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Topological Modal Logics Satisfying Finite Chain Conditions

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Abstract We modify the semantics of *topological modal logic*, a language due to Moss and Parikh. This enables us to study the corresponding theory of further classes of subset spaces. In the paper we deal with spaces where every chain of opens fulfils a certain finiteness condition. We consider both a *local* finiteness condition relevant to points and a *global* one concerning the whole frame. Completeness of the appearing logical systems, which turn out to be generalizations of the well-known modal system **G**, can be obtained in the same manner as in the case of the general subset space logic. It is our main purpose to show that the systems differ with regard to the *finite model property*.

1 Introduction A certain logical framework has been introduced recently by Moss and Parikh [9], which in particular admits *reasoning about knowledge*. Complementarily to earlier approaches to this topic, the authors focus on a certain relationship between *knowledge* and *topology* therein. Let us briefly review the ideas of Moss and Parikh for convenience.

In computer science the modal system **S5** is commonly used to describe knowledge of a single agent formally (see Fagin [3]). Corresponding frames are equivalence relations and the class containing the actual state represents the agent's current view of the world. Now spending *effort* generally results in more knowledge, as it shrinks the set of states the agent considers possible. For instance, the computation of an infinite binary sequence gives more knowledge of the computed function $f : \mathbb{N} \longrightarrow \{0, 1\}$ the more digits are printed, and each output successively halves the set of alternatives. Thus shrinking the set of possible states corresponds to an increasing of the agent's knowledge. In this way knowledge acquisition turns out to be a *topological* phenomenon, namely, an approximation procedure which is modeled by descending within a system of sets.

Accordingly, Moss and Parikh have defined a *modal language with two operators, K* and $\Box$. These modalities represent *knowledge* and *effort*, respectively; however, their semantics differs from that of ordinary modal operators to a large extent:

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in the present case underlying frames consist of a nonempty set X and a distinguished set of subsets of X, $\mathcal{O}$ called the *opens* (although they need not necessarily be open sets in the sense of topology); K then varies over the elements of an open set, whereas $\Box$ captures the shrinking of an open. While K retains its **S5**-like character from the usual logic of knowledge, the $\Box$-modality reflects properties of the inclusion relation and consequently, is **S4**-like.

Various systems of this kind of *topological modal logic*, *TML*, have been studied meanwhile. Sound and complete axiomatizations are available, for example, for the validities of the basic *subset space logic* in which $\mathcal{O}$ may be an arbitrary set of subsets of X (Dabrowski [2]), and for *topologic* where frames actually carry a topology as the set of opens (Georgatos [4], [2]). Moreover, the satisfiability problem is shown to be *decidable* in both cases.

As a further topic, the topological modal theory of *treelike spaces* has been examined in [5]. Since these structures play a certain part in the present paper as well, let us give the basic definition here. A subset frame is called treelike, if and only if $U \subseteq V$ or $V \subseteq U$ or $U \cap V = \varnothing$ holds for all $U, V \in \mathcal{O}$. As it is pointed out in Georgatos [5], treelike spaces are very appropriate to reason about the development of knowledge in computational contexts; additionally, they generalize the computation-tree structure of branching-time semantics. In that paper, a corresponding sound and complete axiom system as well as a decidability proof for the set of derivable theorems is given; actually the finite model property holds for the logic of treelike spaces.

The semantics of *TML* has been changed slightly in the note Heinemann [7], which is a predecessor of this paper. There it has been assumed that an investment of computational resources is always provided with *success*, that is, yields properly more knowledge of the computed object. Assuming this seems to be quite reasonable for several applications, for exapmle, reasoning about programs which output 0-1-streams. The well-known Cantor space of all infinite 0-1-sequences may serve as the underlying computational model then. This model and those programs based on it occur in effective analysis (for example, see Weihrauch [11]). In fact, it has been our original motivation investigating the systems of topological modal logic to eventually provide a framework that is suitable for reasoning about the behavior of effective approximations to (the) infinite objects (occurring in analysis).

Also presently it is presupposed that we get a *proper* shrinking of the actual open in each computation step. This amounts to a **K4**-like modal operator $\Box$ in the formal theory. However, we now take into account the limitedness of resources, computation time in particular. According to this we impose *finiteness conditions* on the trees modeling program executions. Such finiteness requirements read as *finite descent conditions* on the set of opens and those are emphasized in the given paper. We study two appropriate logics. The first system, **D1**, corresponds to frames sharing the following property of $\mathcal{O}$: every descending chain containing a given point x is of bounded length, where the bound depends on x; this constraint on the opens is called *weak bounded chain condition (wbcc)*. We give schemes axiomatizing the topological modal theory of spaces satisfying wbcc and prove a corresponding *soundness* and *completeness* result. Our proceeding here is similar to that for the subset-space logic in the paper [2]: since wbcc does not hold on the canonical model of the system we will construct an "abstract" space satisfying this property and falsifying a given nonderivable formula; in fact, we can even adapt the construction of [2] to a certain extent. The second system, **D2**, is characterized by the class of spaces where the so-called *bounded chain condition (bcc)* is valid. For these spaces *every* chain of opens is of bounded length by definition. The class of finite irreflexive computation trees occurring in [5] appears as a special case. Again we propose a sound and complete axiomatization. However, our main concern is to establish the *finite model property* for the system **D2**. This property does not hold for **D1**. Finally, we prove that even a certain *small model property* is valid for **D2**. This can be done with the aid of filtrations with respect to suitable Kripke structures associated with **D2**.

As in both systems the class of semantical structures coincides with a class of trees which satisfy a certain finiteness condition we actually obtain two generalizations of the well-known modal system **G** to the context of topological modal logic; compare Smorynski [10], 4.9.

The paper is organized as follows. We first define the basic logical language in Section 2. In Section 3 we treat the system **D1**. The main result there is *completeness* of the given list of axioms. Section 4 represents the core of the paper and therein the finite model property for the system **D2** is proved. The small model property announced above is shown in the final section. In Sections 3 and 5 some proofs are not carried out completely. Because of similarities with corresponding proofs in Sections 2.2 and 2.3 of the paper [2], only ideas are sketched, and different proceedings are pointed out. Consequently, the present paper is not completely self-contained, but acquaintance of the reader with §2 of [2] is assumed. As to the basic notions from (multi)modal logic we like to cite Goldblatt [6], but they can clearly be found elsewhere as well, for example, in Chellas [1].

