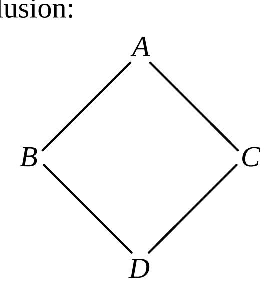
Notre Dame Journal of Formal Logic
Volume 39, Number 3, Summer 1998

Homeomorphism and the Equivalence of Logical Systems

STEPHEN POLLARD

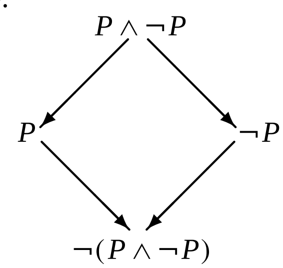
Abstract Say that a property is *topological* if and only if it is invariant under homeomorphism. Homeomorphism would be a successful criterion for the equivalence of logical systems only if every logically significant property of every logical system were topological. Alas, homeomorphisms are sometimes insensitive to distinctions that logicians value: properties such as functional completeness are not topological. So logics are not just devices for exploring closure topologies. One still wonders, though, *how much* of logic is topological. This essay examines some logically significant properties that *are* topological (or are topological in some important class). In the process, we learn something about the conditions under which the meaning of a connective can be "given by the connective's role in inference."

1 Introduction Here is a guessing game. I have in mind a two-valued sentential logic with a denumerable set of well-formed formulas A and a consequence relation $\models$. ($X \models \varphi$ if and only if no interpretation assigns 1 to each member of X and 0 to φ). Say that a subset X of A is *closed* under $\models$ just in case $X = \{\varphi \in A : X \models \varphi\}$ (that is, just in case each consequence of X is a member of X). A itself is closed; and the logic I have in mind has exactly three other closed sets: B, C, and D. All are denumerable. A is the set of consequences of some well-formed formula φ : $A = \{\psi \in A : \{\varphi\} \models \psi\}$. Furthermore, $A = (B \cup C)$ and $D = (B \cap C)$. So the closed sets of my logic form the following lattice under inclusion:

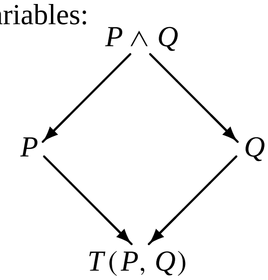


Received July 30, 1997; revised February 12, 1998

So what logic do I have in mind? Well, the question turns out to be unfair. You can infer that my logic has no more than two sentential variables (since the number of variables cannot exceed the width of the lattice of closed sets). But you have no way of knowing whether my logic has as many as two or as few as one. You can prove that my logic has denumerably many tautologies (since $D = \{\psi \in A : \varnothing \models \psi\} = \{\psi \in A : \psi \text{ is assigned 1 by every interpretation}\}$). But you do not know whether my logic has even one unsatisfiable well-formed formula. (For all you know, my logic might allow an interpretation that assigns 1 to *every* well-formed formula.) A result given below assures you that my logic expresses conjunction and the T-constant function. But you have no way of telling whether my logic expresses all truth functions or only a pitiful few. I might have in mind, say, the classical logic of negation, conjunction, and a single sentential variable:



But then, I could just as well be thinking of the classical logic of conjunction, T-constant, and two sentential variables:



The topological and implicational structures are the same in each case.

Now it so happens that any logics (indeed, any closure spaces) with the properties I have attributed to my mystery logic will be homeomorphic to one another. So homeomorphism would be "an excellent criterion for the equivalence of logical systems" (Pollard and Martin [11], p. 127) only if there were no logically significant differences between any logical systems satisfying the conditions of my mystery logic. This, in turn, would require that properties such as functional completeness or the existence of an unsatisfiable well-formed formula not be logically significant. Does this seem right? *My* idea of a logically insignificant property is, say, the use of '$\supset$' rather than '$\rightarrow$' as the symbol for material implication. Should we insist that functional completeness is *that* sort of property? Keen as I am on the closure theoretic approach, I decline to do so. Logics with the same closure topology can differ in logically significant ways. So logic is not *entirely* topological.

That's the bad news. The good news is that, in the case of classical sentential logics, this concession does not amount to much. Consider the property of expressing an unsatisfiable well-formed formula. We saw that this property is not preserved by homeomorphisms: in a suitably impoverished logic, the conjunction of all sentential variables will imply all well-formed formulas and hence, will play the same topological role as a contradiction. Of course, if we make the standard assumption that there are denumerably many sentential variables, then there is (classically) no such thing as the conjunction of all variables. In this setting then, we might hope that no satisfiable well-formed formula can impersonate an unsatisfiable one. This hope is realized: in a classical sentential logic with infinitely many variables, a well-formed formula is unsatisfiable if and only if it has the topological property of implying all well-formed formulas. This happy result generalizes: it is arguable, in fact, that all logically significant properties of classical sentential logics *with infinitely many variables* are preserved by homeomorphisms.[1] Of course, one normally does assume that one's logic has infinitely many variables. So, while the misbehavior of homeomorphisms in the finite case is disappointing, their performance under more standard conditions is a considerable consolation.

2 Preliminaries It will be convenient to concentrate on classical sentential logics whose connectives are all binary. (The restriction to *binary* connectives will help us with some bookkeeping later on. There is no deeper motivation.) More formally, say that a *two-valued logic with binary connectives* is a triple $\langle V, \mathrm{CON}, *\rangle$ where V is a nonempty set of variables, CON is a nonempty set of binary connectives, and $*$ is a function that assigns a binary, two-valued truth function to each member of CON. The well-formed formulas of such a logic are the variables and any expressions $g(\varphi, \psi)$ where φ and ψ are well-formed formulas and g is a connective. An *interpretation* is any $0, 1$-valued homomorphism on well-formed formulas. That is, if $\mathcal{F}$ is an interpretation, then $\mathcal{F}(g(\varphi, \psi)) = g^*(\mathcal{F}(\varphi), \mathcal{F}, (\psi))$. If φ is a well-formed formula and A is a set of well-formed formulas, then, as usual, we say $A \models \varphi$ just in case no interpretation assigns 1 to each member of A and 0 to φ. We let $\mathsf{Cl}(A) = \{\varphi : A \models \varphi\}$.

