Notre Dame Journal of Formal Logic
Volume 41, Number 3, 2000

ON THE ORIGIN AND STATUS OF OUR CONCEPTION OF NUMBER

WILLIAM DEMOPOULOS

Abstract This paper concerns the epistemic status of "Hume's principle"—the assertion that for any concepts F and G, the number of Fs is the same as the number of Gs just in case the Fs and the Gs are in one-one correspondence. I oppose the view that Hume's principle is a stipulation governing the introduction of a new concept with the thesis that it represents the correct analysis of a concept in use. Frege's derivation of the basic laws of arithmetic from Hume's principle shows our pure arithmetical knowledge to arise out of the most common everyday applications we make of the numbers. The analysis of arithmetical knowledge in terms of Hume's principle ties our conception of number to the interconnections of which our concepts of divided reference are capable; in so doing, it locates the origin of our conception of number in the structure of our conceptual framework.

1. The Neo-Fregean Background

In *Frege's Conception of Numbers as Objects* [18], Wright showed that it is possible to extract from Frege's *Grundlagen* [11] a valid second-order proof of the Dedekind infinity of the natural numbers using only a suitable formalization of the following "partial contextual definition" for numerical identity:

> For any concepts F and G, the number of Fs is identical with the number of Gs if and only if the Fs and the Gs are in one-one correspondence.[1]

Wright argued that this proof can form the basis of a defensible, if modified, formulation of Frege's philosophy of arithmetic, one that relies on Frege's thesis that numbers are *objects* but makes no use of the problematic notion of the extension of a concept. Hale's *Abstract Objects* [14], which appeared in 1987, is perhaps the next important landmark in the elaboration of a "neo-Fregean" philosophy of arithmetic along the lines of [18].

Received March 29, 2001; printed September 30, 2002
2001 Mathematics Subject Classification: Primary, 18A15
Keywords: logicism, Frege, philosophy of arithmetic

In an influential series of papers, Boolos clarified and extended Wright's discussion and its relation to the central mathematical argument of *Grundlagen*. Boolos's designation of the partial contextual definition as "Hume's principle" has since become established in the literature as has his suggestion that the theorem that the second-order theory whose sole axiom is Hume's principle (Frege Arithmetic) has a definitional extension which contains Peano Arithmetic be called "Frege's theorem". Both Boolos and Dummett have advanced a number of criticisms of Wright's neo-Fregean program, prompting Wright to clarify his interpretation of Hume's principle, and more generally, his view of the methodology of concept formation via "abstraction principles," of which Hume's principle and our concept of number are the primary examples.[2]

For Wright, the philosophical significance of Frege's theorem is that it shows the Dedekind infinity of the natural numbers to follow from an "explanation" of the concept of number in terms of an abstraction principle. By an "abstraction principle," Wright means the universal closure of an expression of the form

$$\Sigma(X) = \Sigma(Y) \leftrightarrow X \Re Y.$$

where $\Re$ is an equivalence relation, the variables X and Y may be of any order, and the function Σ may be of mixed type. In the case of Hume's principle, the equivalence relation is the (second-order definable) relation on concepts of one-one correspondence, and the "cardinality function" is a map from Fregean concepts to objects. As we will see in greater detail later, it is important for this program that concept formation by means of an abstraction principle be distinguished from the ordinary practice of axiomatization and indeed, that the case be made that not all cases of concept formation by an abstraction principle are on a par.

