denoted by different alphabets from each other so far as is possible. $\Pi_{[\vec{t}/\vec{x}]}$ denotes the deduction obtained from a deduction Π by substituting a term t_i for a free variable x_i in the consequence of Π $(i = 1, \ldots, n)$, where $\vec{x}$ denotes $x_1, \ldots, x_n$ and $\vec{t}$ denotes $t_1, \ldots, t_n$. $\Pi^{[\vec{s}/\vec{y}]}$ denotes the deduction obtained from Π by substituting a term s_i for a free variable y_i in a live assumption of Π $(i = 1, \ldots, m)$, where $\vec{y}$ denotes $y_1, \ldots, y_m$ and $\vec{s}$ denotes $s_1, \ldots, s_m$. For example, if Π is a deduction of the form

$$\dfrac{\dfrac{\dfrac{[\forall x(0 = 1 + x)]}{0 = 1 + z}\ \forall\text{-}E \qquad [(Sy > 0)]}{(0 = 1 + z) \wedge (Sy > 0)}\ \wedge\text{-}I}{0 = 1 + z}\ \wedge\text{-}E\quad,$$

then

$$\Pi_{[n/z]} = \qquad \dfrac{\dfrac{\dfrac{[\forall x(0 = 1 + x)]}{0 = 1 + n}\ \forall\text{-}E \qquad [(Sy > 0)]}{(0 = 1 + n) \wedge (Sy > 0)}\ \wedge\text{-}I}{0 = 1 + n}\ \wedge\text{-}E\quad,$$

$$\Pi^{[m/y]} = \cfrac{\cfrac{\cfrac{[\forall x(0=1+x)]}{0=1+z}\ \forall\text{-}E \qquad [(Sm>0)]}{(0=1+z)\wedge(Sm>0)}\ \wedge\text{-}I}{0=1+z}\ \wedge\text{-}E \ .$$

(We mostly follow [2], but there are some terminologies which are used in different context from those in [2]. Such terminologies are explicitly defined in this paper.)

***2* TRDB** In this section, we define the system **TRDB**, which is defined in [5] and [6]. This system, in particular the language of this, seems to be highly specialized. The reason is that **TRDB** is defined so that one can use the system directly for formalizing accessibility proofs and can construct functional interpretations of formalized accessibility proofs (see [3], [5], and [6]). However, in this paper, these special properties of **TRDB** are not important except the two inference rules: definition by transfinite recursion and bar induction. Therefore, the reader, who is interested not in accessibility proofs but in the strong normalization theorem for constructive arithmetics, may consider **TRDB** as **HA** with definition by transfinite recursion and bar induction. (However, the reader should notice the special logical symbol ρ, which is introduced only for technical reasons. See Remark 2.2 (5).)

In what follows, a word 'integer' means 'non-negative integer'.

Definition 2.1 Preceding to the definition of **TRDB**, we specify a primitive recursive well-ordered set I $(= (I, <_I))$. We identify the domain set I with the set of all integers.

Symbols

1. Countably many n-ary variables, where n is an integer
2. Function constants for primitive recursive functions in function parameters
3. A designated unary function constant c
4. Predicate constants for primitive recursive predicates in function parameters
5. A special predicate constant H
6. Logical symbols $\wedge, \vee, \supset, \forall$ and $\exists$
7. A special logical symbol ρ

Terms

1. Variables and function constants
2. If f is an n-ary term and if $t_1, t_2, \ldots, t_n$ are 0-ary terms, then $f(t_1, t_2, \ldots, t_n)$ is a 0-ary term. We often call 0-ary terms *number terms*.
3. If t is a number term and if $x_1, x_2, \ldots, x_n$ are number variables, then $\lambda x_1 \lambda x_2 \ldots \lambda x_n.t$ is an n-ary term, where λ is the lambda abstraction.

Formulas

1. If p is an n-ary predicate constant and $t_1, \ldots, t_n$ are appropriate terms, then $p(t_1, \ldots, t_n)$ is an atomic formula. In particular, if s and t are number terms, then $H(s, t)$ is an atomic formula.
2. If A and B are formulas, then $A \wedge B$, $A \vee B$, $A \supset B$, and $\forall x A$ are formulas, where x is a variable.

3. If A is a formula, then $\exists x A$ is a formula, where the variable x in A is 0-ary and it does not occur in any H in A.
4. Let $=$ denote a 2-ary predicate constant expressing the equality of integers; let 0 and 1 denote 0-ary function constants expressing the integers 0 and 1 respectively; and let $<_I$ denote a 2-ary predicate constant expressing the order of I. Then $\rho((j <_I i \supset H(j,s)) \wedge ((j <_I i \supset 0 = 1) \supset 0 = 1))$ is a formula, where i, j, and s are arbitrary number terms. We abbreviate this formula by $\rho(j <_I i; H(j,s))$.

Axioms and inference rules

1. **TRDB** contains inference rules of constructive logics formulated in natural deductions as usual: introduction rules $\wedge$-I, $\vee$-I, $\supset$-I, $\forall$-I, $\exists$-I; elimination rules $\wedge$-E, $\vee$-E, $\supset$-E, $\forall$-E, $\exists$-E, $\bot$-E. ('$\bot$' means $0 = 1$, see Definition 3.3 (3) in the next section.)
2. **TRDB** contains axioms and inference rules on constants of $\mathbf{PRA^2}$ (primitive recursive arithmetic with function variables). (See [2] and Girard [1].) We give these axioms and inference rules as follows.

   2.1. For any number terms t and t',

$$t = t, \qquad \frac{\begin{matrix}\vdots & \vdots\\ t = t' & P[t]\end{matrix}}{P[t']},$$

   where $P[t]$ denotes an atomic formula.

   2.2. For any number terms t, t', and t_i ($i = 1, \dots, n$),

$$\frac{\begin{matrix}\vdots\\ 0 = St\end{matrix}}{0 = 1}, \qquad \frac{\begin{matrix}\vdots\\ St = St'\end{matrix}}{t = t'}, \qquad I_i^n(t_1, \dots, t_i, \dots, t_n) = t_i,$$

   where S denotes a function constant expressing the successor function, and I_i^n ($i \leq n$) denotes a function constant expressing the n-place projection function.

   2.3. Let PRF_n denote the set of all n-ary primitive recursive functions in function parameters. Let F be the element of PRF_{n+1} obtained from a $G \in PRF_m$ and a $K_i \in PRF_n$ ($i = 1, \dots, m$) by functional composition:

$$F(x_1, \dots, x_n) = G(K_1(x_1, \dots, x_n), \dots, K_m(x_1, \dots, x_n)).$$

   If f, g, and k_i denote function constants expressing F, G, and K_i respectively, then for any number terms t_i ($i = 1, \dots, n$), u_j ($j = 1, \dots, m$), and v,

$$\frac{\begin{matrix}\vdots & \vdots & & \vdots\\ g(u_1, \dots, u_m) = v & k_1(t_1, \dots, t_n) = u_1 & \cdots & k_m(t_1, \dots, t_n) = u_m\end{matrix}}{f(t_1, \dots, t_n) = v}.$$

   2.4. Let F be the element of PRF_{n+1} obtained from a $K \in PRF_n$ and a $G \in PRF_{n+2}$ by primitive recursion:

$$F(x_1, \dots, x_n, 0) = K(x_1, \dots, x_n),$$

$$F(x_1, \dots, x_n, Sx) = G(x_1, \dots, x_n, x, F(x_1, \dots, x_n, x)).$$

If f, k, and g denote function constants expressing F, K, and G respectively, then for any number terms t_i ($i = 1, \dots, n$), t, u, and v,

$$\frac{\begin{array}{c}\vdots\\ k(t_1, \dots, t_n) = u\end{array}}{f(t_1, \dots, t_n, 0) = u}\ , \qquad \frac{\begin{array}{c}\vdots\\ f(t_1, \dots, t_n, t) = u\end{array} \quad \begin{array}{c}\vdots\\ g(t_1, \dots, t_n, t, u) = v\end{array}}{f(t_1, \dots, t_n, St) = v}\ .$$

2.5. If f denotes a function constant expressing a characteristic function of an n-ary primitive recursive predicate in function parameters expressed by a predicate constant p, then for any number terms t_i ($i = 1, \dots, n$),

$$\frac{\begin{array}{c}\vdots\\ f(t_1, \dots, t_n) = 0\end{array}}{p(t_1, \dots, t_n)}\ , \qquad \frac{\begin{array}{c}\vdots\\ p(t_1, \dots, t_n)\end{array}}{f(t_1, \dots, t_n) = 0}\ .$$

3. Well-foundedness of the set I
4. Monotone and elementary bar induction: proceeding to the definition of this inference rule, we give definitions as the following paragraphs.

   Let $R[a]$ be a formula, where a is a 0-ary variable, and let $R[a]$ contain neither any quantifier, any H, nor any variable except a. Such a formula is said to be *elementary*.

   We consider a bijective function Φ from finite sequences of integers onto integers. It is well known that Φ can be defined primitive recursively. We fix such a bijection Φ which is defined primitive recursively.

   Let $R[a]$ be an elementary formula. Then $R[a]$ is said to be *monotone* if $R[a]$ satisfies the following conditions (i) and (ii).

   (i) For any infinite sequence f, there exists an integer n such that $R[f\lceil n]$ holds, where $f\lceil n$ denotes a number term expressing the integer assigned to the finite sequence $\langle f(0), f(1), \dots, f(n-1)\rangle$ by Φ.

   (ii) For any infinite sequence f and for any integer n, $R[f\lceil n]$ implies $R[f\lceil m]$ for any $m > n$.