Each such function Cl is a *closure operator*. That is, $A \subseteq \mathsf{Cl}(B)$ if and only if $\mathsf{Cl}(A) \subseteq \mathsf{Cl}(B)$.[2] Note though that Cl will not necessarily satisfy all of the Kuratowski closure axioms characteristic of a topological space. There are two reasons. First of all, in some of our logics, the empty set has a nonempty closure: $\{\varphi : \varnothing \models \varphi\} \neq \varnothing$. This just means that some of our logics feature well-formed formulas assigned 1 by every interpretation. (Think of $\mathsf{Cl}(\varnothing)$ as the set of all tautologies.) Secondly, it takes a bit of work to find a logic in which $\mathsf{Cl}(A \cup B) \subseteq (\mathsf{Cl}(A) \cup \mathsf{Cl}(B))$ for all sets (of well-formed formulas) A and B. That is, it is common for $(A \cup B)$ to imply well-formed formulas implied by neither A nor B. So our notion of closure is not really the topological one, but rather its generalization from the more abstract theory of closure spaces. We also employ the closure theoretic, rather than the strictly topological, notion of homeomorphism. Some of us find it natural though to say that properties invariant under (closure theoretic) homeomorphism are "topological." (Cf. Martin and Pollard [8], p. 91.)

If Γ_1 and Γ_2 are logics with closure operators Cl_1 and Cl_2 and if f is a bijection that assigns well-formed formulas of Γ_2 to well-formed formulas of Γ_1, then f is a *homeomorphism* just in case $f[\mathsf{Cl}_1(A)] = \mathsf{Cl}_2(f[A])$ for each set A of Γ_1-well-formed formulas. Just as in topology, a homeomorphism is a continuous bijection whose inverse is continuous. (Cf. [8], p. 85.) Note too, that if f is a homeomorphism, then $\{\varphi_1, \ldots, \varphi_m\} \models \psi$ in Γ_1 if and only if $\{f(\varphi_1), \ldots, f((\varphi_m)\} \models f(\psi)$ in Γ_2. So homeomorphism is isomorphism with respect to the consequence relation. The question then is whether logics should be considered equivalent whenever they have the same implicational structure.

If Φ is an n-ary truth function, Γ is a logic, and φ is a Γ-well-formed formula with occurrences of exactly n variables $P_1, \ldots, P_n$, then φ *expresses* Φ *in* Γ if and only if $\Phi(\mathcal{F}(P_1), \ldots, \mathcal{F}(P_n)) = \mathcal{F}(\varphi)$ for every interpretation $\mathcal{F}$. A truth function is *expressible* in a logic Γ if and only if some Γ-well-formed formula expresses it in Γ. Let $\mathrm{FNC}(\Gamma)$ be the set of truth functions expressible in Γ. Then, for example, if $[\mathrm{CON}]^*$ (the set of truth functions assigned to members of CON by $*$) is any functionally complete set of two-valued truth functions, then $\mathrm{FNC}(\Gamma)$ is the set of *all* two-valued truth functions. If the only members of $[\mathrm{CON}]^*$ are material implication and two-valued conjunction, then $\mathrm{FNC}(\Gamma)$ is the set of Post's β-functions: the two-valued truth functions Φ such that $\Phi(1, 1, \ldots, 1) = 1$.

We say that logics $\Gamma_1 = \langle V_1, \mathrm{CON}_1, *\rangle$ and $\Gamma_2 = \langle V_2, \mathrm{CON}_2, '\rangle$ are *Post-equivalent* just in case $|V_1| = |V_2|$ and $\mathrm{FNC}(\Gamma_1) = \mathrm{FNC}(\Gamma_2)$; that is, just in case the two logics have the same number of variables and express the same truth functions. The term 'Post-equivalent' is meant only to *honor* Post; I do not claim that this was his standard for the equivalence of sentential logics. (Since the logics of [12] all have denumerably many variables, Post would, at the very least, have omitted the clause about cardinality.) Although Post-equivalence does seem an attractive standard, I introduce it here mainly because a comparison between Post-equivalence and homeomorphism helps us to understand homeomorphism.

3 Homeomorphism $\neq$ Post-equivalence Given any fixed V, there are 33 Post-equivalence classes of logics $\langle V, \mathrm{CON}, *\rangle$. The following truth functions will allow us to characterize them.

	$\top$	$\vee$	$=_1$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\underline{\vee}$	$\overline{\rightarrow}$	$\neg_1$	$\downarrow$	$\bot$
11	1	1	1	1	1	1	0	0	0	0	0
10	1	1	1	0	0	0	1	1	0	0	0
01	1	1	0	1	0	0	1	0	1	0	0
00	1	0	0	1	1	0	0	0	1	1	0

The 33 Post-equivalence classes correspond to the logics whose connectives are assigned the following truth functions. (The names listed are borrowed from Post [12]. Post lists 37 "second order" systems. We obtain just 33 equivalence classes because we only consider logics whose connectives are all binary. As earlier noted, this will allow us to simplify some tedious bookkeeping.)