In the case of Hume's principle, Wright's basic idea appears to be the following: Hume's principle is a stipulation which gives truth conditions for a restricted class of statements of numerical identity, namely, those of the form, the number of Fs is identical with the number of Gs. Although the specification of truth conditions is *partial*—it concerns only a restricted class of statements of numerical identity—the resulting *explanation* of the concept of number is *complete* insofar as it suffices for the (second-order) derivation of the basic laws of arithmetic. As Wright [22] has emphasized, whether or not Hume's principle counts as an analytic truth is entirely subordinate to its status as a stipulation governing our application of the concept of number. Moreover, since concept formation by Hume's principle, like concept formation in accordance with any other abstraction principle, involves the introduction of a *new* concept, it would be a mistake to view the principle as *analytic of* the concept of number in the way in which the notion *analytic of* has traditionally been understood. Rather, for Wright, Hume's principle is a trivial consequence of the nature of the concept of number because it is a *stipulation* governing the introduction of that concept (rather than the analysis of a preexisting concept). The stipulative character of the principle is important, since it is this feature which allows it to fulfill (albeit in a restricted and modified sense) logicism's promise to deliver arithmetic from logic plus some species of stipulation. Thus, Wright originally called his view "number-theoretic logicism" because it derives the basic laws of arithmetic from a stipulation governing the concept of number without, however, providing an *explicit definition* of the individual numbers or of *immediately precedes* on the basis of a purely logical vocabulary. Although number-theoretic logicism falls short of the goal of explicitly defining the vocabulary of arithmetic in purely logical terms, its explanation of numerical identity by Hume's principle is achieved in terms of one-one correspondence, and this *is* a concept of pure (second-order) logic.

There can be no question of the importance of Hale's and Wright's contributions and of the vitality they have imparted to a subject long regarded as having at most historical interest. Wright's rediscovery of Frege's theorem has been the major impetus to the reevaluation of Frege's philosophy of arithmetic, and his and Hale's defense of a modified form of Frege's logicism has elicited a critical reaction that has prompted a substantial clarification of the neo-Fregean position. Nevertheless, I think their final view of matters is not entirely successful. My purpose here is to review some of the critical reaction that has been directed at Wright's formulation of the neo-Fregean program with the aim of clarifying where I believe the central difficulty with the position lies. I will then present what I take to be the correct account of the status of Hume's principle and the significance of Frege's theorem.

2. Is Hume's Principle Correctly Represented as a Stipulation?

For reasons that will soon become evident, the first class of objections that have been urged against the neo-Fregean program are known as "bad company" objections. Wright addresses two such bad company objections to his position: First, concept introduction by an abstraction principle can fail, and can fail spectacularly, as the case of the "abstraction principle" expressed by *Grundgesetze*'s Basic Law V showed. But (the objection goes) how can we accept a view which invites us to rely upon a methodology which is known to be seriously flawed? And secondly, given our freedom to stipulate one or another abstraction principle, it is necessary to supplement the account with a criterion capable of governing the choice of one abstraction over another.[3]

Apropos of the first objection, Wright makes a convincing case that the fact that a methodology sometimes leads to flawed conclusions does not mean that it is itself irremediably flawed, and he reminds us that Hume's principle is consistent relative to analysis.[4] Why should it be necessary to show that the procedure will work whatever abstraction principle is employed? One difficulty with this response is that it tends to assimilate concept introduction by abstraction to the case of ordinary axiomatization, and this makes it difficult to appreciate Wright's insistence that what he is proposing is sharply different from what he calls "the mere axiomatic stipulation of existence." The contrast with axiomatic stipulation is important and I will return to it shortly.

However, the consistency of Hume's principle fails to settle the second "bad company" objection which effectively raises the question whether, and in what sense, it is *true*. For suppose we grant that Hume's principle stipulates or lays down the truth conditions for certain statements of numerical identity. Is the truth of Hume's principle *merely* a matter of stipulation? That this is how Wright intends to be understood is suggested by several passages:

> [a] state of affairs is initially given to us as the obtaining of a certain equivalence relation . . . ; but we have the option, by stipulating that the abstraction is to hold, of so reconceiving such states of affairs that they come to constitute a new kind of thing. (Wright [19], p. 208)

That Wright views abstraction principles as stipulations is reinforced by a contrast he draws between the basis of our knowledge of the truth of Hume's principle and our knowledge of the existence of the numbers: Abstract objects are not creations of the mind,

> *brought into being* by a kind of stipulation. What is formed—created—by such an abstraction is rather a *concept*: the effect is merely to fix the truth conditions of identity statements concerning a new kind of thing, and it is quite another question whether those truth conditions are ever realized. ([19], p. 208)

Thus, according to Wright, the representation of Hume's principle as a kind of stipulation does not prejudice the question whether the existence of the numbers is a matter of stipulation. In fact, Wright maintains that the existence of the numbers is something discovered rather than stipulated, while holding that our a priori knowledge of their necessary existence is ultimately derived from a principle whose truth *is* a matter of stipulation.