   Let $*$ denote a 2-ary primitive recursive function constant satisfying the following: $a * b$ expresses the integer assigned to $\langle a_1, \dots, a_n, b\rangle$ by Φ, where $\langle a_1, \dots, a_n\rangle$ is the finite sequence to which Φ assigns a. We give an inference rule called a *BI-rule* as follows:

$$\frac{\begin{array}{c}\vdots\\ \forall z(R[z] \supset A[z])\end{array} \quad \begin{array}{c}\vdots\\ \forall z(\forall x A[z * x] \supset A[z])\end{array}}{A[t]}\ BI\ ,$$

   where $A[a]$ is an arbitrary formula, $R[a]$ is an arbitrary monotone formula and t is an arbitrary number term.
5. Definition by transfinite recursion $TRD(G, I)$: Preceding to the definition of this inference rule, we fix a formula $G[a, b, H[a]]$, where a and b are 0-ary free variables, so that G satisfies the following conditions.

(i) No free variable occurs in G except a or b.

(ii) No H occurs in G except in a subformula of the form $\rho(j <_I a; H(j, s))$, where j and s are some number terms.

(iii) No scope of an $\exists$-quantifier in G contains an H.

Here $G[a, b, H[a]]$ is understood to denote a formula substituted for an abstract formula $H[a]$ of the form

$$\{x, y\}\rho((x <_I a; H(x, y))).$$

We give inference rules called *H-rules* as follows:

$$\frac{\begin{array}{c}\vdots\\ G[i, t, H[i]]\end{array}}{H(i, t)}\ H\text{-}I\ , \qquad \frac{\begin{array}{c}\vdots\\ H(i, t)\end{array}}{G[i, t, H[i]]}\ H\text{-}E\ ,$$

where i and t are arbitrary number terms.

6. ρ-elimination:

$$\frac{\begin{array}{c}\vdots\\ \rho(j <_I i; H(j, s))\end{array}}{(j <_I i \supset H(j, s)) \wedge ((j <_I i \supset 0 = 1) \supset 0 = 1)}\ \rho\text{-}E\ ,$$

where i, j, and s are arbitrary number terms.

Note that ρ-introduction is not defined as an inference rule in **TRDB**.

Remark 2.2

1. **TRDB** is specified with a certain formula G and a certain primitive recursive ordered set I. So, **TRDB** should be written as **TRDB**(G, I) to be precise.
2. Let **TRDB**$^-$ be the system obtained from **TRDB** by removing the inference rule $TRG(G, I)$ and the axiom for the well-foundedness of I. Since the bar induction defined in Definition 2.1 (4) implies the mathematical induction, every formula provable in **HA** is provable in **TRDB**$^-$.
3. In this paper, we confine ourselves to the case where the order type of I is smaller than ε_0. So we can remove the axiom for the well-foundedness of I, since it is provable in **HA**.
4. When a term t is substituted for a variable x in a formula, t must have the same arity as that of x. In what follows, we assume that the above condition is satisfied whenever one considers such a substitution.
5. $\rho(j <_I i; H(j, s))$ and ρ-E entail the following restriction: *no* deduction Π contains an inference rule

$$\frac{\begin{array}{c}\vdots\\ j <_I i \supset H(j, s)\end{array} \qquad \begin{array}{c}\vdots\\ (j <_I i \supset 0 = 1) \supset 0 = 1\end{array}}{\begin{array}{c}(j <_I i \supset H(j, s)) \wedge ((j <_I i \supset 0 = 1) \supset 0 = 1)\\ \vdots\end{array}}\ \wedge\text{-}I\ ,$$

where the conclusion of the $\wedge$-I occurs in a formula $G[i, t, H[i]]$ contained in Π. The reason why $\rho(j <_I i; H(j, s))$ and ρ-E are introduced in the definition of **TRDB** is only that **TRDB** should satisfy the above restriction. In fact, except the restriction, there is no essential difference between **TRDB** and that without $\rho(j <_I i; H(j,s))$ and ρ-E. The restriction essentially has an effect only on Proposition 2.4 (2).

We shall define a degree of a formula of **TRDB**. The definition is taken after the degree of a type-form in [5].

Definition 2.3

1. For a primitive recursive well-ordered set I $(= (I, <_I))$ which we assume in defining **TRDB**, we define $I^* = (I^*, <^*)$ as follows:

$$I^{\sim} = \{i^{\sim}\ ;\ i \in I\}\ ;\ I^* = I \cup I^{\sim} \cup \{\infty\}\ ;\ i <^* i^{\sim} <^* j <^* \infty \text{ when } i <_I j.$$

   Moreover, we define $I_* = (I_*, <_*)$ so that $I_* = \omega^{I^*}$, where we identify I^* with the ordinal type of itself.

2. Let A be an H-formula, that is, a formula which contains the predicate constant H. Let $\overline{H}$ denote an occurrence of the predicate constant H in A. Then we define $r(\overline{H}; A)$ $(\in I^*)$ as the following conditions.

   (i) Suppose that $\overline{H}$ is an occurrence in a subformula of A which is of the form $\rho(j <_I i; H(j,s))$.

   (i-1) If i is closed, then $r(\overline{H}; A) = i$;

   (i-2) If i contains a variable, then $r(\overline{H}; A) = \infty$.

   (ii) Suppose that $\overline{H}$ occurs in a subformula of the form $H(j,s)$ and that $\overline{H}$ does not satisfy (i).

   (ii-1) If j is closed, then $r(\overline{H}; A) = j^{\sim}$;

   (ii-2) If j contains a variable, then $r(\overline{H}; A) = \infty$.

3. Let A_0 be a formula, and let $\overline{A}$ be an occurrence of a subformula in A_0. Then we define the degree of $\overline{A}$ in A_0 denoted by $d(\overline{A}; A_0)$ $(\in I_*)$ as follows:

   (i) if $\overline{A}$ is an atomic formula except an H-formula, $d(\overline{A}; A_0) = 1$;

   (ii) $d(\overline{B \wedge C}; A_0) = \max(d(\overline{B}; A_0), d(\overline{C}; A_0)) + 1$, where $\overline{B}$ and $\overline{C}$ are occurrences in $\overline{B \wedge C}$;

   (iii) $d(\overline{B \vee C}; A_0) = \max(d(\overline{B}; A_0), d(\overline{C}; A_0)) + 1$, where $\overline{B}$ and $\overline{C}$ are occurrences in $\overline{B \vee C}$;

   (iv) $d(\overline{B \supset C}; A_0) = \max(d(\overline{B}; A_0), d(\overline{C}; A_0)) + 1$, where $\overline{B}$ and $\overline{C}$ are occurrences in $\overline{B \supset C}$;

   (v) $d(\overline{\forall x B[x]}; A_0) = d(\overline{B[x]}; A_0) + 1$, where $\overline{B[x]}$ is an occurrence in $\overline{\forall x B[x]}$;

   (vi) $d(\overline{\exists x B[x]}; A_0) = d(\overline{B[x]}; A_0) + 1$, where $\overline{B[x]}$ is an occurrence in $\overline{\exists x B[x]}$;

   (vii) $d(\overline{\rho(j <_I i; H(j,s))}; A_0) =$
   $$d(\overline{(j <_I i \supset H(j,s)) \wedge ((j <_I i \supset 0 = 1) \supset 0 = 1)}; A_0) + 1,$$
   where $\overline{(j <_I i \supset H(j,s)) \wedge ((j <_I i \supset 0 = 1) \supset 0 = 1)}$ is an occurrence in $\overline{\rho(j <_I i; H(j,s))}$;

   (viii) $d(\overline{H(j,s)}; A_0) = \omega^{r(\overline{H}; A_0)}$, where $\overline{H}$ is an occurrence in $\overline{H(j,s)}$.

   Put $d(A_0) = d(\overline{A_0}; A_0)$, and call this the *degree* of A_0.

Proposition 2.4

1. *Let $G[i, t, H[i]]$ be the formula which determines the axiom $TRD(G, I)$. If i is closed, then $d(G[i, t, H[i]]) <_* d(H(i, t))$.*
2. *Let A be a closed formula derived from a formula B by an introduction rule. Then $d(B) <_* d(A)$.*

Proof:

1. By Definitions 2.1 and 2.3, $d(H(i, t)) = \omega^{i^\sim}$ and $d(G[i, t, H[i]]) = \omega^i \cdot m + n$ for some integers m and n. Thus $d(G[i, t, H[i]]) <_* d(H(i, t))$ holds by Definition 2.3.
2. (i) Suppose that A $(= B \wedge C)$ is derived from closed formulas B and C by a $\wedge$-I. By Definition 2.3 (2), $r(\overline{H}; B) = r(\overline{H}; A)$ for any $\overline{H}$ in B, and $r(\overline{H}; C) = r(\overline{H}; A)$ for any $\overline{H}$ in C. Therefore, by Definition 2.3 (3), $d(B)$ and $d(C)$ are smaller than $d(A)$.
   (ii) All other cases can be proved the same way as (i), using the above (1). □

***3 Strong normalization theorem for* TRDB** In this section, we define reductions of **TRDB**-deductions, and we show that every deduction is strongly normalizable for these reductions.

Definition 3.1 A deduction Π is said to be *elementary* if Π has neither any live assumption, any $\forall$-rule, any $\exists$-rule, any H-rule, any ρ-E, any BI-rule, nor any free variable.

Remark 3.2 For any elementary formula R which is closed and true, there exists an elementary deduction whose consequence is R. For any elementary formula R which is closed and true, we fix an elementary deduction Θ_R whose consequence is R.

Definition 3.3 For any deduction Π, we define the contraction of Π in the following (1)–(6). We let '$\Pi \longrightarrow \Sigma$' mean that Π is contracted to Σ.