R_1:	$=_1$	S_1:	$\vee$	P_1:	$\wedge$	A_1:	$\top, \bot, \vee, \wedge$
R_2:	$\top$	S_2:	$=_1, \vee$	P_2:	$=_1, \wedge$	A_2:	$\top, \vee, \wedge$
R_3:	$\bot$	S_3:	$\top, \vee$	P_3:	$\bot, \wedge$	A_3:	$\bot, \vee, \wedge$
R_4:	$\neg_1$	S_4:	$=_1, \top, \vee$	P_4:	$=_1, \bot, \wedge$	A_4:	$\vee, \wedge$
R_6:	$=_1, \top$	S_5:	$\bot, \vee$	P_5:	$\top, \wedge$	C_1:	$\downarrow$
R_8:	$=_1, \bot$	S_6:	$\top, \bot, \vee$	P_6:	$\top, \bot, \wedge$	C_2:	$\wedge, \rightarrow$
R_9:	$\top, \bot$	L_1:	$\leftrightarrow, \neg_1$	F_4^∞:	$\rightarrow$	C_3:	$\vee, \overline{\rightarrow}$
R_{11}:	$=_1, \top, \bot$	L_2:	$\leftrightarrow$	F_8^∞:	$\overline{\rightarrow}$		
R_{13}:	$\top, \neg_1$	L_3:	$\underline{\vee}$				

Homeomorphism yields a different classification. For example, each S_1 logic with, say, two variables is homeomorphic to each S_2 logic with two variables. (These logics fail to be Post-equivalent only because S_2 logics express the projection function $=_1$ while S_1 logics do not.) Logics of types S_3 and S_4, P_1 and P_2, P_3 and P_4 have the same property. So homeomorphism yields only 29 equivalence classes of logics with exactly two variables. Someone who thinks the expressibility of $=_1$ is logically insignificant would score this as a point in favor of the topological outlook. The two lemmas given below offer further comfort to fans of homeomorphism. There are, however, two reasons for concern. Note, first of all, that homeomorphisms are *sometimes* sensitive to the expressibility of $=_1$. (Cf. R_2 and R_6, R_3 and R_8, R_9 and R_{11}.[3])

One wonders why the expressibility of $=_1$ should count as logically significant in some cases, but not in others. Secondly, while homeomorphism yields a more or less reasonable classification of logics *with a certain number of variables*, having exactly n sentential variables is not a topological property. So significantly different logics with different numbers of variables could turn out to be homeomorphic. (In fact, we already saw in §1 that this *will* happen.)

We say that a property is *topological* just in case it is invariant under homeomorphism. (So, if Γ_1 and Γ_2 are homeomorphic logics, then Γ_2 will have every topological property that Γ_1 has.) We say that a property P is *topological in a class W* just in case $\forall x, y \in W((Px \wedge x \text{ is homeomorphic to y}) \rightarrow Py)$ (that is, just in case P is invariant under homeomorphisms between members of W). Let λ be the class of two-valued logics with binary connectives.

Lemma 3.1 *The expressibility of $\wedge$ is topological in λ.*

Proof: Suppose $(P\&Q)$ expresses $\wedge$ in the logic Γ_1. Then $(P\&Q)$ is assigned 1 by exactly those interpretations that assign 1 to both P and Q. Indeed, given any Γ_1-well-formed formulas φ and ψ, $(\varphi\&\psi)$ is assigned 1 by exactly those interpretations that assign 1 to both φ and ψ. It follows that $\mathsf{Cl}_1(\{\varphi\&\psi\}) = \mathsf{Cl}_1(\{\varphi, \psi\})$. So if we let f be a homeomorphism that assigns well-formed formulas of Γ_2 to well-formed formulas of Γ_1, then

$$\mathsf{Cl}_2(\{f(\varphi), f(\psi)\}) = f[\mathsf{Cl}_1(\{\varphi, \psi\})] = f[\mathsf{Cl}_1(\{\varphi\&\psi\})] = \mathsf{Cl}_2(\{f(\varphi\&\psi)\}).$$

So $f(\varphi\&\psi)$ is assigned 1 by exactly those interpretations that assign 1 to $f(\varphi)$ and $f(\psi)$. Let R and S be variables of Γ_2. Then $f(f^{-1}(R)\&f^{-1}(S))$ is assigned 1 by exactly those interpretations that assign 1 to both R and S. So $\wedge$ is expressible in Γ_2. □

Here is a closely related result. Suppose $\Gamma = \langle V, \mathrm{CON}, *\rangle \in \lambda$. Let & be a function that assigns Γ-well-formed formulas to Γ-well-formed formulas. And suppose $\mathsf{Cl}(\{\varphi\&\psi\}) = \mathsf{Cl}(\{\varphi, \psi\})$ for all Γ-well-formed formulas φ and ψ. Then, given any Γ-interpretation $\mathcal{F}$ and any Γ-well-formed formulas φ and ψ, $\mathcal{F}(\varphi\&\psi) = 1$ if and only if $\mathcal{F}(\varphi) = \mathcal{F}(\psi) = 1$. So, in a λ-logic, a function obeys the truth table for $\wedge$ if (and, in fact, only if) the function plays a particular role in the closure topology of that logic. Less formally, if we already know that the intended theory of meaning for a language is classical, then, "The meaning of a connective like 'and' is . . . given by its role in inference" (as Koslow expresses the doctrine of Belnap [1] on p. 26 of his [5]). Well, what other logical expressions are like 'and' in this respect? For a relevant negative result, see Theorem 4.2 below. First, though, here is another positive result.

Lemma 3.2 *The expressibility of $\top$ is topological in λ.*

Proof: Suppose $\tau(P, Q)$ expresses $\top$ in the logic Γ_1. Then $\tau(P, Q)$ is assigned 1 by every interpretation and hence, $\varnothing \models \tau(P, Q)$ and hence, $\tau(P, Q) \in \mathsf{Cl}_1(\varnothing)$. If f is a homeomorphism, then $f[\mathsf{Cl}_1(\varnothing)] = \mathsf{Cl}_2(f[\varnothing]) = \mathsf{Cl}_2(\varnothing)$. So $f(\tau(P, Q)) \in \mathsf{Cl}_2(\varnothing)$ and hence, $\varnothing \models f(\tau(P, Q))$ and hence, $f(\tau(P, Q))$ is assigned 1 by every interpretation. I note without proof that a 1-constant function is expressible in a system of binary connectives only if the binary 1-constant function is expressible. $\square$

Suppose $\Gamma = \langle V, \mathrm{CON}, *\rangle \in \lambda$. Let τ be a function that assigns Γ-well-formed formulas to Γ-well-formed formulas. And suppose $\tau(\varphi, \psi) \in \mathsf{Cl}(\varnothing)$ for all Γ-well-formed formulas φ and ψ. Then, given any Γ-interpretation $\mathcal{F}$ and any Γ-well-formed formulas φ and ψ, $\mathcal{F}(\tau(\varphi, \psi)) = 1$. So, in a λ-logic, a function obeys the truth table for T-constant if (and, in fact, only if) the function plays a particular role in the closure topology of that logic. Less formally, if we already know that the intended theory of meaning for a language is classical, then we can tell whether an expression obeys the truth table for T-constant by examining its role in inference.