This is entirely credible as a view of conventional *explicit* definitions but it cannot be so easily maintained in the case of a contextual definition like Hume's principle. It is precisely because ordinary definitions are not creative that we can say that anything, expressed in unabbreviated notation, which is established or discovered on their basis does not really depend on them and, in consequence, does not share the conventional character of their epistemic basis. But even if Hume's principle can be regarded as a stipulation, it is certainly not a stipulation of this character, since it is creative over the theory to which it is added.[5] Indeed, this is precisely the point where the analogy between "concept introduction via an abstraction principle" and the methodology of concept introduction by ordinary explicit definition can break down with the result that concept introduction by abstraction becomes difficult to distinguish from axiomatic stipulation.

In [23], Hale and Wright emphasize the difference between treating Hume's principle as a stipulation—something which they hold to be unproblematic—and treating the existence of the numbers as a matter of stipulation, which they agree *is* problematic. Their point is that Hume's principle merely lays down partial satisfaction conditions for the relation of numerical identity, thereby establishing the existence of the relation in analogy with the use of axioms as *implicit* definitions of the terms they introduce. Hale and Wright insist that it is not part of their position that the stipulation of such satisfaction conditions should secure the existence of the objects related by the relation of numerical identity; rather, their existence is secured by a proof, the proof that there are objects related by the relation thus introduced. (The discovery of this proof gives the sense in which the existence of the numbers is "discovered.") Thus understood, their methodology differs from the "axiomatic stipulation of existence" that is embodied in, for example, the set-theoretic practice of simply laying down a comprehension axiom and claiming, on its basis, to have secured the existence of whatever sets—whatever "objects"—the axiom allows while excluding whatever sets the axiom disallows. But the methodology Hale and Wright are here reporting *is* a straightforward variation on another thoroughly familiar and unexceptionable set-theoretic practice. In standard axiomatic set theory, we freely lay down a defining condition and then proceed to verify—to prove, on the basis of our axioms of set existence—that our definition is not *vacuous* and that we have not simply redefined the empty set. For example, we are free to define the power set of a set A by the familiar use of class-abstract notation,

$$P(A) = \{B : B \subseteq A\}.$$

But the proof that the set we have defined is the intended one and not just the empty set requires an axiom—the power set axiom, $\exists B \forall C(C \in B \leftrightarrow C \subseteq A)$—to ensure the existence of the appropriate set. It should be clear that in the absence of an account of our basis for believing the set-theoretic axioms, this methodology does not give—and does not pretend to give—an account of the basis for our beliefs about the existence of sets with the intended membership, so that if the set-theoretic axioms are represented as stipulations, this is naturally assumed to transfer to the basis for our belief in the sets whose existence and nonvacuousness we are able to prove on their basis. In the absence of Hume's principle, we can, following Frege, lay down defining conditions for all of the individual numbers. What we cannot do without Hume's principle is prove that the numbers we have defined are distinct from zero and from one another. Now except for the fact that they are working in Frege Arithmetic rather than Zermelo-Fraenkel set theory, this is exactly the methodological situation described by Hale and Wright. Why should their claim to have given us an account of the basis for our belief in the numbers be viewed any differently? In particular, if the set theoretic situation supports the idea that the basis for our belief in the existence of sets with the intended membership is the same as the basis for our believing the truth of the axioms—the same, that is, in the sense that by stipulating the truth of the axioms, we leave the existence of the intended sets resting on a stipulation—then if Hume's principle is represented as a stipulation, the existence of the numbers must also be regarded as resting on a stipulation.