2 The logical language In this section we define a logical language called *topological modal tree language, TMTL*. This is done in the following way. The *syntax* of *TMTL* is based upon a suitable alphabet, which contains in particular symbols in order to define a recursive set of *propositional variables*, PV. Then the set $\mathcal{F}$ of *TMTL-formulas* is defined by the following clauses:

1. $\mathrm{PV} \subseteq \mathcal{F}$;
2. $\alpha, \beta \in \mathcal{F} \Longrightarrow \neg\alpha, K\alpha, \Box\alpha, (\alpha \wedge \beta) \in \mathcal{F}$;
3. no other strings belong to $\mathcal{F}$.

We omit brackets whenever possible, and we use the following abbreviations besides the usual ones from propositional logic:

$L\alpha$ for $\neg K \neg \alpha$, and
$\Diamond\alpha$ for $\neg\Box\neg\alpha$.

The *semantical domains* of *TMTL* are generally triples (X, O, σ), where X is a nonempty set, O is a set of nonempty subsets of X (specified further if need be), and $\sigma : \mathrm{PV} \times X \to \{0, 1\}$ is a mapping called *X-valuation*. The pair $\mathcal{S} = (X, O)$ is named a *subset frame* from now on. The elements of O mostly are called the *opens of* $\mathcal{S}$.

We concentrate on some special types of subset frames presently, which are introduced in the subsequent definition.

Definition 2.1 (Chain conditions) Let $\mathcal{S} = (X, O)$ be a subset frame.

1. $\mathcal{S}$ satisfies the *weak bounded chain condition (wbcc)*, if and only if for all $x \in X$ there is an $n \in \mathbb{N}$ such that every descending $\subset$-chain of opens containing x is at most n in length; here '$\subset$' means *proper* inclusion.
2. $\mathcal{S}$ satisfies the *finite chain condition (fcc)*, if and only if every $\subset$-chain of opens is of finite length.
3. $\mathcal{S}$ satisfies the *bounded chain condition (bcc)*, if and only if there is an $n \in \mathbb{N}$ such that every $\subset$–chain of opens is at most n in length.
4. Let σ be an X-valuation. Then $\mathcal{M} = (X, \mathcal{O}, \sigma)$ is called a *subset space* or a *model* (based on $\mathcal{S}$). If $\mathcal{S}$ satisfies wbcc (fcc, bcc), then this attribute is also attached to every model based on $\mathcal{S}$.

We now define *validity* of *TMTL*-formulas in models based on subset frames. As motivated in the introduction, our definition differs slightly from the usual one in [9], [4], [5], and [2].

Definition 2.2 (Semantics of *TMTL*) Let $\mathcal{S} = (X, \mathcal{O})$ be a subset frame and let $\mathcal{M} = (X, \mathcal{O}, \sigma)$ be a model based on $\mathcal{S}$.

1. $X \otimes \mathcal{O} := \{(x, U) \mid U \in \mathcal{O},\ x \in U\}$ is called the set of *neighborhood situations* of $\mathcal{S}$; subsequently we designate elements of $X \otimes \mathcal{O}$ simply without brackets.
2. The *validity* of a *TMTL*-formula *in* $\mathcal{M}$ *at a neighborhood situation* x, U is defined by recursion on the structure of formulas:

$$\begin{array}{lcl}
x, U \models_{\mathcal{M}} A & :\Longleftrightarrow & \sigma(A, x) = 1 \\
x, U \models_{\mathcal{M}} \neg\alpha & :\Longleftrightarrow & x, U \not\models_{\mathcal{M}} \alpha \\
x, U \models_{\mathcal{M}} \alpha \wedge \beta & :\Longleftrightarrow & x, U \models_{\mathcal{M}} \alpha \text{ and } x, U \models_{\mathcal{M}} \beta \\
x, U \models_{\mathcal{M}} K\alpha & :\Longleftrightarrow & (\forall y \in U)\ y, U \models_{\mathcal{M}} \alpha \\
x, U \models_{\mathcal{M}} \Box\alpha & :\Longleftrightarrow & (\forall\, V \in \mathcal{O})[\, V \subset U \text{ and } x \in V \\
 & & \qquad\qquad \Longrightarrow x, V \models_{\mathcal{M}} \alpha\,]
\end{array}$$

   for all $A \in \mathrm{PV}$ and formulas $\alpha, \beta \in \mathcal{F}$.
3. We say that a formula $\alpha \in \mathcal{F}$ *holds in* $\mathcal{M}$ and denote this by $\models_{\mathcal{M}} \alpha$, if and only if it holds in $\mathcal{M}$ at every neighborhood situation.

If there is no ambiguity, we omit the index $\mathcal{M}$ in the following. Examples of various subset frames and valid formulas with respect to the usual semantics are given in [5] and [2].

***3 The system* D1** Giving a list of axioms and rules we present a first logical system **D1** which is an extension of the system **G** from ordinary modal logic. Our aim in this section is to show that the **D1**-theorems are precisely the $TMTL$-formulas holding in every model which satisfies wbcc.

Axioms

1. All $\mathcal{F}$-instances of propositional tautologies
2. $(A \to \Box A) \wedge (\neg A \to \Box\neg A)$
3. $K(\alpha \to \beta) \to (K\alpha \to K\beta)$
4. $K\alpha \to (\alpha \wedge KK\alpha)$

5. $L\alpha \rightarrow KL\alpha$
6. $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$
7. $\Box(\Box\alpha \rightarrow \alpha) \rightarrow \Box\alpha$
8. $K\Box\alpha \rightarrow \Box K\alpha$,

for all $A \in$ PV and $\alpha, \beta \in \mathcal{F}$.

Rules

$$\frac{\alpha \rightarrow \beta, \alpha}{\beta} \qquad \text{(modus ponens)}$$

$$\frac{\alpha}{K\alpha} \qquad \text{(K-necessitation)}$$

$$\frac{\alpha}{\Box\alpha} \qquad \text{($\Box$-necessitation)}$$

Some remarks on the axioms seem to be convenient. All but axiom 7 appear in the list of Moss and Parikh axiomatizing the subset space logic. As to comments on the interesting axioms 2 and 8, which go beyond the standard systems involved in the present one, we refer the reader to Moss and Parikh [9]. Two schemes of that list are missing here:

$$\Box\alpha \rightarrow \Box\Box\alpha \quad \text{and} \quad \Box\alpha \rightarrow \alpha.$$

It is well known that the first one can be derived from (6) and (7) with the aid of modus ponens and $\Box$-necessitation (see [6], p. 26, for example); the second scheme is cancelled without compensation because of the modified semantics. (7) is the famous scheme **W** from common modal logic which represents the essential ingredient of the Löb system **G** = **KW**. The latter system plays an important part in investigations relating modal logic to the notion of *provability*; for a closer explanation see Smorynski [10]. Currently it corresponds to the fact that only bounded descent of the subset-component of a neighborhood situation is possible.