4 Logic $\nsubseteq$ topology Within λ, homeomorphisms are sensitive to the expressibility of $\wedge$ and $\top$. Are there any other binary, two-valued truth functions whose expressibility is topological in λ? After some preliminary definitions and a lemma, we shall see that there are not!

Say that $\varphi \approx \psi$ (φ is *closure equivalent* to ψ) if and only if $\mathsf{Cl}(\{\varphi\}) = \mathsf{Cl}(\{\psi\})$. (So $\varphi \approx \psi$ if and only if $\{\varphi\} \models \psi$ and $\{\psi\} \models \varphi$.) Let $u(\varphi) = \{\psi : \varphi \approx \psi\}$. (So each $u(\varphi)$ is a closure equivalence class.) Let $U = \{u(\varphi) : \varphi \text{ is a well-formed formula}\}$. (So U is the partition consisting of all closure equivalence classes.) Say that $A \lesssim B$ if and only if $\mathsf{Cl}(A) \subseteq \mathsf{Cl}(B)$. Then $u(\varphi) \lesssim u(\psi)$ if and only if $\mathsf{Cl}(\{\varphi\}) \subseteq \mathsf{Cl}(\{\psi\})$. Say that a set is *closed* if and only if it contains its own closure (that is, if and only if every consequence of the set is a member of the set). Finally, say that a logic is *conjunctive* if and only if each of its closed sets is the closure of a singleton. This would mean that every set of well-formed formulas is equivalent to (has exactly the same consequences as) some well-formed formula (or, more correctly, a set whose only member is some well-formed formula). A more familiar phenomenon would be a *finite* set's equivalence to the conjunction of its members.

Theorem 4.1 *If Γ_1 and Γ_2 are conjunctive logics with $\approx$-partitions U_1 and U_2, and if there is a $\lesssim$-isomorphism on U_1 onto U_2 that preserves cardinality, then Γ_1 is homeomorphic to Γ_2.*

Proof: Let h be the $\lesssim$-isomorphism. Suppose A is a set of well-formed formulas closed in Γ_1. Let $A = \mathsf{Cl}_1(\{\alpha\})$. Then $\psi \in \bigcup_{\varphi\in A} h(u(\varphi))$ if and only if $\psi \in h(u(\varphi))$ for some $\varphi \in A$ if and only if $\psi \in h(u(\varphi))$ for some $u(\varphi) \lesssim u(\alpha)$ if and only if $\psi \in h(u(\varphi))$ for some $h(u(\varphi)) \lesssim h(u(\alpha))$ if and only if $\psi \in \mathsf{Cl}_2(h(u(\alpha)))$. So $\bigcup_{\varphi\in A} h(u(\varphi))$ is closed. Similarly, $\bigcup_{\varphi\in B} h^{-1}(u(\varphi))$ is closed whenever B is a set of well-formed formulas closed in Γ_2. Theorem 5.5 of [8] guarantees the existence of a homeomorphism under exactly these conditions. (See Lemma 7.2 below.) □

Theorem 4.2 *$\wedge$ and $\top$ are the only binary, two-valued truth functions whose expressibility is topological in λ.*

Proof: Let Γ_1 be a P_5 logic with two variables and let Γ_2 be a C_1 logic with one variable. The closure equivalence classes of Γ_1 form a Boolean lattice of type 2^2, as do those of Γ_2. All the closure equivalence classes of these logics are denumerable. Furthermore, both logics are conjunctive. So, by Theorem 4.1, these logics are homeomorphic. But Γ_2 is functionally complete, while $\top$, $\wedge$, $=_1$, and $=_2$ are the only binary truth functions expressible in Γ_1. ($=_2$ is the second binary projection function; that is, the binary, two-valued truth function that always returns its second argument.) So the expressibility of the 12 remaining binary truth functions is not a topological property. As already noted, an S_1 logic with two variables will be homeomorphic to any S_2 logic with two variables. So the expressibility of $=_1$ and $=_2$ is not topological either.[4] □

Should one still insist that homeomorphism is an excellent criterion for the equivalence of logical systems? A diehard homeomorphophile might argue that expressibility is not *really* a logically significant notion and that, hence, Theorem 4.2 is of no concern. But that hardly seems right. Such a substantial rearrangement of the way most people think about logic could be justified only by an even more substantial payoff. Since there seems no prospect of one, it is better to concede that homeomorphism is sometimes a *poor* criterion for the equivalence of logical systems.

With that concession behind us, we can begin to determine which logically significant properties *are* topological (or are topological in some important class).

5 Logics with infinitely many variables Let λ^∞ be the class of two-valued logics with binary connectives and infinitely many variables. It turns out that many important properties of logical systems are topological in λ^∞. Note, first, that each of the following properties is topological in λ. (Indeed, the first six are topological *simpliciter*.)