1. If Π is an axiom or a live assumption, then Π is not contracted.
2. If Π has a logical inference rule as the last inference rule, then we follow Definition 4.1.3. in [2], that is, we give the contraction of Π as follows.
   2.1. Proper contraction:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \Pi_1\\ A_1\end{matrix}\qquad\begin{matrix}\vdots\ \Pi_2\\ A_2\end{matrix}}{A_1\wedge A_2}\wedge\text{-}I}{A_i}\wedge\text{-}E\ \to\ \begin{matrix}\vdots\ \Pi_i\\ A_i\end{matrix}\ ;\qquad \dfrac{\dfrac{\begin{matrix}[\not A]\\ \vdots\ \Delta\\ B\end{matrix}}{A\supset B}\supset\text{-}I\qquad\begin{matrix}\vdots\ \Gamma\\ A\end{matrix}}{B}\supset\text{-}E\ \to\ \begin{matrix}\vdots\ \Gamma\\ A\\ \vdots\ \Delta\\ B\end{matrix}\ ;\qquad \dfrac{\dfrac{\begin{matrix}\vdots\ \Gamma\\ A[a]\end{matrix}}{\forall x A[x]}\forall\text{-}I}{A[t]}\forall\text{-}E\ \to\ \begin{matrix}\vdots\ \Gamma_{[t/a]}\\ A[t]\end{matrix}\ ;$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \Delta\\ A_i\end{matrix}}{A_1\vee A_2}\vee\text{-}I\qquad\begin{matrix}[\not A_1]\\ \vdots\ \Gamma_1\\ B\end{matrix}\qquad\begin{matrix}[\not A_2]\\ \vdots\ \Gamma_2\\ B\end{matrix}}{B}\vee\text{-}E\ \to\ \begin{matrix}\vdots\ \Delta\\ A_i\\ \vdots\ \Gamma_i\\ B\end{matrix}\ ;\qquad \dfrac{\dfrac{\begin{matrix}\vdots\ \Delta\\ A[t]\end{matrix}}{\exists x A[x]}\exists\text{-}I\qquad\begin{matrix}[\not A[a]]\\ \vdots\ \Gamma\\ B\end{matrix}}{B}\exists\text{-}E\ \to\ \begin{matrix}\vdots\ \Delta\\ A[t]\\ \vdots\ \Gamma^{[t/a]}\\ B\end{matrix}\ .$$

Here $i = 1$ or 2.

2.2. Permutative contraction:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\Delta\\ A_1\vee A_2\end{matrix}\quad\begin{matrix}[A_1]\\ \vdots\,\Gamma_1\\ B\end{matrix}\quad\begin{matrix}[A_2]\\ \vdots\,\Gamma_2\\ B\end{matrix}}{B}\vee\text{-}E\quad\begin{matrix}\vdots\,\Xi_1\\ C_1\end{matrix}\ \cdots\ \begin{matrix}\vdots\,\Xi_n\\ C_n\end{matrix}}{D}\,R\quad\to$$

$$\dfrac{\begin{matrix}\vdots\,\Delta\\ A_1\vee A_2\end{matrix}\quad\dfrac{\begin{matrix}[A_1]\\ \vdots\,\Gamma_1\\ B\end{matrix}\quad\begin{matrix}\vdots\,\Xi_1\\ C_1\end{matrix}\ \cdots\ \begin{matrix}\vdots\,\Xi_n\\ C_n\end{matrix}}{D}\,R\quad\dfrac{\begin{matrix}[A_2]\\ \vdots\,\Gamma_2\\ B\end{matrix}\quad\begin{matrix}\vdots\,\Xi_1\\ C_1\end{matrix}\ \cdots\ \begin{matrix}\vdots\,\Xi_n\\ C_n\end{matrix}}{D}\,R}{D}\vee\text{-}E\quad;$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\Delta\\ \exists x A[x]\end{matrix}\quad\begin{matrix}[A[a]]\\ \vdots\,\Gamma\\ B\end{matrix}}{B}\exists\text{-}E\quad\begin{matrix}\vdots\,\Xi_1\\ C_1\end{matrix}\ \cdots\ \begin{matrix}\vdots\,\Xi_n\\ C_n\end{matrix}}{D}\,R\quad\to\quad\dfrac{\begin{matrix}\vdots\,\Delta\\ \exists x A[x]\end{matrix}\quad\dfrac{\begin{matrix}[A[a]]\\ \vdots\,\Gamma\\ B\end{matrix}\quad\begin{matrix}\vdots\,\Xi_1\\ C_1\end{matrix}\ \cdots\ \begin{matrix}\vdots\,\Xi_n\\ C_n\end{matrix}}{D}\,R}{D}\exists\text{-}E\quad.$$

Here R is an elimination rule; B is the major premise of R; $n = 0, 1$ or 2.

2.3. Immediate simplification:

$$\dfrac{\begin{matrix}\vdots\,\Delta\\ A_1\vee A_2\end{matrix}\quad\begin{matrix}\vdots\,\Gamma_1\\ B\end{matrix}\quad\begin{matrix}\vdots\,\Gamma_2\\ B\end{matrix}}{B}\vee\text{-}E\quad\to\quad\begin{matrix}\vdots\,\Gamma_i\\ B\end{matrix}\ ;\qquad\dfrac{\begin{matrix}\vdots\,\Delta\\ \exists x B[x]\end{matrix}\quad\begin{matrix}\vdots\,\Gamma\\ B\end{matrix}}{A}\exists\text{-}E\quad\to\quad\begin{matrix}\vdots\,\Gamma\\ B\end{matrix}\ .$$

Here Γ_i and Γ have no live assumption which is discharged by the last inference rule of Π. Such an elimination is said to be *redundant*.

3. If Π has a $\bot$-E as the last inference rule, then Π is contracted as follows:

$$\dfrac{\begin{matrix}\vdots\,\Delta\\ 0=1\end{matrix}}{A\wedge B}\bot\text{-}E\ \to\ \dfrac{\dfrac{\begin{matrix}\vdots\,\Delta\\ 0=1\end{matrix}}{A}\bot\text{-}E\quad\dfrac{\begin{matrix}\vdots\,\Delta\\ 0=1\end{matrix}}{B}\bot\text{-}E}{A\wedge B}\wedge\text{-}I\ ,\qquad\dfrac{\begin{matrix}\vdots\,\Delta\\ 0=1\end{matrix}}{A\vee B}\bot\text{-}E\ \to\ \dfrac{\dfrac{\begin{matrix}\vdots\,\Delta\\ 0=1\end{matrix}}{A}\bot\text{-}E}{A\vee B}\vee\text{-}I\ ,$$

$$\dfrac{\begin{matrix}\vdots\,\Delta\\ 0=1\end{matrix}}{A\supset B}\bot\text{-}E\ \to\ \dfrac{\dfrac{\begin{matrix}\vdots\,\Delta\\ 0=1\end{matrix}}{B}\bot\text{-}E}{A\supset B}\supset\text{-}I\ ,\quad\dfrac{\begin{matrix}\vdots\,\Delta\\ 0=1\end{matrix}}{\forall x A[x]}\bot\text{-}E\ \to\ \dfrac{\dfrac{\begin{matrix}\vdots\,\Delta\\ 0=1\end{matrix}}{A[x]}\bot\text{-}E}{\forall x A[x]}\forall\text{-}I\ ,\quad\dfrac{\begin{matrix}\vdots\,\Delta\\ 0=1\end{matrix}}{\exists x A[x]}\bot\text{-}E\ \to\ \dfrac{\dfrac{\begin{matrix}\vdots\,\Delta\\ 0=1\end{matrix}}{A[0]}\bot\text{-}E}{\exists x A[x]}\exists\text{-}I\ .$$

4. If Π has an H-E as the last inference rule, then we give the contraction of Π as follows:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\Delta\\ G[i,t,H[i]]\end{matrix}}{H(i,t)}H\text{-}I}{G[i,t,H[i]]}H\text{-}E\qquad\to\qquad\begin{matrix}\vdots\,\Delta\\ G[i,t,H[i]].\end{matrix}$$

5. If Π has a *BI*-rule as the last inference rule and if Π is of the form

$$\dfrac{\begin{matrix}\vdots\,\Delta\\ \forall z(R[z]\supset A[z])\end{matrix}\quad\begin{matrix}\vdots\,\Gamma\\ \forall z(\forall x A[z*x]\supset A[z])\end{matrix}}{A[t]}\,BI\ ,$$

where t is closed, then Π satisfies the following properties (5.1) and (5.2).

5.1. If $R[t]$ is true, then

$$\Pi \quad \to \quad \dfrac{\dfrac{\begin{matrix}\vdots\ \Delta \\ \forall z(R[z] \supset A[z])\end{matrix}}{R[t] \supset A[t]}\ \forall\text{-}E \qquad \begin{matrix}\vdots\ \Theta_{R[t]} \\ R[t]\end{matrix}}{A[t]}\ \supset\text{-}E,$$

where $\Theta_{R[t]}$ is the deduction fixed as in Remark 3.2.