Soundness of the axioms with respect to the intended structures can rather easily be established.

Proposition 3.1 *Axioms 1 – 8 hold in every model satisfying wbcc.*

Proof: We only show the validity of (7). Let a subset space $\mathcal{M} = (X, O, \sigma)$ that satisfies wbcc and a neighborhood situation x, U of (X, O) be given. Moreover, assume that

$$x, U \models \Box(\Box\alpha \rightarrow \alpha)$$

holds. We have to prove that $x, U \models \Box\alpha$ holds, that is, that $x, V \models \alpha$ is valid for all opens $V \subset U$ containing x. So let V be such an element of O. Since the subset space satisfies wbcc, x is contained in certain opens $W \subseteq V$ which are minimal among those elements of O which contain x and are properly contained in U. For these W,

$$x, W \models \Box\alpha \rightarrow \alpha$$

holds by assumption. The minimality of W also implies $x, W \models \Box\alpha$. Hence $x, W \models \alpha$. Now the boundedness of every chain of opens between W and V allows us to lift the validity of α to each neighborhood situation x, V' such that $W \subseteq V' \subseteq V$. Consequently, $x, V \models \alpha$. This proves soundness of the scheme 7. $\Box$

As to *completeness*, we proceed via the canonical model $\mathcal{M}_{\mathbf{D1}}$ of **D1**. Although we do not have closure of the system under substitution its canonical model can be built up in the usual way (see [6], §5). That is in particular, the carrier set C of $\mathcal{M}_{\mathbf{D1}}$ equals the set of all maximal **D1**-consistent sets of formulas, and the accessibility relations corresponding to the modal operators K and $\Box$ are defined by

$$s \xrightarrow{L} t \quad :\Longleftrightarrow \quad \{\alpha \in \mathcal{F} \mid K\alpha \in s\} \subseteq t$$

$$s \xrightarrow{\Diamond} t \quad :\Longleftrightarrow \quad \{\alpha \in \mathcal{F} \mid \Box\alpha \in s\} \subseteq t,$$

for all $s, t \in C$, respectively. Note that every $s \in C$ contains all axioms, and if $L\alpha \in s$, then there exists $t \in C$ such that $s \xrightarrow{L} t$ and $\alpha \in t$ (analogously for $\Diamond\alpha$); to show the latter already the properties of the basic modal logic **K** are sufficient, which are present for both modalities.

With the aid of $\mathcal{M}_{\mathbf{D1}}$ one can now construct a subset space validating all of the above axioms, but falsifying a given formula α which is not **D1**-derivable. For this purpose the construction of [2], Section 2.2, can be adapted with only minor modifications concerning the relation of *proper* containment. (Note that no chain condition is demanded at the moment.) Especially, a corresponding *truth lemma* is valid (compare [2], Lemma 2.5), so that we in fact get the following theorem.

Theorem 3.2 *Let $\alpha \in \mathcal{F}$ be a formula which is not derivable in the system* **D1**. *Then there exist a subset space $\mathcal{X} = (X, \mathcal{O}^{\mathcal{X}}, \sigma)$ and a point $x \in X$ such that*

1. *all axioms of* **D1** *hold in $\mathcal{X}$,*
2. *$X \in \mathcal{O}^{\mathcal{X}}$ and $x, X \not\models \alpha$.*

To turn the theorem to good account we must have a somewhat closer look at the construction involved in its proof. Nevertheless, we need not be too detailed here, but report the main issues only. In order to obtain the structure $\mathcal{X}$ one defines suitably a set of points X, and an order-reversing injection i from a certain partially ordered set $(P, \leq)$ into the set of nonempty subsets of X (ordered by inclusion). There is a least element $\perp$ of the partial order $(P, \leq)$, and the mapping i satisfies $i(\perp) = X$. A further property of $(P, \leq)$ is that the set $\{q \mid q \leq p\}$ is linearly ordered, for every $p \in P$. Then a triple (X, P, i) is yielded as the limit of a sequence (X_n, P_n, i_n) of finite *approximations* satisfying for all $n \in \mathbb{N}$

1. $X_n \subseteq X_{n+1}$ and $X_{n+1} \setminus X_n$ contains at most one point,
2. P_{n+1} is an *end* extension of P_n by at most one point,
3. $i_{n+1}(p) \cap X_n = i_n(p)$ for all $p \in P_n$.

Moreover, an element $t(y, p) \in C$ of the canonical model is associated to every pair $(y, p) \in X \times P$ during the construction. This is done by *realizing* every *existential* formula $\Diamond\beta$ and $L\beta$, respectively, such that

$$\text{for all} \quad \gamma \in \mathcal{F} : \gamma \in t(y, p) \quad \text{iff} \quad y, i(p) \models \gamma$$

holds in the final model $\mathcal{X}$, which is based on the frame $(X, i(P))$. Providing instances in this way in fact causes the creation of a "new" open $U \in i(P)$ in case of a '$\Diamond$'-formula δ, which depends on this formula as well as on some already constructed point x. So, the pair (x, δ) may be called the *reason* for U; in particular, we designate $x = r(U)$. It is important to note that one proceeds in accordance with the requirement that no point chosen before x is contained in U; consequently, r is well defined.

As we show by the subsequent theorem it is possible to single out suitable opens then so that a space satisfying wbcc results. We let $\mathrm{sf}(\alpha)$ denote the set of subformulas of $\alpha \in \mathcal{F}$.