1. The set of all well-formed formulas is the closure of a singleton. (That is, some well-formed formula implies all well-formed formulas. Every logic with an unsatisfiable well-formed formula has this property.)
2. No closed set is empty. (So, in particular, the empty set itself has a nonempty closure. So some well-formed formulas are implied by the empty set of well-formed formulas. So some well-formed formulas are assigned 1 by every interpretation.)
3. The lattice of closed sets has no atoms. (An atom is a closed set whose only closed proper subset is $\mathsf{Cl}(\varnothing)$.)
4. There are 2^k maximally consistent sets, where k is the number of well-formed formulas. (A maximally consistent set is a closed set whose only closed proper superset is the set of all well-formed formulas.)
5. If $\mathsf{Cl}(\{\varphi\})$ and $\mathsf{Cl}(\{\psi\})$ are atoms and $\mathsf{Cl}(\{\varphi, \psi\})$ is not the set of all well-formed formulas, then $\mathsf{Cl}(\{\varphi, \psi\})$ properly contains no closed sets other than

$\mathsf{Cl}(\varnothing)$, $\mathsf{Cl}(\{\varphi\})$, and $\mathsf{Cl}(\{\psi\})$. (This obscure property allows us to distinguish between L_1 and R_{13} logics. The idea is that, in an L_1 logic, $\mathsf{Cl}(\{P, Q\})$ will properly contain $\mathsf{Cl}(\varnothing)$, $\mathsf{Cl}(\{P\})$, $\mathsf{Cl}(\{Q\})$, *and* $\mathsf{Cl}(\{P \longleftrightarrow Q\})$, whereas, in an R_{13} logic, $\mathsf{Cl}(\{P, Q\})$ will properly contain no closed sets other than $\mathsf{Cl}(\varnothing)$, $\mathsf{Cl}(\{P\})$, and $\mathsf{Cl}(\{Q\})$.)

6. If A is an atom, then $A\backslash\mathsf{Cl}(\varnothing) = \{\varphi \in A : \varphi \notin \mathsf{Cl}(\varnothing)\}$ is infinite. (That is, each atom has infinitely many members that are not tautologies.)
7. $\wedge$ is expressible.

Lemma 5.1 *Each C_1 logic in λ^∞ has the seven properties listed above.*

Proof: Let $\Gamma[C_1]$ be a C_1 logic in λ^∞. We confirm that $\Gamma[C_1]$ has all seven properties.

1. Just consider $\mathsf{Cl}(\{\bot(P, Q)\})$.
2. Note that $\top(P, Q) \in \mathsf{Cl}(\varnothing)$.
3. Note first that an atom is always the closure of a singleton. (For suppose A is closed and $\psi \in A\backslash\mathsf{Cl}(\varnothing)$. Then A contains $\mathsf{Cl}(\{\psi\})$, while $\mathsf{Cl}(\{\psi\})$ properly contains $\mathsf{Cl}(\varnothing)$. So, if A is an atom, then $A = \mathsf{Cl}(\{\psi\})$.) Now given any $\Gamma[C_1]$-well-formed formula $\varphi \notin \mathsf{Cl}(\varnothing)$, we can always pick a $\Gamma[C_1]$ well-formed formula ψ such that $(\varphi \vee \psi) \notin \mathsf{Cl}(\varnothing)$ and $\varphi \notin \mathsf{Cl}(\{\psi\})$. Then $\mathsf{Cl}(\{\varphi\})$ properly contains $\mathsf{Cl}(\{\varphi \vee \psi\})$, while $\mathsf{Cl}(\{\varphi \vee \psi\})$ properly contains $\mathsf{Cl}(\varnothing)$. So $\mathsf{Cl}(\{\varphi\})$ cannot be an atom.
4. In $\Gamma[C_1]$, there is a pairing between maximally consistent sets and sets of variables: given any $A \subseteq V$, just consider $\mathsf{Cl}(A \cup \{\neg\varphi : \varphi \in V\backslash A\})$.

5,6. Since $\Gamma[C_1]$ has no atoms, all of its atoms have the required properties.

7. Each C_1 logic is functionally complete. □

An inventory of the 33 Post-equivalence classes reveals that the C_1 logics are the only members of λ^∞ with all seven properties. (They are, in fact, the only members of λ^∞ with the first four properties.)

	R_1	R_2	R_3	R_4	R_6	R_8	R_9	R_{11}	R_{13}	S_1	S_2	S_3	S_4	S_5	S_6	P_1
1	−	−	+	−	−	+	+	+	+	−	−	−	−	+	+	−
2	−	+	−	−	+	−	+	+	+	−	−	+	+	−	+	−
3	−	−	−	−	−	−	−	−	−	+	+	+	+	+	+	−
4	−	−	−	+	−	−	−	−	+	−	−	−	−	−	−	−
5	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
6	+	−	−	+	+	+	−	+	+	+	+	+	+	+	+	+
7	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+

P_2	P_3	P_4	P_5	P_6	A_1	A_2	A_3	A_4	L_1	L_2	L_3	C_1	C_2	C_3	F_4^∞	F_8^∞
−	+	+	−	+	+	−	+	−	+	−	+	+	−	+	−	+
−	−	−	+	+	+	+	−	−	+	+	−	+	+	−	+	−
−	−	−	−	−	+	+	+	+	−	−	−	+	+	+	+	−
−	−	−	−	−	−	−	−	−	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	−	−	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	−	−	−	+	+	+	−	+

Since all seven properties are topological in λ, the property of being a C_1 logic (and hence, the property of being functionally complete) is topological in λ^∞. Say that a type of logic is topological in λ^∞ if and only if membership in that type is topological in λ^∞. Then, more briefly, C_1 is topological in λ^∞. Indeed, inspection of our table allows us to say more.

Lemma 5.2 25 *out of* 33 *types of logic are topological in* λ^∞*: the only exceptions are* S_1, S_2, S_3, S_4, P_1, P_2, P_3, *and* P_4.

Theorem 5.3 $=_1$ *and* $=_2$ *are the only binary, two-valued truth functions whose expressibility is not topological in* λ^∞.