5.2. If $R[t]$ is false, then

$$\Pi \quad \to \quad \dfrac{\dfrac{\begin{matrix}\vdots\ \Gamma \\ \forall z(\forall x A[z * x] \supset A[z])\end{matrix}}{\forall x A[t * x] \supset A[t]}\ \forall\text{-}E \qquad \dfrac{\dfrac{\begin{matrix}\vdots\ \Delta \\ \forall z(R[z] \supset A[z])\end{matrix} \quad \begin{matrix}\vdots\ \Gamma \\ \forall z(\forall x A[z * x] \supset A[z])\end{matrix}}{A[t * a]}\ BI}{\forall x A[t * x]}\ \forall\text{-}I}{A[t]}\ \supset\text{-}E.$$

6. Otherwise, any deduction cannot be contracted.

Definition 3.4

1. Suppose that Γ is a subdeduction of a deduction Π, Δ is a deduction with $\Gamma \to \Delta$, and that Σ is the deduction obtained from Π by replacing Γ by Δ. Then we say that Π is (*one-step-*) *reduced* to Σ. The last inference rule of the above deduction Γ is called the *reduction point* of $\Pi \rightsquigarrow \Sigma$, where $\Pi \rightsquigarrow \Sigma$ means that Π is one-step-reduced to Σ.
2. If there exists a finite sequence of deductions such that $\Pi = \Pi_0 \rightsquigarrow \cdots \rightsquigarrow \Pi_n = \Sigma$, then we say that Π is reduced to Σ. $\Pi \rightsquigarrow\!\!\rightsquigarrow \Sigma$ means that Π is reduced to Σ or $\Pi = \Sigma$. If $\{\Pi_n\}_{n<M}$ $(0 < M \leq \omega)$ is a sequence such that $\Pi_0 = \Pi$ and $\forall n\ (n + 1 < M \Longrightarrow \Pi_n \rightsquigarrow \Pi_{n+1})$, then we call this sequence a *reduction sequence* of Π.

Definition 3.5 A deduction Π is said to be *strongly normalizable* if every reduction sequence of Π is finite. $\Pi \in SN$ means that Π is strongly normalizable. If there exists a deduction Σ such that $\Pi \rightsquigarrow\!\!\rightsquigarrow \Sigma$ and if there is no deduction to which Σ is reduced, then we call Σ a *normal form* of Π.

We will subsequently prove the strong normalizability in **TRDB**.

Theorem 3.6 *Any deduction in* **TRDB** *is strongly normalizable.*

Preceding the proof, we give definitions and results. Using induction on the structure of a deduction, we establish the following definition.

Definition 3.7 For any deduction Π, Π is said to be *closed* if Π satisfies the following conditions.

1. If Π consists only of a live assumption $[A]$ or an axiom A, then A is a closed formula.
2. If Π is of the form

$$\dfrac{\begin{matrix}\vdots\ \Sigma \\ A[a]\end{matrix}}{\forall x A[x]}\ \forall\text{-}I\ ,$$

then $\begin{array}{c}\vdots\ \Sigma_{[t/a]}\\ A[t]\end{array}$ is closed for any closed term t.

3. If Π is of the form

$$\dfrac{\begin{array}{c}\vdots\ \Sigma\\ \exists x B[x]\end{array}\quad\begin{array}{c}[B[b]]\\ \vdots\ \Delta\\ A\end{array}}{A}\ \exists\text{-}E\ ,$$

then Σ is closed and $\begin{array}{c}[B[t]]\\ \vdots\ \Delta^{[t/b]}\\ A\end{array}$ is closed for any closed term t.

4. If Π is not of the form in (1) – (3) and if Π is of the form

$$\dfrac{\begin{array}{c}\vdots\ \Pi_1\\ A_1\end{array}\quad\cdots\quad\begin{array}{c}\vdots\ \Pi_n\\ A_n\end{array}}{A}\ R\ ,$$

then A is a closed formula and Π_i is a closed deduction for any $i \leq n$.

Definition 3.8 Let Π be a deduction, and let a be a free variable contained in Π. If a is not an eigenvariable for any inference rule in Π, then a is said to be *strictly free* in Π.

Lemma 3.9

1. *If Π is a closed deduction, then there is not any strictly free variable in Π, in particular, $Cnsq(\Pi)$ and all live assumptions of Π are closed formulas.*
2. *For any deduction Π, there exists a closed deduction $\overline{\Pi}$ obtained from Π by substituting suitable closed terms for all strictly free variables in Π. We call $\overline{\Pi}$ a closure of Π.*
3. *If Π and Σ are closed deductions, and if Π has a live assumption which is of the form $[Cnsq(\Sigma)]$, then $\Pi[\Sigma/Cnsq(\Sigma)]$ is a closed deduction, where $\Pi[\Sigma/Cnsq(\Sigma)]$ is the deduction obtained from Π by substituting Σ for the live assumption $[Cnsq(\Sigma)]$.*
4. *If Π is closed and $\Pi \rightsquigarrow \Sigma$, then Σ is closed.*

Proof: Using induction on the structure of Π, this lemma can be proved easily. □

Lemma 3.10

1. *Let Π and Σ be deductions with $\Pi \rightsquigarrow \Sigma$. Then any strictly free variable in Π is not an eigenvariable for any inference rule in Σ.*
2. *Let Π and Σ be deductions with $\Pi \rightsquigarrow \Sigma$, let a be a strictly free variable in Π, let $\Pi[t/a]$ be the deduction obtained from Π by substituting a term t for a, and let Σ^* be the deduction to which $\Pi[t/a]$ is reduced by the same reduction as $\Pi \rightsquigarrow \Sigma$. Then Σ^* is the deduction obtained from Σ by substituting t for a.*
3. *Let Π be a deduction, let a be a strictly free variable in Π, and let $\Pi[t/a]$ be the deduction obtained from Π by substituting a term t for a. If $\Pi[t/a] \in SN$, then $\Pi \in SN$.*

Proof: (1) is trivial. (2) can be proved easily by induction on the structure of Π. Let $\{\Pi_i\}_{i<M}$ be a reduction sequence of Π. By (2) in this lemma, there exists a reduction sequence $\{\Pi_i[t/a]\}_{i<M}$ where $\Pi_i[t/a]$ is the deduction obtained from Π_i by substituting the term t for the free variable a. So, M is finite. □

Lemma 3.11 *If every closed deduction is strongly normalizable, then so is every deduction.*

Proof: Let Π be a deduction. We prove $\Pi \in SN$ by induction on the number k of strictly free variables in Π.

(i) If $k = 0$, then $\Pi \in SN$ since Π is a closed deduction by Lemma 3.9 (2).

(ii) Suppose $k > 0$. Let $\Pi[t/a]$ be the deduction obtained from Π by substituting a closed term t for a strictly free variable a in Π. By the induction hypothesis, $\Pi[t/a] \in SN$. Therefore, $\Pi \in SN$ by Lemma 3.10 (3). □

Definition 3.12 (Troelstra [2]) Let $\{S_i\}_{i\leq n}$ be a sequence of (occurrences of) formulas in a deduction Π. This sequence is called a *segment* if it satisfies the following conditions.

(i) S_1 is not the conclusion of an $\exists$-E or a $\vee$-E.

(ii) If $i < n$, then S_i is the minor premise of an $\exists$-E or a $\vee$-E whose conclusion is S_{i+1}.

(iii) S_n is not the minor premise of an $\exists$-E or a $\vee$-E.

If there is a segment $\{S_i\}_{i\leq n}$ such that $S_n = Cnsq(\Pi)$, we call this an *end segment* of Π. If a deduction Π does not end with an I-rule, Π is said to be *neutral*.

We define a *reducibility* for a deduction, referring to Definition 4.1.9. in [2] and Definition 5.2. in [5].

Definition 3.13 For any closed formula A, we define a *reducibility set* $Red(A)$ which is a set of deductions whose consequence is A. The definition of $\Pi \in Red(A)$ is primarily given by transfinite induction on the degree of A; for deductions Π with $A = Cnsq(\Pi)$ of fixed complexity, the definition of $\Pi \in Red(A)$ takes the form of a generalized inductive definition.

1. Suppose that Π is of the form

$$\dfrac{\begin{matrix}\vdots\ \Pi_1 \\ A_1\end{matrix} \quad \cdots \quad \begin{matrix}\vdots\ \Pi_n \\ A_n\end{matrix}}{A}\ R\text{-}I\ ,$$

where R-I is an introduction rule. Then $\Pi \in Red(A)$ if Π satisfies the following conditions (1.1) – (1. 4).

1.1. If R is $\wedge$, $\vee$, or H, then $\Pi_1 \in Red(A_1), \ldots, \Pi_n \in Red(A_n)$.

1.2. If Π is of the form

$$\dfrac{\begin{matrix}[B] \\ \vdots\ \Pi_1 \\ C\end{matrix}}{B \supset C}\supset\text{-}I\ ,$$

then $\Pi_1[\Sigma/B] \in Red(C)$ for any deduction Σ with $\Sigma \in Red(B)$.

1.3. If Π is of the form

$$\dfrac{\begin{matrix}\vdots\ \Pi_1\\ B[b]\end{matrix}}{\forall x B[x]}\ \forall\text{-}I\ ,$$

then $\Pi_{1[t/b]} \in Red(B[t])$ for any closed term t.

1.4. If Π is of the form

$$\dfrac{\begin{matrix}\vdots\ \Pi_1\\ B[t]\end{matrix}}{\exists x B[x]}\ \exists\text{-}I\ ,$$

then $\Pi_{1[\bar{t}/t]} \in Red(B[\bar{t}])$ for any closure $\bar{t}$ of t.

2. Suppose that Π is neutral. Then $\Pi \in Red(A)$ if Π satisfies the following conditions (2.1) – (2.3).

2.1. For any deduction Σ with $\Pi \rightsquigarrow \Sigma$, $\Sigma \in Red(A)$.

2.2. If Π is of the form

$$\dfrac{\begin{matrix}\\ \vdots\ \Gamma\\ A_1 \vee A_2\end{matrix}\quad \begin{matrix}[A_1]\\ \vdots\ \Delta_1\\ A\end{matrix}\quad \begin{matrix}[A_2]\\ \vdots\ \Delta_2\\ A\end{matrix}}{A}\ \vee\text{-}E\ ,$$

then Π satisfies the following: ($\vee$1) $\Gamma \in SN$; ($\vee$2) $\Delta_1, \Delta_2 \in Red(A)$; ($\vee$3) for any Γ_1 with $\Gamma \rightsquigarrow\!\!\rightsquigarrow \Gamma_1$ and for any Γ_2 which is a subdeduction immediately above an end segment of Γ_1 with $Cnsq(\Gamma_2) = A_i$, $\Delta_i[\Gamma_2/A_i] \in Red(A)$.