Theorem 3.3 *A formula $\alpha \in \mathcal{F}$ is derivable in the system* **D1**, *if and only if α holds in all subset spaces satisfying wbcc.*

Proof: The 'only if' part is an immediate consequence of Proposition 3.1. Now let α be not **D1**-derivable. Consider the model $\mathcal{X}$ and the neighborhood situation x, X from Theorem 3.2. The model we are looking for has carrier X and X-valuation σ, as $\mathcal{X}$. Its set of opens O is constructed inductively in stages with the aid of P:

stage 0:

Let $\bar{P}_0 := \{\bot\} \subseteq P$.

stage $n+1$:

For every $p \in \bar{P}_n$, $y \in i(p)$, and subformula $\Box\beta$ of α such that $\Diamond\neg\beta \in t(y, p)$, choose an element $q \in P$, $q > p$, satisfying $\neg\beta \wedge \Box\beta \in t(y, q)$. Let $\bar{P}_{n+1}$ be the collection of all those q.

Note that the existence of the elements q is always guaranteed by axiom 7 and the model construction described above. Now define

$$\bar{P} := \bigcup_{n \in \mathbb{N}} \bar{P}_n, \quad O := i(\bar{P}).$$

Furthermore, let $\mathcal{M} := (X, O, \sigma)$. Then $\mathcal{M}$ satisfies wbcc.

This can be seen in the following way: enumerating the points of X as they are obtained, that is, in "chronological" order, $X = \{x_0, x_1, x_2, \ldots, x_k, \ldots\}$, one proves by induction on k that there is some $m \in \mathbb{N}$ such that every descending chain $U_0 \supset U_1 \supset U_2 \supset \cdots$ of opens containing x_k has length at most m. In the induction step one uses the remark concerning the *reason* of an open immediately preceding this theorem. Accordingly, and due to the induction hypothesis, there is some $l \in \mathbb{N}$ such that $x_i \notin U_l$ for $0 \leq i < k$, and all members of the chain below U_l have reason x_k. We observe that there is only a finite number of subformulas $\Box\beta$ of α, and each of these can contribute at most one set to the chain because of the definition of O. Thus wbcc is in fact satisfied.

We now prove by induction on the structure of formulas:

For all $\beta \in \mathcal{F}$ and neighborhood situations y, U of (X, O) :

$$\beta \in \mathrm{sf}(\alpha) \Longrightarrow [\, y, U \models_{\mathcal{X}} \beta \iff y, U \models_{\mathcal{M}} \beta \,].$$

The case β a propositional variable is clear from the definitions. If $\beta = \neg\gamma$, $\beta = \gamma \wedge \delta$, or $\beta = K\gamma$, the induction hypothesis directly applies. The implication '$\Longrightarrow$' in case $\beta = \Box\gamma$ follows easily from the induction hypothesis, since $O \subseteq O^{\mathcal{X}}$. In order to prove the reverse direction let y, U be a neighborhood situation of (X, O) such that

$y, U \not\models_{\mathcal{X}} \Box\gamma$ ($\Box\gamma$ a subformula of α). By construction of $\mathcal{M}$, there is some $n \in \mathbb{N}$ and some $p \in \bar{P}_n \subseteq P$ such that $U = i(p)$. Since $y, U \not\models_{\mathcal{X}} \Box\gamma$ implies $y, U \models_{\mathcal{X}} \Diamond\neg\gamma$, we get $\Diamond\neg\gamma \in t(y, p)$. But in step $n+1$ of the above construction a $q \in P$, $q > p$, was chosen satisfying $\neg\gamma \wedge \Box\gamma \in t(y, q)$. Moreover, $V := i(q) \in \mathcal{O}$, and $V \subset U$ because of $q > p$. Thus we obtain $y, V \not\models_{\mathcal{X}} \gamma$. By induction hypothesis, $y, V \not\models_{\mathcal{M}} \gamma$. Consequently, $y, U \not\models_{\mathcal{M}} \Box\gamma$. This ends the induction.

Since $x, X \not\models_{\mathcal{X}} \alpha$ by Theorem 3.2, the assertion just proved yields $x, X \not\models_{\mathcal{M}} \alpha$, as desired. □

For later purposes it is important to remark once again (and more explicitly) a property of the partial order $(P, \leq)$ mentioned above already: for every two points $p, q \in P$ which are not comparable with respect to $\leq$ there does not exist an $r \in P$ satisfying $p \leq r$ and $q \leq r$. For the associated set of subsets $\mathcal{O}$ this means that no two elements $U, V \in \mathcal{O}$ which are incomparable with respect to set inclusion can contain a common $W \in \mathcal{O}$. Let us call subset spaces sharing this property *pseudo-tree-like*.

We finish this section by showing that for the logic of spaces satisfying wbcc the finite model property is *not* valid; that is, there are nonderivable formulas which cannot be falsified in a finite model of the axioms.

Proposition 3.4 *The logic of subset spaces satisfying wbcc lacks the finite model property.*

Proof: Let $X := \mathbb{N}$. For every $i \in \mathbb{N}$ let $U_i := \mathbb{N} \setminus \{0, \ldots, i\}$, and define $\mathcal{O} := \{U_i \mid i \in \mathbb{N}\}$. Then $(X, \mathcal{O})$ obviously satisfies wbcc. Define an X-valuation σ by $\sigma(A, j) := 1$, for all $A \in \mathrm{PV}$ and $j \in \mathbb{N}$. The following formula α, in which $A \in \mathrm{PV}$ is arbitrarily chosen, holds in $\mathcal{M} := (X, \mathcal{O}, \sigma)$:

$$LA \wedge K(A \to L\Diamond A) \wedge K\Box(A \to L\Diamond A).$$

Clearly, this formula cannot hold at any neighborhood situation of some finite subset space. It follows that $\neg\alpha$ holds in every finite model. This implies the lack of the finite model property, as $\models_{\mathcal{M}} \alpha$. □

4 The system D2 Subsequently we introduce the system **D2**, for which we want to have soundness and completeness with respect to subset spaces satisfying bcc. It turns out that even the finite model property holds for **D2**.

The system **D2** is essentially determined by the following scheme:

9. $K\Box(K\Box\alpha \to \alpha) \to K\Box\alpha \quad (\alpha \in \mathcal{F})$.

This axiom corresponds to that stronger finiteness condition on the set of opens, bcc. The scheme (7) from the list presented in Section 3 may now be weakened to its transitivity part; that is,

7′. $\Box\alpha \to \Box\Box\alpha \quad (\alpha \in \mathcal{F})$

substitutes the former scheme 7. The remaining axioms are retained. So, let **D2** consist of the axiom schemes 1 – 6, 7′, 8, 9, and the previous rule schemes.