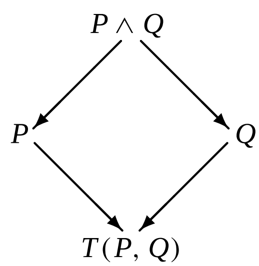
Proof: The expressibility of $=_1$ and $=_2$ cannot be topological in λ^∞, since, for example, S_1 and S_2 logics with the same infinite number of variables are homeomorphic. On the positive side, we already know that the expressibility of $\wedge$ and $\top$ is topological in λ. Furthermore, a λ^∞ logic expresses $\perp$ if and only if it has property 1; and a λ^∞ logic expresses $\vee$ if and only if it has property 3. We can characterize the remaining truth functions by listing the logics in which they are expressible. For example, a λ^∞ logic expresses $\downarrow$ if and only if it is of type C_1. □

An example may help to convey why this theorem is so welcome to proponents of the closure theoretic approach. Say that a set is *dense* if and only if its closure is the set of all well-formed formulas. And suppose $\neg$ is a function that assigns a well-formed formula to every well-formed formula. Then $\neg$ is a *closure theoretic classical negation* (CTCN for short) if and only if, for each well-formed formula φ and each set of well-formed formulas A, $\{\varphi, \neg\varphi\}$ is dense and $(\mathsf{Cl}(A \cup \{\varphi\}) \cap \mathsf{Cl}(A \cup \{\neg\varphi\})) \subseteq \mathsf{Cl}(A)$. The existence of a CTCN is a topological property, but the expressibility of $\neg_1$ is not even topological in λ. So, since each logic that expresses $\neg_1$ has a CTCN, there must be logics that do not express $\neg_1$ but do have a CTCN. (For example, in a P_5 logic with four variables P, Q, R, and S, the conjunction of Q, R, and S behaves like the negation of P.) The expressibility of $\neg_1$ is topological in λ^∞. So one can hope that the closure theoretic notion of negation will agree with the more usual semantic conception within λ^∞. In fact, this turns out to be the case.

Suppose Γ is a λ^∞ logic with a CTCN $\neg$. Then $\{\varphi\}$ is dense if and only if $\neg\varphi \in \mathsf{Cl}(\varnothing)$; and $\varphi \in \mathsf{Cl}(\varnothing)$ if and only if $\{\neg\varphi\}$ is dense. (Cf. [11], pp. 119–20.) So the set of all well-formed formulas is the closure of a singleton if and only if no closed set is empty. That is, Γ either has both property 1 and property 2 (from the list of seven properties at the beginning of this section) or it has neither. This rules out 18 of 33 λ^∞ logics. Of the remaining 15, six (R_1, S_1, S_2, P_1, P_2, and A_4) have no finite, dense set of well-formed formulas, while five (R_9, R_{11}, S_6, P_6, and A_1) have only one maximally consistent set (whereas Theorems 6.32 and 7.33 of [8] imply that the maximally consistent sets form a closed basis in Γ). This leaves only the four logics that express $\neg_1$. So a λ^∞ logic has a CTCN if and only if it expresses $\neg_1$. Furthermore, if θ expresses $\neg_1$, then, by Theorem 9.1 of [8], any CTCN $\neg$ will agree with θ up to closure equivalence. That is, $\theta(\varphi, \psi) \approx \neg\varphi$ for all well-formed formulas φ and ψ. But this, in turn, means that $\neg\varphi$ is assigned 1 by exactly those interpretations that assign 0 to φ. So, within λ^∞, each CTCN has the semantic properties of classical negation. In this domain, then, the closure theorist's use of the term 'classical negation' seems fully justified. (Here the meaning of 'not' really is 'given by its role in inference'.) Elsewhere the case is not so clear. (See the discussion of Koslow below.)

Theorem 5.3 assures us that homeomorphism and Post-equivalence impose *almost* the same partition on λ^∞. They differ only because homeomorphism is sometimes insensible to the expressibility of the binary projection functions. Since these truth functions are, arguably, of no logical interest, this might count as evidence of homeomorphism's *superiority*. In any case, it is hardly clear that homeomorphism is inferior to Post-equivalence in this regard. Nor is it too outlandish to claim that homeomorphism is an "excellent" criterion for the equivalence of λ^∞ logics.

6 Appendix I We now briefly consider Koslow's structuralist approach to logic in order to forestall a seductive, but erroneous, interpretation of some of Koslow's claims. Consider again a P_5 logic with two sentential variables. Call it Γ. If $\models$ is the consequence relation of Γ, then naturally we say that $\varphi_1, \ldots, \varphi_n$ (jointly) *imply* ψ if and only if $\{\varphi_1, \ldots, \varphi_n\} \models \psi$. So we obtain the implicational structure



where each well-formed formula φ represents the denumerably many well-formed formulas equivalent to φ. Recall (from §2) that our logics come supplied with a theory of meaning encoded in a class of interpretations. The Γ-interpretations are the four 0, 1- valued homomorphisms (on the algebra of Γ-well-formed formulas) that respect the classical matrices for conjunction and $\top$- constant. (The consequence relation $\models$ is defined in terms of these interpretations.) Since one of the Γ-interpretations assigns 1 to every Γ-well-formed formula, there are no well-formed formulas φ and ψ such that φ is assigned 1 by exactly those Γ-interpretations that assign 0 to ψ. That is, *from the point of view of* Γ*'s underlying theory of meaning*, there are no Γ-well-formed formulas that stand to one another in the semantic relationship of a well-formed formula to its classical negation. Nonetheless, Koslow proceeds, " . . . to sort out, for any [well-formed formula], those elements in the structure that are the negations of [that well-formed formula]" (Koslow [5], p. 91). Say that a Γ-well-formed formula φ is a *Koslowian negation* of a Γ-well-formed formula ψ if and only if φ and ψ satisfy the following two conditions:

N1 φ, ψ (jointly) imply every Γ-well-formed formula;
N2 if θ, ψ imply every Γ-well-formed formula, then θ implies φ.

Now suppose N is an operator that assigns Γ-well-formed formulas to Γ-well-formed formulas. Then we say that N is a *Koslowian classical negation operator* on Γ if and only if for each Γ-well-formed formula φ,

CN1 $N(\varphi)$ is a Koslowian negation of φ;
CN2 N(N(φ)) implies φ.

It is easy to confirm that there are Koslowian classical negation operators on Γ. Consider: $N(P) = Q$; $N(Q) = P$; $N(P \wedge Q) = T(P, Q)$; $N(T(P, Q)) = (P \wedge Q)$. But we already know that however we define N and whatever Γ-well-formed formula φ we might consider, φ and $\mathrm{N}(\varphi)$ will both be assigned 1 by a Γ-interpretation and hence, *from the perspective of* Γ*'s intended theory of meaning*, $\mathrm{N}(\varphi)$ will not behave semantically like the classical negation of φ.