2.3. If Π is of the form

$$\dfrac{\begin{matrix}\\ \vdots\ \Gamma\\ \exists x B[x]\end{matrix}\quad \begin{matrix}[B[b]]\\ \vdots\ \Delta\\ A\end{matrix}}{A}\ \exists\text{-}E\ ,$$

then Π satisfies the following: ($\exists$1) $\Gamma \in SN$; ($\exists$2) $\Delta \in Red(A)$; ($\exists$3) for any Γ_1 with $\Gamma \rightsquigarrow\!\!\rightsquigarrow \Gamma_1$ and for any Γ_2 which is a subdeduction immediately above an end segment of Γ_1 with $Cnsq(\Gamma_2) = B[t]$, $\Delta^{[t/b]}[\Gamma_2/B[t]] \in Red(A)$.

For a deduction Π, $\Pi \in Red$ means that $\Pi \in Red(A)$ for some closed formula A. Π is said to be *reducible* if $\Pi \in Red$.

Proposition 3.14 *The reducibility set Red is well defined.*

Proof: It can be proved by transfinite induction over the definition of *Red*, using Proposition 2.4. □

Lemma 3.15 *For any deductions Π and Σ, the following properties hold.*

(CR 1) $\Pi \in Red$ *implies* $\Pi \in SN$.

(CR 2) *If* $\Pi \in Red$ *and* $\Pi \rightsquigarrow \Sigma$, *then* $\Sigma \in Red$.

Proof: The proof goes similarly to that of Lemma 4.1.12. and to that of Theorem 4.1.13. in [2]. We prove this lemma by transfinite induction over the definition of *Red*.

Case 1 (CR 1): Suppose that $\Pi \in Red$.

1. If Π is of the form
$$\dfrac{\begin{matrix}\vdots\ \Pi_1\\ A[a]\end{matrix}}{\forall x A[x]}\ \forall\text{-}I\ ,$$
then $\Pi_{1[t/a]} \in Red(A[t])$ for any closed term t. Since $d(A[t]) <_* d(\forall A[x])$, $\Pi_{1[t/a]} \in SN$ by the induction hypothesis. By Lemma 3.10 (3), $\Pi_1 \in SN$. Therefore, $\Pi \in SN$ because the last inference of Π is an introduction.
2. If Π has an introduction rule except $\forall$-I, then the proof goes similarly to that of (1).
3. If Π is neutral, then $\forall\Sigma(\Pi \rightsquigarrow \Sigma \Longrightarrow \Sigma \in Red)$. Then it holds that $\forall\Sigma(\Pi \rightsquigarrow \Sigma \Longrightarrow \Sigma \in SN)$ by the induction hypothesis. So $\Pi \in SN$.

Case 2 (CR 2): Suppose that $\Pi \rightsquigarrow \Sigma$ and $\Pi \in Red$.

1. If Π is of the form
$$\dfrac{\begin{matrix}\vdots\ \Pi_1\\ A[a]\end{matrix}}{\forall x A[x]}\ \forall\text{-}I\ ,$$
then Σ is of the form
$$\dfrac{\begin{matrix}\vdots\ \Sigma_1\\ A[a]\end{matrix}}{\forall x A[x]}\ \forall\text{-}I\ ,$$
where $\Pi_1 \rightsquigarrow \Sigma_1$. By the definition of the reducibility, $\Pi_{1[t/a]} \in Red$ for any closed term t. By the induction hypothesis, $\Sigma_{1[t/a]} \in Red$ for any closed term t. Therefore $\Sigma \in Red$ by the definition.
2. If Π has an introduction rule except $\forall$-I, then the proof goes similarly to that of (1).
3. If Π is neutral, then this result holds trivially. □

If $\Pi \in SN$, we can construct a well-founded tree T_Π consisting of reduction sequences of Π. For any node t in T, the number of branches of t is finite, and hence, as is well known, T is a finite tree. So, for any deduction Π with $\Pi \in SN$ we let $\nu(\Pi)$ denote the number of nodes in T_Π.

Lemma 3.16 *Let Π be a deduction of the form*

$$\dfrac{\begin{matrix}\vdots\ \Pi_1\\ A_1\end{matrix}\quad \cdots \quad \begin{matrix}\vdots\ \Pi_n\\ A_n\end{matrix}}{A}\ R\ ,$$

where A is a closed formula and R is not an introduction rule nor a BI-rule. Then Π is reducible if the following conditions are satisfied.

(i) $\Pi_1, \ldots, \Pi_n \in SN$.

(ii) If R is either a $\wedge$-E, an $\supset$-E, a $\forall$-E, a $\bot$-E or an H-E, then $\Pi_1, \ldots, \Pi_n \in Red$.
(iii) If R is a $\vee$-E, then Π satisfies Definition 3.13 (2.2).
(iv) If R is an $\exists$-E, then Π satisfies Definition 3.13 (2.3).

Proof: The proof goes similarly to that of Lemma 4.1.16. in [2]. To a deduction Π satisfying the above conditions, we assign an *induction value* $(\alpha, \beta, \gamma, \delta)$ as follows:

(a) α is the degree o $Cnsq(\Pi)$;
(b) $\beta = \nu(\Pi_1)$ if R is an elimination rule; $\beta = 0$ otherwise;
(c) γ is the number of inference rules of Π_1 if R is an elimination rule; $\gamma = 0$ otherwise;
(d) δ is the sum of $\nu(\Pi_1), \ldots, \nu(\Pi_n)$.

Let $\prec$ be the lexicographical order on the induction values. We prove the lemma by induction on the order $\prec$. By the conditions (iii) and (iv), it suffices to show that

$$\forall \Sigma (\Pi \rightsquigarrow \Sigma \Longrightarrow \Sigma \in Red(A)).$$

We deal only with the case where R is an H-E. All other cases can be proved in the same way as in the proof of Lemma 4.1.16. in [2].

1. Suppose that Π is of the form

$$\dfrac{\begin{array}{c}\vdots\ \Pi_1 \\ H(i,t)\end{array}}{G[i,t,H[i]]}\ H\text{-}E\ ,$$

   and that Σ is of the form

$$\dfrac{\begin{array}{c}\vdots\ \Sigma_1 \\ H(i,t)\end{array}}{G[i,t,H[i]]}\ H\text{-}E\ ,$$

   where $\Pi_1 \rightsquigarrow \Sigma_1$. Since $\Pi_1 \in Red$, $\Sigma_1 \in Red$ by (CR 2) in Lemma 3.15. Let ε $(= (\alpha, \beta, \gamma, \delta))$ be the induction value of Π, and let ε' $(= (\alpha', \beta', \gamma', \delta'))$ be the induction value of Σ. Then $\varepsilon' \prec \varepsilon$ since $\alpha = \alpha'$ and $\beta' < \beta$. Therefore, by the induction hypothesis, $\Sigma \in Red$.
2. Suppose that the following scheme holds,

$$\Pi \quad = \quad \dfrac{\begin{array}{c}\vdots\ \Pi_1 \\ H(i,t)\end{array}}{G[i,t,H[i]]}\ H\text{-}E \quad = \quad \dfrac{\dfrac{\begin{array}{c}\vdots\ \Gamma \\ G[i,t,H[i]]\end{array}}{H(i,t)}\ H\text{-}I}{G[i,t,H[i]]}\ H\text{-}E\ ,$$

   and that $\Sigma = \Gamma$. Since $\Pi_1 \in Red$, $\Sigma \in Red$ by Definition 3.13 (1.1). □

Lemma 3.17 *Every elementary deduction is reducible.*

Proof: It suffices to show the following proposition $\mathcal{P}$.

$\mathcal{P}$: Let Π be a closed deduction with live assumptions $[A_1], \ldots, [A_n]$, which has neither any $\forall$-rule, any $\exists$-rule, any H-rule, any ρ-E, nor any *BI*-rule, and let Π_i $(i = 1, \ldots, n)$ be a reducible deduction with $Cnsq(\Pi_i) = A_i$. Then $\Pi[\Pi_1/A_1, \ldots, \Pi_n/A_n]$ is reducible, where $\Pi[\Pi_1/A_1, \ldots, \Pi_n/A_n]$ is the deduction obtained from Π by substituting $\Pi_1, \ldots, \Pi_n$ for $[A_1], \ldots, [A_n]$.

Using induction on the structure of Π and Lemma 3.16, we can show $\mathcal{P}$ easily. □

Lemma 3.18 *Let* Π *be a deduction of the form*

$$\dfrac{\begin{array}{c}\vdots\ \Pi_1\\ \forall z(R[z] \supset A[z])\end{array} \quad \begin{array}{c}\vdots\ \Pi_2\\ \forall z(\forall x A[z * x] \supset A[z])\end{array}}{A[t]}\ BI\ ,$$

where $A[t]$ *is a closed formula. Then* Π *is reducible if* Π_1 *and* Π_2 *are reducible.*

Proof: We fix formulas $R[a]$ and $A[a]$, where $R[a]$ is a monotone formula (see Definition 2.1 (4)) and $A[a]$ does not contain any free variable except a. Let $\mathcal{A}[s]$ denote the following unary predicate, where s ranges over finite sequences of integers.