First we show that (7) can actually be derived with the aid of (9). Formal derivability is designated $\vdash$, indexed by the system (if need be).

Lemma 4.1 $\vdash_{\mathbf{D2}} \Box(\Box\alpha \to \alpha) \to \Box\alpha$.

Proof: We omit the index **D2** during the proof of the lemma. According to [10], 2.11, it suffices to establish the so-called Unformalized Löb Theorem

$$\frac{\Box\alpha \to \alpha}{\alpha}$$

as a derived **D2**-rule. So let $\vdash \Box\alpha \to \alpha$ be valid. Due to

$$\vdash (\Box\alpha \to \alpha) \to (K\Box\alpha \to \alpha),$$

which can be seen with the aid of axiom scheme 4 and propositional reasoning, we obtain

$$\vdash K\Box\alpha \to \alpha$$

by assumption. Necessitation with respect to each modality yields

$$\vdash K\Box(K\Box\alpha \to \alpha).$$

Applying modus ponens to this and the scheme 9 gives

$$\vdash K\Box\alpha;$$

hence, again by (4),

$$\vdash \Box\alpha$$

follows. Now the assumption comes into play a second time proving

$$\vdash \alpha,$$

as desired. □

The proceeding in the above proof utilizes the presence of axiom 7′. This scheme is also used implicitly below when we proceed as we did in connection with Theorem 3.2. Soundness with respect to the target structures is again easy to see. Thus a corresponding proof is omitted here. The completeness proof for the new system starts as in the case of **D1**, see Theorem 3.2; there is only one modification concerning the notion of consistency, which is now understood with respect to **D2**. However, we obtain a subset space initially that only satisfies fcc, but not necessarily bcc.

Theorem 4.2 *Every formula $\alpha \in \mathcal{F}$ which is not* **D2**-*derivable can be falsified in a subset space satisfying fcc.*

Proof: To begin with, we repeat the model construction underlying the proof of Theorem 3.2, but consider the canonical model $\mathcal{M}_{\mathbf{D2}}$ instead of $\mathcal{M}_{\mathbf{D1}}$; later on in the proof we also take up some notations introduced in Section 3. In this way we obtain a subset space $\mathcal{X} = (X, \mathcal{O}^{\mathcal{X}}, \sigma)$ and a point $x \in X$ such that all **D2**-axioms hold in $\mathcal{X}$ and $x, X \not\models \alpha$; here α denotes the given non-**D2**-derivable formula.

We now select suitable opens from $\mathcal{O}^{\mathcal{X}}$ to get the desired space satisfying fcc. Let

$$\begin{aligned} \mathcal{O}' &:= \{U \in \mathcal{O}^{\mathcal{X}} \mid (\exists \Box\beta \in sf(\alpha))(\exists y \in U)\ y, U \models_{\mathcal{X}} K\Box\beta \wedge \neg\beta\}, \text{ and} \\ \mathcal{O} &:= \{V \in \mathcal{O}^{\mathcal{X}} \mid (\exists U \in \mathcal{O}')\ U \subseteq V\}. \end{aligned}$$

Then every $\subset$-chain in $\mathcal{O}'$ as well as in $\mathcal{O}$ is finite. The first finiteness-condition follows easily from the definition of $\mathcal{O}'$ and the fact that there are only finitely many subformulas of α. The latter holds because of the following reason: $p \in P$ corresponding to $U \in \mathcal{O}'$ (by means of i) has been chosen in step k of the model construction for some $k \in \mathbb{N}$, and no $q \in P$ has been inserted below p in a step $l > k$ since P_{n+1} is an end extension of P_n for every $n \in \mathbb{N}$. Thus $\mathcal{M} := (X, \mathcal{O}, \sigma)$ is a model satisfying fcc. The following assertion can be proved by a structural induction (see the proof of Theorem 3.3).

$$\text{For all } \beta \in \mathcal{F} \text{ and neighborhood situations } y, U \text{ of } (X, \mathcal{O}):$$
$$\beta \in \mathrm{sf}(\alpha) \Longrightarrow [\, y, U \models_{\mathcal{X}} \beta \iff y, U \models_{\mathcal{M}} \beta\,].$$

We do not carry out this induction here except the direction '$\Longleftarrow$' in case $\beta = \Box\gamma$. So let y, U be a neighborhood situation of $(X, \mathcal{O})$ such that $y, U \not\models_{\mathcal{X}} \Box\gamma$. Then there exists a set $V \in \mathcal{O}^{\mathcal{X}}$ satisfying $V \subset U$ and $y, V \models_{\mathcal{X}} \neg\gamma$. If $y, V \models_{\mathcal{X}} K\Box\gamma$ holds additionally, then V belongs to $\mathcal{O}'$ according to the definition of $\mathcal{O}'$, and we are done because of the induction hypothesis. Otherwise, $y, V \models_{\mathcal{X}} L\Diamond\neg\gamma$ is valid. With the aid of axiom 9 we conclude that

$$y, V \models_{\mathcal{X}} L\Diamond(K\Box\gamma \wedge \neg\gamma)$$

holds. Thus, in particular, there exists an open $W \subset V$ such that

$$v, W \models_{\mathcal{X}} K\Box\gamma \wedge \neg\gamma \text{ for some } v \in W.$$

Hence $W \in \mathcal{O}'$ follows. Consequently, we get $V \in \mathcal{O}$. So the induction hypothesis applies again. This means that $y, V \models_{\mathcal{M}} \neg\gamma$ is implied, and we are done in the present case as well.

Since we have $x, X \not\models_{\mathcal{X}} \alpha$, the above assertion yields $x, X \not\models_{\mathcal{M}} \alpha$. This proves Theorem 4.2. $\Box$

Note that the model $\mathcal{M}$ constructed in the proof of Theorem 4.2 not only satisfies fcc, but is pseudo-tree-like additionally (see the remark following Theorem 3.3 in Section 3).

Next we want to construct a model falsifying α which is even *treelike* (see Section 1).