We could, of course, concoct a *different* semantic scheme under which N is better behaved. Let the Γ-N-INTERPRETATIONS be the two Γ-interpretations that assign distinct values to the variables P and Q. Then, given any Γ-well-formed formula φ, φ will be assigned 1 by exactly those Γ-N-INTERPRETATIONS that assign 0 to $\mathrm{N}(\varphi)$. So there is a (nonstandard) scheme for distributing truth values that supplies N with the semantic properties of classical negation. (Theorem 8.24 of [8] gives a sufficient condition for the existence of such schemes. See also ch. 19 of [5].) Of course, if a logic with a Koslowian classical negation operator is characterized semantically (as are our logics), there is no guarantee that the *intended* semantics will treat the operator classically. Evidently, when Koslow "sorts out the classical negations," he means only to identify operators whose implicational properties *sometimes* imply that *some* distributions of truth values treat the operators like classical negation. This is a worthy task. (Cf. the discussion of Tarski-style conditions in ch. 2 of Wójcicki [15].) We should be aware, though, that these distributions of truth values need not conform to any *intended* semantics; they might exclude some standard assignments of truth values to variables; and they might even fail to be homomorphisms on the algebraic structure of well-formed formulas.

Our motto might be: the semantic properties of a logic do not always supervene on its implicational structure.[5] And this is so even if we already know that the intended theory of meaning is classical. It is some consolation, though, that many significant semantic properties of λ^∞ logics are discernible from the closure theoretic structure of those logics. For example, if we already know that a structure is a λ^∞-logic, we can tell from its closure topology whether the underlying theory of meaning treats any operator as classical negation. (Alas, the property of being a λ^∞-logic is not itself topological.)

7 Appendix II This section features (1) a variation on Theorem 4.1 that should be somewhat more widely applicable and (2) a theorem that helps us determine when the conditions of Theorem 4.1 are satisfied. The proofs of some helpful lemmas are sufficiently routine to be omitted.

We assume that $\langle S, \mathrm{Cl} \rangle$ is a *closure space*; that is, Cl is a closure operator that assigns a subset of S to each subset of S. A closure space is *finitely conjunctive* if and only if each of its finite sets has the same closure as a singleton. A closure space is *finitary* if and only if a point x belongs to the closure of a finite subset of a set A whenever x belongs to the closure of A. $W \subseteq U$ is an *ideal* (in the structure $\langle U, \lesssim \rangle$) if and only if (1) $A \in W$ whenever $A \in U$, $B \in W$, and $A \lesssim B$; and (2) each finite subset of W has an upper bound in W. (U and $\lesssim$ are defined in §4.)

Lemma 7.1 *The following are equivalent: (i)* $\langle S, \mathrm{Cl} \rangle$ *is finitary and finitely conjunctive; (ii)* $\cup W$ *is closed if and only if* W *is an ideal.*

Lemma 7.2 *If h is a bijection on U_1 onto U_2 that preserves cardinality, if $\bigcup_{x\in A} h(u(x))$ is closed whenever A is a closed subset of S_1, and if $\bigcup_{x\in B} h^{-1}(u(x))$ is closed whenever B is a closed subset of S_2, then $\langle S_1, \mathsf{Cl}_1\rangle$ is homeomorphic to $\langle S_2, \mathsf{Cl}_2\rangle$.*

Lemma 7.2 is just Theorem 5.5 of [8].

Theorem 7.3 *If the closure spaces $\langle S_1, \mathsf{Cl}_1\rangle$ and $\langle S_2, \mathsf{Cl}_2\rangle$ are both finitary and finitely conjunctive and if there is a $\lesssim$-isomorphism on U_1 onto U_2 that preserves cardinality, then $\langle S_1, \mathsf{Cl}_1\rangle$ is homeomorphic to $\langle S_2, \mathsf{Cl}_2\rangle$.*

Proof: Let h be the $\lesssim$-isomorphism. Suppose A is a closed subset of S_1. Then, by Lemma 7.1, $\{u(x) : x \in A\}$ is an ideal. So $\{h(u(x)) : x \in A\}$ is also an ideal and hence, by Lemma 7.1, $\cup\{h(u(x)) : x \in A\}$ is closed. By similar reasoning, $\cup\{h^{-1}(u(x)) : x \in B\}$ is closed whenever B is a closed subset of S_2. Now apply Lemma 7.2. □

It is common for logics to be both finitary and finitely conjunctive. (Every logic that expresses conjunction and at least one logical truth is finitely conjunctive. The usual deductive systems, with proofs all of finite length, induce finitary closure spaces.) Since conjunctiveness (as required by Theorem 4.1) is a more unusual property, one would expect 7.3 to be the more useful theorem.

A chain of closed sets K is *maximal* in a set B if and only if (1) B is an upper bound of K; but (2) B is not an upper bound of any chain that properly contains K. A set is *finitely axiomatizable* if and only if it has the same closure as a finite set.

Lemma 7.4 *If some finite chain is maximal in $\mathsf{Cl}(B)$, then B is finitely axiomatizable.*

The converse of Lemma 7.4 is false. Let our closed sets be $\omega + 1$ together with all the finite ordinals. Then $\omega + 1 = \mathsf{Cl}(\{\omega\})$, but no finite chain is maximal in $\omega + 1$. A closure space has the *finite rank property* if and only if some finite chain is maximal in B whenever B is closed.

Corollary 7.5 *In a closure space with the finite rank property, each set is finitely axiomatizable.*

The finite rank property does not imply finitariness. Let our closed sets be $\varnothing$, $\{2\}$, ω, and all finite sets of odd numbers. Then 2 is in the closure of the set of all odd numbers, but it is not in the closure of any finite set of odd numbers.

Theorem 7.6 *Each finitely conjunctive closure space with the finite rank property is conjunctive.*

Proof: Just apply Corollary 7.5. □

Conjunctiveness may be an unusual property, but Theorem 7.6 at least helps us to detect it (and hence, helps us to determine when Theorem 4.1 can be applied).