$\mathcal{A}[s]$: For any closed term t expressing the integer assigned to s by the bijection Φ fixed in Definition 2.1 (4), for any reducible deduction Σ with $Cnsq(\Sigma) = \forall z(R[z] \supset A[z])$ and for any reducible deduction Δ with $Cnsq(\Delta) = \forall z(\forall x A[z * x] \supset A[z])$, a deduction Π_t of the form

$$\dfrac{\begin{array}{c}\vdots\ \Sigma\\ \forall z(R[z] \supset A[z])\end{array} \quad \begin{array}{c}\vdots\ \Delta\\ \forall z(\forall x A[z * x] \supset A[z])\end{array}}{A[t]}\ BI$$

is reducible.

Since $\forall s\mathcal{A}[s]$ implies our result, we show $\forall s\mathcal{A}[s]$. Let $\mathcal{R}[s]$ be a unary predicate, where s ranges over finite sequences of integers, such that $\mathcal{R}[s]$ is equivalent to $R[t]$ for any finite sequence of integers s and for any closed term t expressing the integer assigned to s by Φ. By using (informal) bar induction on s, in order to show $\forall s\mathcal{A}[s]$, it suffices to establish the following properties.

Hyp 1: $\forall f \exists n \mathcal{R}[f\lceil n]$.
Hyp 2: $\forall f \forall n\, (\mathcal{R}[f\lceil n] \Longrightarrow \forall m > l\ \mathcal{R}[f\lceil m])$.
Hyp 3: $\forall s\, (\mathcal{R}[s] \Longrightarrow \mathcal{A}[s])$.
Hyp 4: $\forall s\, (\forall n \mathcal{A}[s \bar{*} n] \Longrightarrow \mathcal{A}[s])$.

Here, $s \bar{*} n$ denotes the finite sequence $\langle s_1, \ldots, s_m, n\rangle$ for any finite sequence s $(= \langle s_1, \ldots, s_m\rangle)$ and any integer n. Since Hyp 1 and Hyp 2 are obvious from the condition of the monotone formula $R[a]$, we show Hyp 3 and Hyp 4.

Case 1 (Hyp 3): Suppose that $\mathcal{R}[s]$ is true. Then $R[t]$ is true for any closed term t corresponding to s. We prove that $\Pi_t \in Red$ by induction on $\nu(\Sigma) + \nu(\Delta)$.

3.1. Suppose $\nu(\Sigma) + \nu(\Delta) = 0$. For any deduction Γ with $\Pi_t \rightsquigarrow \Gamma$, Γ is of the form

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \Sigma\\ \forall z(R[z] \supset A[z])\end{array}}{R[t] \supset A[t]}\ \forall\text{-}E \quad \begin{array}{c}\vdots\ \Theta_{R[t]}\\ R[t]\end{array}}{A[t]}\ \supset\text{-}E\ .$$

Since $\Theta_{R[t]}$ is an elementary deduction (see Definition 3.3 (5.1)), $\Theta_{R[t]}$ is reducible by Lemma 3.17. So, by Lemma 3.16, $\Gamma \in Red$. Therefore, $\Pi_t \in Red$ by Definition 3.13.

3.2. Suppose $\nu(\Sigma) + \nu(\Delta) > 0$. For any deduction Γ with $\Pi_t \rightsquigarrow \Gamma$, Γ is either in (3.1) or of the form

$$\dfrac{\begin{matrix}\vdots\ \Sigma' \\ \forall z(R[z] \supset A[z])\end{matrix} \quad \begin{matrix}\vdots\ \Delta' \\ \forall z(\forall x A[z * x] \supset A[z])\end{matrix}}{A[t]}\ BI\ ,$$

where ($\Sigma \rightsquigarrow \Sigma'$ and $\Delta = \Delta'$) or ($\Sigma = \Sigma'$ and $\Delta \rightsquigarrow \Delta'$). If Γ is of the form in (3.1), the proof goes the same way as in (3.1). Otherwise, $\Gamma \in Red$ follows from the induction hypothesis on $\nu(\Sigma) + \nu(\Delta)$. Therefore, $\Pi_t \in Red$. By (3.1) and (3.2) in this lemma, we have shown that $\mathcal{A}[s]$.

Case 2 (Hyp 4): Suppose $\forall n \mathcal{A}[s \bar{*} n]$. Let t be a closed term corresponding to s. We prove that $\Pi_t \in Red$ by induction on $\nu(\Sigma) + \nu(\Delta)$.

4.1. Suppose $\nu(\Sigma) + \nu(\Delta) = 0$. Let Γ be a deduction with $\Pi \rightsquigarrow \Gamma$. Γ is either of the same form as in (3.1) or is of the form

$$\Gamma = \dfrac{\dfrac{\begin{matrix}\vdots\ \Delta \\ \forall z(\forall x A[z * x] \supset A[z])\end{matrix}}{\forall x A[t * x] \supset A[t]}\ \forall\text{-}E \qquad \dfrac{\begin{matrix}\vdots\ (\Pi_t)_{[t*a/t]} \\ A[t * a]\end{matrix}}{\forall x A[t * x]}\ \forall\text{-}I}{A[t]}\ \supset\text{-}E\ .$$

If Γ is of the same form as in (3.1), the proof goes the same way as in (3.1). Otherwise, by the hypothesis $\forall n \mathcal{A}[s \bar{*} n]$, $(\Pi_t)_{[t'/t]} \in Red$ for any integer n and for any closed term t' corresponding to $s \bar{*} n$. By Definition 3.13 (1.3), and by Lemma 3.16, $\Gamma \in Red$. Therefore, $\Pi_t \in Red$.

4.2. $\nu(\Sigma) + \nu(\Delta) > 0$. The proof goes the same way as in (3.2).

By (4.1) and (4.2) in this lemma, we can show $\mathcal{A}[s]$. □

Lemma 3.19 *Let Π be a deduction whose consequence is closed, and let a be a strictly free variable in Π. If $\Pi[t/a] \in Red$ for any closed term t, then $\Pi \in Red$. Here, $\Pi[t/a]$ is the deduction obtained from Π by substituting t for a.*

Proof: We consider a deduction Π which satisfies the following condition (i) or (ii): (i) $\Pi \in Red$; (ii) $Cnsq(\Pi)$ is closed and there exists a strictly free variable a in Π such that $\Pi[t/a] \in Red$ for any closed term t. Such a deduction Π is said to be *prereducible*. To a prereducible deduction Π, we assign an induction value $\varepsilon(\Pi) = (\alpha,\ \beta,\ \gamma)$ as follows:

1. α is the degree of $Cnsq(\Pi)$;
2. $\beta = \nu(\Pi)$;
3. γ is the number of inference rules of Π.

Note that since $\Pi[t/a] \in SN$ by (CR 1), $\Pi \in SN$ by Lemma 3.10 (3). Let $\prec$ be the lexicographical order of the induction values. We prove that every prereducible deduction is reducible by induction on the order $\prec$.

1. Suppose that Π is not neutral.

1.1 Suppose that Π is of the form

$$\dfrac{\begin{array}{c}[\not{B}] \\ \vdots\ \Sigma \\ C\end{array}}{B \supset C} \supset\text{-}I \ .$$

Since $\Pi[t/a] \in Red$, $\Sigma[t/a][\Gamma/B] \in Red$ for any closed term t and for any $\Gamma \in Red(B)$. Since $\varepsilon(\Sigma[\Gamma/B]) \prec \varepsilon(\Pi)$, $\Sigma[\Gamma/B] \in Red$ by the induction hypothesis. So, $\Pi \in Red$.

1.2 Suppose that Π is of the form

$$\dfrac{\begin{array}{c}\vdots\ \Sigma \\ B[s]\end{array}}{\exists x B[x]} \exists\text{-}I \ .$$

1.2.1 If s does not contain a as a free variable, then $(\Sigma[t/a])_{[\bar{s}/s]} = (\Sigma_{[\bar{s}/s]})[t/a]$ and $(\Sigma[t/a])_{[\bar{s}/s]} \in Red$ for any closed term t and for any closure $\bar{s}$ of s. Since $\varepsilon(\Sigma_{[\bar{s}/s]}) \prec \varepsilon(\Pi)$, and by the induction hypothesis, $\Sigma_{[\bar{s}/s]} \in Red$ for any closure $\bar{s}$ of s. So, $\Pi \in Red$.

1.2.2 Suppose that s contains a as a free variable in s. For the term s $(= s[a])$, let $\bar{s}[a]$ denote a term obtained from s by substituting closed terms for all free variables except a. If $s[t]$ denotes $(s[a])[t/a]$ and $\bar{s}[t]$ denotes $(\bar{s}[a])[t/a]$, then $(\Sigma[t/a])_{[\bar{s}[t]/s[t]]} \in Red$ for any closed term t and for any $\bar{s}[a]$, since $\Pi[t/a] \in Red$ for any closed term t. In this case, $(\Sigma[t/a])_{[\bar{s}[t]/s[t]]} = (\Sigma_{[\bar{s}[a]/s]})_{[t/a]} = \Sigma_{[\bar{s}[t]/s]}$. Therefore, $\Sigma_{[\bar{s}/s]} \in Red$ for any closure $\bar{s}$ of s. So, $\Pi \in Red$.

1.3 The other cases where Π is not neutral can be proved in the same way as in (1.1) and (1.2) in this lemma.

2. Suppose that Π is neutral.

2.1 We show $\forall\Sigma(\Pi \rightsquigarrow \Sigma \Longrightarrow \Sigma \in Red)$. Let Σ be a deduction with $\Pi \rightsquigarrow \Sigma$. If a is a strictly free variable in Π, then a is not an eigenvariable in Σ by Lemma 3.10 (1). For any closed term t, $\Sigma[t/a]$ can be obtained from $\Pi[t/a]$ by the same reduction as $\Pi \rightsquigarrow \Sigma$. So, by (CR 2), $\Sigma[t/a] \in Red$ for any closed term t. Since $\varepsilon(\Sigma) \prec \varepsilon(\Pi)$, $\Sigma \in Red$ by the induction hypothesis.