Lemma 4.3 *Let $\mathcal{M} = (X, \mathcal{O}, \sigma)$ be a pseudo-tree-like subset space satisfying fcc, and let $X \in \mathcal{O}$. Then there exist a treelike model $\mathcal{M}' = (X', \mathcal{O}', \sigma')$ satisfying fcc and a surjection $\varphi : X' \to X$ which induces an inclusion-preserving bijection from $\mathcal{O}'$ onto $\mathcal{O}$ such that*

$$(\forall \beta \in \mathcal{F})[\, y', U' \models_{\mathcal{M}'} \beta \iff \varphi(y'), \varphi(U') \models_{\mathcal{M}} \beta\,]$$

holds for all neighborhood situations y', U' of $(X', \mathcal{O}')$.

Proof: The idea of the proof is to separate overlapping opens. For this purpose we index every element of X by the open sets containing it. Since the structure $\mathcal{M}$ we start with is a pseudo-tree-like subset space our approach will be successful.

Let $y_U := (y, U)$ and $X' := \{y_U \mid y \in X;\ y, U \in X \otimes \mathcal{O}\}$. We claim that X' is the carrier set of the desired model. For every open $U \in \mathcal{O}$ let $\check{U} := \{y_U \mid y \in U\}$. Now define

$$U' := \check{U} \cup \bigcup_{V \subset U} \check{V}$$

and let $\mathcal{O}' := \{U' \mid U \in \mathcal{O}\}$. Then $(X', \mathcal{O}')$ is a treelike subset frame because $(X, \mathcal{O})$ is pseudo-tree-like.

To complete the specification of a model it remains to define an appropriate X'-valuation σ'. But the definition of σ' is canonical:

$$\sigma'(A, y_U) := \sigma(A, y)$$

for all $A \in \mathrm{PV}$ and $y_U \in X'$.

The mapping φ is likewise defined in a natural manner:

$$\varphi(y_U) := y \text{ for all } y_U \in X'.$$

Surjectivity of φ holds due to the construction of $\mathcal{M}'$; note that $X \in \mathcal{O}$. We get $\varphi(U') = U$ and $V \subseteq U$ if and only if $V' \subseteq U'$ for all $U, V \in \mathcal{O}$. Thus $\mathcal{M}'$ in particular satisfies fcc.

So far we have defined $\mathcal{M}'$ and φ. The final assertion of the lemma can be proved by induction on the structure of β. As the details are routine they are not carried out here. □

Note that φ reminds one of what is called a *p-morphism* in ordinary modal logic [6]. We now provide for *finite ramification* of the tree obtained right now. We proceed as follows.

Let $\mathcal{M} = (X, \mathcal{O}, \sigma)$ be any treelike model satisfying fcc and $X \in \mathcal{O}$. Then, according to the *distance* of an open V from X with respect to proper reverse inclusion, a partition

$$\mathcal{O} = \bigcup_{i \in \mathbb{N}} \mathcal{O}_i$$

of $\mathcal{O}$ is induced such that for all $x \in X$ there is at most one $V_x^i \in \mathcal{O}_i$ satisfying $x \in V_x^i$; especially, $\mathcal{O}_0 = \{X\}$. If $V \in \mathcal{O}_i$, then we call i the *level* of V.

Let $\alpha \in \mathcal{F}$ be given. With respect to α we introduce the following equivalence relation on the set of opens:

$$\begin{aligned} V \sim_\alpha W :\quad \iff \quad & (\forall\, K\beta \in \mathrm{sf}(\alpha))(\forall x \in V,\ y \in W) \\ & x, V \models K\beta \iff y, W \models K\beta. \end{aligned}$$

Clearly, $\sim_\alpha$ is well defined and there is only a finite number of equivalence classes. The class of $V \in \mathcal{O}$ is designated $[V]_\alpha$, and we let $\mathcal{U}$ be the set of all such equivalence classes.

We now define inductively a finite partition $\widehat{\mathcal{U}}_j$ of $\mathcal{O}_j$ for all $0 \leq j$.

$j = 0$:

Let $\widehat{\mathcal{U}}_0 := \{\{X\}\}$.

$j = n + 1$:

Let $\widehat{\mathcal{U}}_n$ be already defined. If $\widehat{\mathcal{U}}_n \neq \varnothing$, then take any element $\mathcal{V} \in \widehat{\mathcal{U}}_n$ and let

$$\mathcal{V}_{n+1} := \{W \in O_{n+1} \mid W \subset V \text{ for some } V \in \mathcal{V}\}.$$

If $\mathcal{V}_{n+1}$ is empty, then nothing has to be done. Otherwise consider

$$\mathcal{V}' := \{[V]_\alpha \cap \mathcal{V}_{n+1} \mid [V]_\alpha \in \mathcal{U}\} \setminus \{\varnothing\}.$$

If the cardinality of $\mathcal{V}'$ is ≥ 2 or $\bigcup\bigcup \mathcal{V}' \subset \bigcup \mathcal{V}$, then let tentatively $\mathcal{V}'$ be a subset of $\widehat{\mathcal{U}}_{n+1}$. Otherwise choose a nontrivial partition $\{\mathcal{V}_1, \mathcal{V}_2\}$ of the single class belonging to $\mathcal{V}'$, and let $\{\mathcal{V}_1', \mathcal{V}_2'\} \subseteq \widehat{\mathcal{U}}_{n+1}$ likewise tentatively. (Such a partition $\{\mathcal{V}_1, \mathcal{V}_2\}$ exists whenever the present case occurs.) Now look at level $n+2$. Split up each of the temporary members of $\widehat{\mathcal{U}}_{n+1}$ into two sets such that all elements of the first one intersect with O_{n+2} and none of the second does. Eliminate $\{\varnothing\}$ if need be. Proceeding in this way for every $\mathcal{V} \in \widehat{\mathcal{U}}_n$ yields $\widehat{\mathcal{U}}_{n+1}$ definitely.

Let $\widehat{O} := \{\bigcup \mathcal{V} \mid \mathcal{V} \in \widehat{\mathcal{U}}_j \text{ for some } j \geq 0\}$. Given any open $V \in O$ there exists a unique $j \in \mathbb{N}$ such that for exactly one $\mathcal{V} \in \widehat{\mathcal{U}}_j$ it holds that $V \in \mathcal{V}$. Consequently, the function $\psi : O \longrightarrow \widehat{O}$ determined by

$$\psi(V) := \bigcup \mathcal{V}$$

is well defined. Furthermore, ψ is surjective and preserves (strict) inclusions; even $U \subset V$ if and only if $\psi(U) \subset \psi(V)$ is valid.

Using these notations we get the following lemma.