Acknowledgments Norman Martin's insight into this topic was indispensable. Since this essay represents a partial retreat from an earlier position, I should note that the dogmatic tone of Pollard and Martin [11] was mainly my doing. Martin himself expressed a more moderate and, as it turns out, more defensible view in the Preface of Martin and Pollard [8]. The comments of two scrupulous referees helped me to eliminate a host of unnecessary obscurities.

NOTES

1. I mean homeomorphisms between said logics. The closure space of, say, a group of integers need not have the semantic properties of a logic to which it is homeomorphic.

2. For a pioneering study of closure operators see Moore [9], pp. 59–60. Closure operators have received attention in both lattice theory and universal algebra. See, for example, Birkhoff [2] and Cohn [4]. Logicians may be more familiar with the closure theoretic investigations of Tarski [14] and its progeny. Note though, that Tarski actually studies the narrower notion of finitary closure operators on a countable domain. [3], [8], and [15] are booklength applications of closure theoretic notions to logic.

3. These logics are, however, syntactically equivalent in the sense of Segerberg [13], p. 43. (See also Pelletier [10], p. 424.) Perhaps this counts as a point in favor of syntactic equivalence. Note though that homeomorphic logics are always syntactically equivalent. So syntactic equivalence is at least as insensitive to logically important distinctions as is homeomorphism. While syntactic equivalence is a relatively loose standard, Martin's notion of *R*-equivalence is a relatively tight one. *R*-equivalence relations are homeomorphisms that preserve certain aspects of logical form: for example, they pair sentential variables with sentential variables. Although P_5 logics with two variables are homeomorphic (and hence, syntactically equivalent) to C_1 logics with one variable, they are not *R*-equivalent. Cf. [7], pp. 25–26. For further references and a discussion of some other notions of equivalence, see [15], pp. 66–71.

4. Theorem 4.2 may seem crazy in light of facts such as the following. Suppose $\Gamma = \langle V, \mathrm{CON}, *\rangle \in \lambda$ and $g \in \mathrm{CON}$. Then $g^* = \vee$ if and only if $\mathsf{Cl}(\{g(\varphi, \psi)\}) = (\mathsf{Cl}(\{\varphi\}) \cap \mathsf{Cl}(\{\psi\}))$ for all Γ-well-formed formulas φ and ψ. So a connective will express disjunction in a λ-logic if and only if the connective plays a particular role in the closure topology of that logic. But if the topological structure of a λ-logic determines whether a connective expresses disjunction, how is it possible that the expressibility of disjunction is not topological in λ? The trick is that topological structure determines whether a member of CON expresses disjunction only because each member of CON is guaranteed to express some classical truth function. (The result does not hold for arbitrary functions.) Since the property of expressing a classical truth function is not topological, it is no great paradox that the expressibility of ∨ fails to be topological in λ. Still, if we have somehow determined that a word expresses a classical truth function, we may be able to determine from its "role in inference" whether it expresses two-valued disjunction.

5. For an example of homeomorphic modal logics that differ substantially in their semantic properties, see Pelletier [10]. For an example of homeomorphic modal logics that differ substantially in their lattice theoretic properties, see Makinson [6].

REFERENCES

[1] Belnap, N. D., "Tonk, plonk, and plink," *Analysis*, vol. 22 (1962), pp. 130–34. 3

[2] Birkhoff, G., *Lattice Theory*, American Mathematical Society Colloquium Publications, vol. 25, Providence, 1967. Zbl 0153.02501 MR 37:2638 7

[3] Cleave, J. P., *A Study of Logics*, Oxford University Press, Oxford, 1991. Zbl 0763.03003 MR 93c:03001 7

[4] Cohn, P. M., *Universal Algebra*, Harper and Row, New York, 1965. Zbl 0141.01002 MR 31:224 7

[5] Koslow, A., *A Structuralist Theory of Logic*, Cambridge University Press, Cambridge, 1992. Zbl 0813.03001 MR 93i:03003 3, 6, 6

[6] Makinson, D., "A warning about the choice of primitive operators in modal logic," *Journal of Philosophical Logic*, vol. 2 (1973), pp. 193–96. Zbl 0266.02016 MR 54:2424 7

[7] Martin, N. M., *Systems of Logic*, Cambridge University Press, Cambridge, 1989. Zbl 0752.03001 MR 91c:03001 7

[8] Martin, N. M., and S. Pollard, *Closure Spaces and Logic*, Kluwer Academic Publishers, Dordrecht, 1996. Zbl 0855.54001 MR 97m:03022 2, 2, 5, 5, 6, 7, 7, 7

[9] Moore, E. H., *Introduction to a Form of General Analysis*, American Mathematical Society Colloquium Publications, vol. 2, New Haven, 1910. 7

[10] Pelletier, F. J., "Six problems in 'translational equivalence'," *Logique et Analyse*, vol. 27 (1984), pp. 423–34. Zbl 0589.03006 MR 86j:03015 7, 7

[11] Pollard, S., and N. M. Martin, "Closed bases and closure logic," *The Monist*, vol. 79 (1996), pp. 117–27. 1, 5, 7

[12] Post, E. L., *The Two-Valued Iterative Systems of Mathematical Logic*, Princeton University Press, Princeton, 1941. Zbl 0063.06326 MR 2,337a 2, 3

[13] Segerberg, K., *Classical Propositional Operators*, Oxford University Press, Oxford, 1982. Zbl 0491.03003 MR 83i:03001 7

[14] Tarski, A., "Fundamentale begriffe der methodologie der deduktiven wissenschaften I," *Monatshefte für Mathematik und Physik*, vol. 37 (1930), pp. 361–404. 7

[15] Wójcicki, R., *Theory of Logical Calculi*, Kluwer Academic Publishers, Dordrecht, 1988. MR 90j:03001 6, 7, 7

Division of Social Science
Truman State University
Kirksville, Missouri 63501
email: spollard@truman.edu