2.2 Suppose that Π is of the form

$$\dfrac{\begin{array}{c}\vdots\ \Sigma \\ \exists x B[x]\end{array} \quad \begin{array}{c}[B[\not{b}]] \\ \vdots\ \Delta \\ A\end{array}}{A} \exists\text{-}E \ .$$

We show the following: ($\exists$1) $\Sigma \in SN$; ($\exists$2) $\Delta \in Red(A)$; ($\exists$3) for any Σ_1 such that $\Sigma \rightsquigarrow\!\!\rightsquigarrow \Sigma_1$ and for anyΣ_2 which is a subdeduction immediately above an end segment $\{S_i\}_{i \le n}$ of Σ_1 with $Cnsq(\Sigma_2) = B[s]$, $\Delta^{[s/b]}[\Sigma_2/B[s]] \in Red(A)$.

Proof of (∃1): Since $\Pi \in SN$, $\Sigma \in SN$.

Proof of (∃2): Since $\varepsilon(\Delta) \prec \varepsilon(\Pi)$, $\Delta \in Red$ by the induction hypothesis.

Proof of (∃3): We fix Σ_1, Σ_2 and $\{S_i\}_{i \le n}$, and let Σ_1 be of the form

$$\dfrac{\begin{array}{c}\vdots\ \Gamma_{n-1}\\ C_{n-1}\end{array} \quad \cdots \quad \begin{array}{c} \dfrac{\begin{array}{c}\vdots\ \Gamma_1\\ C_1\end{array}\quad\cdots\quad \dfrac{\begin{array}{c}\vdots\ \Sigma_2\\ B[s]\end{array}}{S_1}\,\exists\text{-}I \quad\cdots}{S_2}\, Q_1\text{-}E \\ \vdots \\ S_{n-1}\end{array} \quad\cdots}{S_n}\, Q_{n-1}\text{-}E\ ,$$

where $n \ge 1$ and Q_i-E is either a $\vee$-E or an $\exists$-E for any $1 \le i < n$. Note that S_i has the same form as $\exists x B[x]$ for any $i \le n$. Let Γ be a deduction obtained from Π by replacing Σ by Σ_1, and let Γ^* be a deduction obtained from Γ by $(n-1$ times) permutative contractions along $\{S_1, \ldots, S_n\}$. Then

$$\Pi \rightsquigarrow\rightsquigarrow \Gamma \rightsquigarrow\rightsquigarrow \Gamma^*,$$

and Γ^* is of the form

$$\dfrac{\begin{array}{c}\vdots\ \Gamma_{n-1}\\ C_{n-1}\end{array} \quad \cdots \quad \begin{array}{c} \dfrac{\begin{array}{c}\vdots\ \Gamma_1\\ C_1\end{array}\quad\cdots\quad \dfrac{\dfrac{\begin{array}{c}\vdots\ \Sigma_2\\ B[s]\end{array}}{S_1}\,\exists\text{-}I \quad \begin{array}{c}[B[b]]\\ \vdots\ \Delta\\ A\end{array}}{A}\,\exists\text{-}E \quad\cdots}{A}\, Q_1\text{-}E \\ \vdots \\ A\end{array} \quad\cdots}{A}\, Q_{n-1}\text{-}E\ .$$

If a is a strictly free variable in Π, then a is not an eigenvariable in Γ^* by Lemma 3.10 (1), and hence, a is not an eigenvariable in $\Delta^{[s/b]}[\Sigma_2/B[s]]$. Let t be a closed term. Since $\Pi[t/a] \in Red$, $\Gamma^*[t/a] \in Red$ by (CR 2). So, $\Delta^{[s/b]}[\Sigma_2/B[s]][t/a] \in Red$ by (CR 2) and Definition 3.13 (2.3). Therefore, since $\varepsilon(\Delta^{[s/b]}[\Sigma_2/B[s]]) \prec \varepsilon(\Gamma^*) \preceq \varepsilon(\Gamma) \preceq \varepsilon(\Pi)$, $\Delta^{[s/b]}[\Sigma_2/B[s]] \in Red$ by the induction hypothesis.

2.3 If Π has a $\vee$-E as the last inference rule, then the proof goes similarly to that of (2.2) in this lemma. □

Lemma 3.20 *Let Π be a deduction whose consequence is closed, and let $\vec{a}$ $(= a_1, \ldots, a_n)$ be strictly free variables in Π. Then $\Pi \in Red$ whenever $\Pi[\vec{t}/\vec{a}] \in Red$ for any closed terms $\vec{t}$ $(= t_1, \ldots, t_n)$, where $\Pi[\vec{t}/\vec{a}]$ denotes the deduction obtained from Π by substituting t_i for a_i $(i = 1, \ldots, n)$.*

Proof: Using induction on n, we can easily show this lemma from Lemma 3.19. □

Remark 3.21 Let Π be a deduction whose consequence is closed, and let $\vec{x}$ be the set of all strictly free variables in Π. By Lemma 3.20, in order to show $\Pi \in \mathit{Red}$, it suffices to find a subset $\vec{a}$ of $\vec{x}$ such that $\Pi[\vec{t}/\vec{a}] \in \mathit{Red}$ for any closed terms $\vec{t}$. Applying this property, we show the following lemma.

Lemma 3.22 *Let Π be a closed deduction which has live assumptions $[A_1], \ldots, [A_n]$. If Π_i ($i = 1, \ldots, n$) is a deduction such that $\mathit{Cnsq}(\Pi_i) = A_i$ and $\Pi_i \in \mathit{Red}$, then $\Pi[\Pi_1/A_1, \ldots, \Pi_n/A_n]$ is reducible.*

Proof: We prove this lemma by induction on the structure of Π.

1. If Π consists only of a live assumption $[A]$ or an axiom A, then it is immediate.
2. If Π is not neutral, then it follows immediately from Definition 3.13 (1).
3. Suppose that Π is neutral and that Π has a rule R as the last inference rule.

   3.1. If R is a BI-rule, then it follows immediately from Lemma 3.18.

   3.2. Suppose that Π is of the form

$$\dfrac{\begin{array}{c} \vdots\ \Sigma \\ \exists x B[x] \end{array} \quad \begin{array}{c} [B[b]] \\ \vdots\ \Delta \\ A \end{array}}{A}\ \exists\text{-}E\ .$$

Let $\Sigma^* = \Sigma[\Pi_1/A_1, \ldots, \Pi_n/A_n]$, and let $\Delta^* = \Delta[\Pi_1/A_1, \ldots, \Pi_n/A_n]$. By Lemma 3.16, in order to show $\Pi[\Pi_1/A_1, \ldots, \Pi_n/A_n] \in \mathit{Red}$, it suffices to show the following: ($\exists$1) $\Sigma^* \in SN$; ($\exists$2) $\Delta^* \in \mathit{Red}(A)$; ($\exists$3) for any Σ_1 such that $\Sigma^* \rightsquigarrow\!\rightsquigarrow \Sigma_1$ and for any Σ_2 which is a subdeduction immediately above an end segment of Σ_1 with $\mathit{Cnsq}(\Sigma_2) = B[s]$, $\Delta^{*[s/b]}[\Sigma_2/B[s]] \in \mathit{Red}(A)$.

Proof of ($\exists$1): By the induction hypothesis, $\Sigma^* \in \mathit{Red}$. Therefore, $\Sigma^* \in SN$ by (CR 1).

Proof of ($\exists$2): Let $[B[b]]$ be the live assumption of Δ discharged by the last inference rule $\exists$-E of Π. Since b is the eigenvariable of the last inference rule $\exists$-E of Π, b is not an eigenvariable in Δ or Δ^*. By Definition 3.13, $\Delta^{[t/b]}$ is a closed deduction for any closed term t, where $\Delta^{[t/b]}$ is the deduction obtained from Δ by substituting t for the free variable b in the live assumption $[B[b]]$. Therefore, by the induction hypothesis, $\Delta^{[t/b]}$ satisfies this lemma. So, $\Delta^{[t/b]}[\Pi_1/A_1, \ldots, \Pi_n/A_n] \in \mathit{Red}$ for any closed term t. Since $\Delta^{[t/b]}[\Pi_1/A_1, \ldots, \Pi_n/A_n] = \Delta^{*[t/b]}$, $\Delta^{*[t/b]} \in \mathit{Red}$ for any closed term t. Therefore, by Lemma 3.20, $\Delta^* \in \mathit{Red}$.

Proof of ($\exists$3): By (CR 2), and by Definition 3.13 (2.3), $\Sigma_{2[\bar{s}/s]} \in \mathit{Red}$ for any closure $\bar{s}$ of s, whereas for the closed term $\bar{s}$, $\Delta^{[\bar{s}/b]}$ is the closed deduction. Therefore, by the induction hypothesis, $\Delta^{[\bar{s}/b]}$ satisfies this lemma. So, $\Delta^{*[\bar{s}/b]}[\Sigma_{2[\bar{s}/s]}/B[\bar{s}]] \in \mathit{Red}$ for any closure $\bar{s}$ of s. Since any free variable in the term s is not an eigenvariable in $\Delta^{*[s/b]}[\Sigma_2/B[s]]$, by Lemma 3.20, $\Delta^{*[s/b]}[\Sigma_2/B[s]] \in \mathit{Red}$.