Lemma 4.4 *Let $\mathcal{M} = (X, O, \sigma)$ be any treelike space such that fcc is satisfied and $X \in O$. Furthermore, let $\alpha \in \mathcal{F}$ be given, and let $\widehat{\mathcal{M}} := (X, \widehat{O}, \sigma)$ be the model constructed from $\mathcal{M}$ and α in the way just described. Then, for all subformulas β of α, $V \in \widehat{O}$, and $x \in V$, it holds that*

$$x, V \models_{\mathcal{M}} \beta \text{ iff } x, \psi(V) \models_{\widehat{\mathcal{M}}} \beta.$$

Proof: The proof is by induction on the structure of the formulas. The propositional cases are evident.

$\beta = \Box\gamma$: We first prove '$\Longrightarrow$'. Let $x, \psi(V) \not\models_{\widehat{\mathcal{M}}} \Box\gamma$. Then there is an open $W \in \widehat{O}$ such that $x, W \not\models_{\widehat{\mathcal{M}}} \gamma$. Because of the surjectivity of ψ it follows that $W = \psi(V')$ for some $V' \in O$. As our above construction respects the levels of opens, V' may be chosen such that $V' \subset V$. According to the induction hypothesis, $x, V' \not\models_{\mathcal{M}} \gamma$. Hence $x, V \not\models_{\mathcal{M}} \Box\gamma$. The reverse direction is clear by the induction hypothesis and the fact that ψ preserves proper containment.

$\beta = K\gamma$: In this case the assertion follows because $\sim_\alpha$ was defined adequately and the above defined partitions $\widehat{\mathcal{U}}_j$ respect the equivalence classes. $\Box$

Applying the above construction to the non-**D2**-derivable formula α we started with and the model $\mathcal{M}'$ satisfying fcc which we obtained by Lemma 4.3, we have actually reduced the number of opens to only finitely many. In fact, this is a consequence of König's Lemma. Thus we can state our desired completeness result right now.

Theorem 4.5 *A formula $\alpha \in \mathcal{F}$ is derivable in the system* **D2**, *if and only if α holds in all subset spaces satisfying bcc.*

We introduce a further equivalence relation depending on α, this time on the set X of points. Our aim is to arrive at a *finite model* falsifying α in this way.

$$x \sim y :\iff (\forall U \in \widehat{\mathcal{O}})(\forall A \in \mathrm{PV} \text{ occurring in } \alpha)$$
$$[\, x \in U \iff y \in U \,] \text{ and } [\, \sigma(A, x) = 1 \iff \sigma(A, y) = 1 \,]$$

Let $[x]_\sim$ denote the equivalence class of x with respect to the relation $\sim$. Define a model $[\mathcal{M}] = ([X], [\mathcal{O}], [\sigma])$ in the following manner:

1. $[X] := \{[x]_\sim \mid x \in X\}$,
2. $[\mathcal{O}] := \{[U] \mid U \in \widehat{\mathcal{O}}\}$, where $[U] := \{[x]_\sim \mid x \in U\}$,
3. $[\sigma](A, [x]_\sim) := 1 \iff (\exists y \in [x]_\sim)\ \sigma(A, y) = 1$,

for all $A \in \mathrm{PV}$ and $[x]_\sim \in [X]$, where $(X, \widehat{\mathcal{O}}, \sigma)$ is the model $\widehat{\mathcal{M}}$ from Lemma 4.4. Then $[\mathcal{M}]$ is a treelike space, too, and $[X]$ is obviously finite.

We omit the straightforward induction proving the following lemma.

Lemma 4.6 *For all subformulas β of α and every neighborhood situation $x, U \in X \otimes \widehat{\mathcal{O}}$ we have that*

$$x, U \models_{\widehat{\mathcal{M}}} \beta \iff [x], [U] \models_{[\mathcal{M}]} \beta.$$

Combining the results obtained by Theorem 4.5 and Lemma 4.6 yields the finite model property of the system **D2**.

Proposition 4.7 *Let $\alpha \in \mathcal{F}$ be given. Then α is* **D2***-derivable if and only if α holds in all finite models of the axioms.*

As a corollary of the proof of the finite model property we get that the system **D2** is also sound and complete with respect to the smaller model class of finite treelike spaces.

Corollary 4.8 *A formula $\alpha \in \mathcal{F}$ is derivable in the system* **D2** *if and only if α holds in all finite treelike spaces.*

We conclude this section with two remarks concerning the system **D2**.

1. Since every space satisfying bcc in particular satisfies wbcc, all **D1**-derivable formulas are **D2**-derivable as well. This follows from Theorem 3.3 and Theorem 4.5. Especially, $\Box(\Box\alpha \to \alpha) \to \Box\alpha$ is **D2**-derivable, as we have shown in Lemma 4.1.
2. The system **D2** strongly resembles the above-mentioned system **G** of usual modal logic. The latter system is complete with respect to finite trees (see [10], 4.9, e.g.), whereas **D2** is complete with respect to finite treelike spaces, as we proved above. Thus (9) seems to be the appropriate generalization of **W** to the context of topological modal logic (and not simply adopting that scheme as it was done in Section 3).

5 Small model property Since we used a compactness argument in the proof of the finite model property, we did not obtain a bound of the model size depending on the given formula α which determines the number of structures that have to be checked in order to validate or falsify α. In this section we show that this is nevertheless possible, although only with respect to certain bimodal Kripke structures associated with spaces satisfying bcc.

Definition 5.1 (**D2**-frame, **D2**-model)

1. Let $\mathcal{R} := (W, \{R, S\})$ be a bimodal frame (i.e., W is a nonempty set and R, S are binary relations on W). Then $\mathcal{R}$ is called a **D2**-*frame*, if and only if
   (a) R is an equivalence relation on W;
   (b) S is irreflexive and transitive;
   (c) $(\forall s, t, u \in W)(s\,S\,t \wedge t\,R\,u \Longrightarrow (\exists v \in W)[\,s\,R\,v \wedge v\,S\,u\,]$;
   (d) there is an $n \in \mathbb{N}$ such that every S-chain has length $\leq n$.
2. A model $\mathcal{M} := (W, \{R, S\}, \sigma)$ based on a **D2**-frame $(W, \{R, S\})$ is called a **D2**-*model*, if and only if

$$(\forall s, t \in W)(\forall A \in \mathrm{PV})[\,s\,S\,t \Longrightarrow (\sigma(A, s) = 1 \iff \sigma(A, t) = 1)\,]\,.$$

It is not difficult to see that the axiom system determining **D2** is sound and complete with respect to the just defined structures.