Perhaps the stipulative character of Hume's principle should be thought of on the model of a reference-fixing stipulation, after the manner of Kripke's example of the introduction of a term like 'meter'. The idea would be that we stipulate that at time t, bar b is a meter long, from which it follows that at t, the length of b falls under the concept *meter*. Although we have laid down a stipulation (a reference-fixing stipulation), in doing so we have also succeeded in making a factual assertion, namely, that b has a particular length at t—something that obtained before the convention was laid down and something that will continue to obtain even if the convention is withdrawn. Might not something similar hold for concept introduction by an abstraction, so that it, too, may be seen to consist in laying down a stipulation while at the same time having a factual content? Couldn't Wright argue that the opposition—fact versus convention—has been overdrawn, obscuring the fact that the stipulative character of Hume's principle, on which he has correctly insisted, does not preclude it from having a factual character as well? The difficulty with this response is that while Kripke's model of reference-fixing may suffice to show that it is not *in general* true that stipulations are incapable of a factual content which is independent of the stipulations themselves, it is of no help in clarifying how the stipulative character of concept-introduction by an abstraction is capable of accommodating the factual content of a principle like Hume's. This is because the factual content of the assertion that bar b is a meter long is carried by the possibility of displaying or ostending, independently of the reference-fixing stipulation, the length to which the concept of a meter is to be applied. But this possibility is precisely what is lacking in the account of the number-theoretic case in terms of concept introduction by abstraction.

The idea that the truth of Hume's principle is a matter of a stipulation that is unconstrained by any "antecedently determinate truth" allows Wright to sidestep the difficulties which confront showing that it is true or that it correctly captures our preanalytic notion of numerical identity. However, this comes at a cost: the problem is that there are many abstractions, all of them satisfiable but relative to certain assumptions not necessarily *mutually* satisfiable. For example, in [2], Boolos introduced several sentences with very different model-theoretic properties from one another but with an equal claim to being treated as abstraction principles that introduce a class of abstract singular terms. In [19], Wright considers a modification of one of Boolos's examples in order to motivate a constraint on "admissible" abstraction principles. The example involves a type-lowering function which takes two concepts to the same object just in case their symmetric difference is finite.[6] It can be shown that the resulting abstraction principle, which Wright calls NP (for Nuisance Principle), holds in finite domains but fails if the domain of individuals is infinite and the range of the concept variables is the full power set of this domain.[7]

To see the difficulty that such a "bad company" example poses, assume (for whatever reason) that we are committed to a standard interpretation of second-order logic. Then were we to adopt the stipulation embodied in NP, we would be restricted to models having only finitely many objects. *Since Hume's principle holds only in infinite domains, our adoption of* NP *would preclude us from stipulating that numerical singular terms should be used in accordance with Hume's principle*. But if we take seriously the idea that abstraction principles are merely stipulations governing the use of the singular terms they introduce—that is, that they are conventions which we freely lay down—we might easily defeat the truth of Hume's principle by an "incorrect" initial choice of abstraction principle, one satisfiable only in finite domains. This would occur if, for example, we had first chosen the stipulation embodied in NP. But if abstraction principles are stipulations, and thus, Hume's principle just one stipulation among many, how can we make sense of the idea that there is a *right* initial choice? If the truth of an abstraction is a matter of stipulation, then the existence of a domain sufficiently large to contain the numbers would seem to depend upon which abstraction happened to be laid down first, so that whether the domain of objects contains a subdomain capable of modeling the basic laws of arithmetic would come to rest on an arbitrary decision of ours.

This is clearly a conclusion Wright does not wish to endorse and he recognizes that what might be called the "quasi-conventionalist" features of his approach make it incumbent upon him to articulate a principled division among abstractions. One proposal Wright has explored—there are many other conditions which Wright and Hale have investigated in their insightful search for a theory of good abstractions—is that an abstraction is acceptable only if it satisfies a "conservativeness" requirement according to which an abstraction principle is acceptable if it does not constrain the cardinality of concepts with whose introduction it is not itself explicitly concerned. But to know that Hume's principle constrains only the cardinality of the numbers, we must know of the cardinality function, not only that it associates concepts with objects, so that, in particular, nonequinumerous concepts are associated with distinct objects, we must also know that the objects thus associated are numbers. Otherwise, how are we to maintain, in accordance with the conservativeness condition, that it constrains only the numbers?[8] The problem of demarcating Hume's principle from "bad" abstraction principles on the basis of conservativeness would seem to presuppose a solution to the problem of demarcating the numbers from other objects on the basis of principles internal to number-theoretic logicism—the so-called Julius Caesar problem. Wright could respond[9] to this charge by observing that it assumes we possess an unreconstructed concept of number to which Hume's principle is responsible.