3.3. If R is a $\vee$-E, then the proof goes similarly to that of (3.2).

3.4. If R is the other rule, then it follows immediately from Lemma 3.16. □

Proof of the strong normalization theorem: By Lemma 3.22, every closed deduction is reducible. So every closed deduction is strongly normalizable by (CR 1). By Lemma 3.11, every deduction is strongly normalizable. □

Remark 3.23 In [5], Yasugi and Hayashi introduced the term-system **TRM** (the system of **T**e**RM**). **TRM** consists of parametric types called *type-forms* and terms called *term-forms*, which are used to carry out a certain abstraction of computation to proofs formalized in **TRDB**. The authors of [5] also proved the strong normalization theorem of type-forms and term-forms in **TRM**. In order to prove the strong normalization theorem for **TRM**, the authors of [5] needed a kind of restriction: $\mathcal{R}$*-strategy* and ρ*-strategy* for reductions of type-forms; $\mathcal{B}$*-strategy* and σ*-strategy* for reductions of term-forms. However, the proof in this paper needs neither.

It is known that the reductions in Definition 3.3 do not satisfy the Church-Rosser property. In fact, the immediate signification of $\vee$-E can make a deduction reduce in two ways (see [2]). We can, however, avoid this shortcoming by applying a suitable restriction, for instance, removing immediate signification of $\vee$-E. If we confine ourselves to such a case, the Church-Rosser property holds in **TRDB**.

Lemma 3.24 *Let* Π *be any deduction, and let* Σ *and* Δ *satisfy* $\Pi \rightsquigarrow\rightsquigarrow \Sigma$ *and* $\Pi \rightsquigarrow\rightsquigarrow \Delta$*. Then there exists a deduction* Γ *such that* $\Sigma \rightsquigarrow\rightsquigarrow \Gamma$ *and* $\Delta \rightsquigarrow\rightsquigarrow \Gamma$*.*

Proof: The proof goes the same way as the well-known method. □

From this lemma, we immediately obtain the following theorem.

Theorem 3.25 *For any deduction* Π *in* **TRDB**, Π *is uniquely normalized to a deduction.*

4 Consistency of TRDB

In this section, we establish the consistency of **TRDB**, using the strong normalization theorem, Theorem 3.6, and *paths* used to establish the consistency of **HA** in [2]. We also prove the existence property and the disjunction property of **TRDB**, using the strong normalization theorem and the paths.

Definition 4.1 (Troelstra [2])

1. For a deduction Π, a finite sequence $\{A_i\}_{i\leq n}$ consisting of (occurrences of) formulas in Π is called a *path* of Π if it satisfies the following conditions.
   - (i) A_1 is either a live assumption, an axiom, an assumption discharged by an $\supset$-I, or the conclusion of a *BI*-rule.
   - (ii) For any $i < n$, A_i is neither $Cnsq(\Pi)$, the minor premise of any $\supset$-E, any premise of any *BI*-rule, the major premise of any $\vee$-E which is redundant, nor the major premise of any $\exists$-E which is redundant.
   - (iii) For any $i < n$, if A_i is not the major premise of an $\vee$-E or an $\exists$-E, then A_{i+1} is a formula that occurs immediately below A_i in Π.

(iv) For any $i < n$, if A_i is the major premise of a $\vee$-E or an $\exists$-E, then A_{i+1} is one of the assumptions discharged by the elimination.

(v) A_n is either $Cnsq(\Pi)$, the minor premise of an $\supset$-E, one of the premises of a *BI*-rule, the major premise of a $\vee$-E which is redundant, or the major premise of an $\exists$-E which is redundant.

2. A path of Π whose end formula is $Cnsq(\Pi)$ is called an *end path* of Π.

For a path $\{A_i\}_{i \leq n}$, if there exists an i $(\leq n)$ such that A_i is either the conclusion or a premise of an inference R, then we say that $\{A_i\}_{i \leq n}$ contains R.

For any deduction Π, we let $\overline{\Pi}_N$ denote a normal form of a closure of Π, and let $r_{\mathcal{B}}(\Pi)$ denote the number of end paths of Π whose initial formulas are conclusions of *BI*-rules.

Lemma 4.2 *Let Π be a deduction whose consequence is an atomic formula except an H-formula. Then $r_{\mathcal{B}}(\overline{\Pi}_N) = 0$.*

Proof: Suppose that $\overline{\Pi}_N$ has an end path $\{A_i\}_{i \leq n}$ such that A_1 is the conclusion of a *BI*-rule. Then there exists a *BI*-rule $\mathcal{B}$ such that $\overline{\Pi}_N$ is of the following form

$$\dfrac{\begin{array}{c}\vdots\ \Delta \\ \forall z(R_i(z) \supset A_i[z])\end{array} \quad \begin{array}{c}\vdots\ \Gamma \\ \forall z(\forall x A_i[z * x] \supset A_i[z])\end{array}}{\begin{array}{c}A[t] \\ \vdots\ \Sigma \\ Cnsq(\overline{\Pi}_N)\end{array}} \mathcal{B}$$

and that A_1 is the conclusion $A[t]$ of $\mathcal{B}$. If t is a closed term, then $\overline{\Pi}_N$ is not a normal form since $\mathcal{B}$ is a reduction point of $\overline{\Pi}_N$. So, t contains a free variable. By Lemma 3.9 (4), $\overline{\Pi}_N$ is a closed deduction. So, t contains an eigenvariable for a $\forall$-I or an $\exists$-E by Lemma 3.9 (1). Since $A[t]$ occurs below $\mathcal{B}$, t does not contain any eigenvariable for $\exists$-E. So, t must contain an eigenvariable for a $\forall$-I. However, $\{A_i\}_{i \leq n}$ does not contain any introduction rule, because Σ is a normal form and $Cnsq(\overline{\Pi}_N)$ is an atomic formula except an H-formula. This yields a contradiction. □

Lemma 4.3 *Let Π be a deduction. For any end path $\{A_i\}_{i \leq n}$ of $\overline{\Pi}_N$ which does not contain any $\forall$-I rule, A_1 is not the conclusion of a BI-rule.*

Proof: The proof goes the same way as Lemma 4.2. □

Definition 4.4 Let A be an atomic formula except an H-formula. Then A is said to be *absurd* if A satisfies the following:

$$\textbf{TRDB} - (BI + TRD(G, I)) \vdash A \supset 0 = 1.$$

Theorem 4.5 **TRDB** *is consistent.*

Proof: Let Π be a deduction whose consequence is $0 = 1$. By Lemma 4.2, no end path of $\overline{\Pi}_N$ contains a *BI*-rule or an introduction rule. Therefore, there exists an end path whose initial formula A_1 satisfies the following conditions.

(i) A_1 is not any assumption discharged by a $\supset$-I.

(ii) A_1 contains an H-formula or an absurd formula, that is, A_1 is not an axiom formula.

So, $\overline{\Pi}_N$ has at least one live assumption, and hence, Π has at least one live assumption. □

Theorem 4.6 (The existence property and the disjunction property of **TRDB**)

1. *If a closed formula* $\exists x A[x]$ *is provable in* **TRDB**, *then there exists a closed term* t *such that* $A[t]$ *is provable in* **TRDB**.
2. *If a closed formula* $A \vee B$ *is provable in* **TRDB**, *then* A *or* B *is provable in* **TRDB**.

Proof: (1) Let Π be a deduction which has a closed formula $\exists x A[x]$ as the consequence, and let Π have no live assumption. Then $\overline{\Pi}_N$ also has $\exists x A[x]$ as the consequence and also has no live assumption. We show that $\overline{\Pi}_N$ has an introduction rule as the last inference rule.

(i) Since the consequence of $\overline{\Pi}_N$ is not an atomic formula, $\overline{\Pi}_N$ is not an axiom. Since the conclusion of every inference rule defined in Definition 2.1 (2.1) – (2.5) is an atomic formula, the last inference rule of $\overline{\Pi}_N$ is either a logical inference rule, a $\perp$-E, a *BI*-rule, an H-I, an H-E or ρ-E.

(ii) The consequence of $\overline{\Pi}_N$ is a closed formula, the last inference rule is not a *BI*-rule.

(iii) Suppose that $\overline{\Pi}_N$ has an elimination rule except a $\vee$-E or an $\exists$-E, as the last inference rule. Then every end path contains no introduction rule. So, by Lemma 4.3, $r_{\mathcal{B}}(\overline{\Pi}_N) = 0$. So, for any end path $\{A_i\}_{i \leq n}$, A_1 is an axiom formula. However, in this case, A_1 is an absurd formula or an H-formula whenever A_1 is an atomic formula. This yields a contradiction.

(iv) Suppose that $\overline{\Pi}_N$ has a $\vee$-E or an $\exists$-E rule as the last inference rule. Then the subdeduction Σ of $\overline{\Pi}_N$, whose consequence is the major premise of the last inference rule of $\overline{\Pi}_N$, has neither any $\vee$-E nor any $\exists$-E as the last inference rule. Note that Σ is a closed deduction and a normal form, whose consequence is a closed formula of the form $\exists y B[y]$ or $B \vee C$. By (i) – (iii), Σ has an introduction rule as the last inference rule. Since $\overline{\Pi}_N$ is a normal form, it yields a contradiction.

By (i) – (iv), $\overline{\Pi}_N$ has an introduction rule as the last inference rule. Since the outermost logical symbol of the consequence of $\overline{\Pi}_N$ is an $\exists$-quantifier, the last inference rule is an $\exists$-I. Moreover, since $\overline{\Pi}_N$ is a closed deduction, there exists a closed term t such that $A[t]$ is provable with the subdeduction obtained from $\overline{\Pi}_N$ by removing the last inference rule.

(2) The proof goes the same way as (1). □

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