Proposition 5.2 *A formula $\alpha \in \mathcal{F}$ is* **D2**-*derivable if and only if it holds in every* **D2**-*model.*

Proof: In fact, every subset space $\mathcal{M} = (X, \mathcal{O}, \sigma)$ satisfying bcc gives rise to a **D2**-model $\widetilde{\mathcal{M}} = (W, \{R, S\}, \tilde{\sigma})$ in the following way:

1. $W := X \otimes \mathcal{O}$;
2. $(x, U)\,R\,(y, V) :\iff U = V$;
3. $(x, U)\,S\,(y, V) :\iff x = y \wedge V \subset U$;
4. $\tilde{\sigma}(A, (x, U)) = 1 :\iff \sigma(A, x) = 1$.

An easy induction shows that for all $\alpha \in \mathcal{F}$ it holds that

$$(\forall\, x, U \in W)\left(x, U \models_{\mathcal{M}} \alpha \iff \widetilde{\mathcal{M}} \models_{x,U} \alpha\right);$$

here on the right-hand side usual multimodal satisfaction is denoted (see [6], §5). Now the completeness assertion follows with the aid of Theorem 4.5. The soundness part is easy to see. □

In the next step we show that every formula which holds in the canonical model $\mathcal{M}_{\mathbf{D2}}$ at some point is also satisfied in some finite **D2**-model. Let C denote the carrier set of the canonical model, and let $\alpha \in \mathcal{F}$ satisfy

$$\mathcal{M}_{\mathbf{D2}} \models_s \alpha \text{ for some } s \in C.$$

Let Γ be the set of all subformulas of α joined with the set of all negated subformulas of α,

$$\tilde{\Gamma} := \Gamma \cup \{\beta \mid \beta \text{ is a finite conjunction of distinct elements of } \Gamma\},$$

and $\Delta := \tilde{\Gamma} \cup \{L\beta \mid \beta \in \tilde{\Gamma}\}$. Then the following *filtration lemma* is valid.

Lemma 5.3 *Let $\mathcal{M} := (W, \{R, S\}, \sigma)$ be a Δ-filtration of $\mathcal{M}_{\mathbf{D2}}$ such that R and S are the minimal filtrations of the respective accessibility-relations in $\mathcal{M}_{\mathbf{D2}}$. Then $\mathcal{M}$ satisfies all* **D2***-model properties up to the irreflexivity of S, possibly.*

Proof: The Δ-filtration and the Moss-Parikh filtration introduced in [2], Section 2.3, coincide, and the assertion of the lemma corresponds to that of [2], Lemma 2.10. The proofs are similar, too, so we need not give more details here. It should only be mentioned that, in particular, the transitivity of the relation S, which corresponds to the accessibility relation $\stackrel{\diamond}{\longrightarrow}$ determined by the modal operator $\Box$ on the canonical model, can be established using the completeness of the system with respect to **D2**-models (Proposition 5.2). In fact, it is not hard to see that the bimodal structure $J \oplus_{\Delta} K$, which is defined in the same way as in the proof of [2], Lemma 2.10, satisfies all **D2**-frame properties. This shows the crucial part of the lemma in the present case. $\Box$

As $\mathcal{M} \models (W, \{R, S\}, \sigma)$ is a Δ-filtration of $\mathcal{M}_{\mathbf{D2}}$, W is a finite set having at most $2^{|\Delta|}$ many elements. The carrier W consists of certain equivalence classes $\bar{s}$ of points $s \in C$, and for all $\beta \in \Delta$ and $s \in C$ we have that

$$\mathcal{M}_{\mathbf{D2}} \models_s \beta \text{ iff } \mathcal{M} \models_{\bar{s}} \beta.$$

Thus α is satisfied in the finite model $\mathcal{M}$. Unfortunately, $\mathcal{M}$ is not entirely of the type we are looking for. But self-referential connections may simply be "forgotten":

Lemma 5.4 *Let Δ and $\mathcal{M}$ be as above. Let $\mathcal{M}' := (W, \{R, S'\}, \sigma)$, where $S' = S \setminus \{(x, x) \mid x \in W\}$. Then $\mathcal{M}'$ is a* **D2***-model, and*

$$(\forall \beta \in \Delta)(\forall v \in W)[\,\mathcal{M} \models_v \beta \iff \mathcal{M}' \models_v \beta\,].$$

Proof: We induct on β. The induction is trivial except for the '$\Longleftarrow$'–direction in the case $\beta = \Box\gamma$. So let $v \in W$ be given such that $\mathcal{M} \not\models_v \Box\gamma$. Then v is the class of some point s of the canonical model and we have $\mathcal{M}_{\mathbf{D2}} \not\models_s \Box\gamma$. Because of the **D2**-derivability of $\Box(\Box\gamma \to \gamma) \to \Box\gamma$ (see Lemma 4.1) we get

$$\mathcal{M}_{\mathbf{D2}} \not\models_s \Box(\Box\gamma \to \gamma).$$

Hence there exists a $\stackrel{\diamond}{\longrightarrow}$–successor $t \in C$ of s such that

$$\mathcal{M}_{\mathbf{D2}} \models_t \Box\gamma \wedge \neg\gamma.$$

Let u be the equivalence class of t. Then

$$v\,S\,u \text{ and } \mathcal{M} \models_u \Box\gamma \wedge \neg\gamma$$

holds. Since $\mathcal{M} \models_u \Box\gamma$ we get $u \neq v$, and since $\mathcal{M} \models_u \neg\gamma$ we obtain $\mathcal{M}' \models_u \neg\gamma$ by the induction hypothesis. Consequently, $\mathcal{M}' \not\models_v \Box\gamma$. $\Box$

Summarizing the above results we obtain the following theorem.

Theorem 5.5 *The system* **D2** *fulfils the small model property with respect to* **D2***-models.*

Let us conclude with two remarks. First, the *complexity* of deciding **D2**-satisfiability should be mentioned. It is presumably high. In fact, one can modify the construction of Ladner [8], Section 3, to obtain PSPACE-hardness of this problem. Second, it should be stated that a decidability proof for the set of **D1**-validities is still missing; we guess that it can be done by means of a modification of the Δ-filtration considered above.

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