But the whole point of concept introduction by abstraction is to explain how numbers may be introduced *without* incurring any such obligation. There is therefore a certain internal coherence to Wright's program that makes it resistant to criticisms of this sort. Nevertheless, even if it should turn out to be possible to draw a principled division between good and bad abstractions, the point of developing a theory of good abstractions rests on the assumption that an abstraction principle as rich as Hume's is adequately represented as a stipulation.[10]

3. Securing the Truth of Hume's Principle

Now the key *philosophical* idea of *Grundlagen* that underlies the revival of Frege's program is the notion that numerical singular terms are referential because with Hume's principle we are in possession of a clear criterion by which we can say when the same number has been "given to us" in two different ways, as the number of one or another concept. That is, given the truth of a suitable criterion of identity, we are entitled by the context principle (*Only in the context of a sentence does a word have reference* [Bedeutung]) to infer that numerical singular terms refer. Our "access" to the numbers is supposed to be mediated by our recognition of the truth of Hume's principle, and this same principle also serves as the only substantive premise in the proof of their infinity. Frege's account of our access or reference to the numbers thus relies fundamentally on our recognition of the *truth* of Hume's principle. What is needed therefore is an account of the sense in which this principle is true and an account of how it is known to be true, since this is what is required to infer, by the context principle, that numerical singular terms refer.

To my mind, number-theoretic logicism has not adequately addressed the consequences of treating Hume's principle as a stipulation which we freely lay down. Instead it has concentrated on the question whether the surface grammar of the left-hand side of the principle can be taken seriously, whether, in other words, the apparent deployment of numerical singular terms as singular terms is justified. As Wright puts it, one has "to read the left-hand sides of the appropriate abstraction principles not merely as notational variants of the right-hand sides, but in a way which is constrained by their surface syntax" (Wright [21], p. 404). Once this is established, Wright argues, no further question can arise over whether such terms genuinely refer, and there is "no good sense in which their reference could be stigmatized as semantically idle" ([21], p. 404). But the difficulty, as I see it, is with the transition from the characterization of the truth of Hume's principle as a simple stipulation to its deployment as a truth having significant existential presuppositions. Since it is the latter characterization of Hume's principle that its use in conjunction with the context principle relies upon, this transition is clearly required; nevertheless, it is deeply perplexing and constitutes the chief stumbling block to accepting number-theoretic logicism's claim to have secured or explained the reference of numerical singular terms.

However sound the contrast of the methodology of concept introduction by an abstraction principle with ordinary axiomatization, the characterization of a fundamental principle like Hume's as a kind of stipulation is certainly reminiscent of Hilbert's account of his axioms as implicit definitions. In his correspondence with Hilbert and in (the first of) his series of essays on the foundations of geometry [13], Frege argued that by treating axioms as definitions Hilbert precluded taking them to express truths in the intended and "generally accepted" sense, since if Hilbert's axioms are genuine axioms, and therefore for Frege, are not merely true but known to be true, then the reference of their constituent expressions must be settled independently of the axioms themselves. Hilbert's response consisted in resisting the notion that his axioms express truths, arguing that they need only be susceptible of an interpretation under which (as we might say) "they come out true."[11] A curious feature of this debate is that Frege's own suggestion for approaching the question of reference to abstract objects in terms of the context principle and the use of one or another contextual definition seems vulnerable to the same circularity with which he charged Hilbert over the matter of treating axioms as implicit definitions.[12] There is even a sense in which Frege's application of the context principle to contextual definitions, requiring, as it does, that contextual definitions are true simpliciter (absolutely true), is a more appropriate target of these criticisms than Hilbert's use of the doctrine of implicit definition. And indeed, the role of Hume's principle in Frege's analysis of our reference to the numbers appears no less ambiguous than the role Hilbert assigned to axioms. Frege had argued, against Hilbert, that treating axioms as definitions puts us in the position of having a single equation with two unknowns; but we might equally well ask, regarding a comparable circularity in Frege's own methodology, how it is possible to know of Hume's principle that it is true if we have not first specified the reference of *its* constituent expressions.

The difficulty confronting a development of Frege's philosophy of arithmetic which relies on the contextual definition and the context principle is that it requires Hume's principle to fulfill roles that are typically in tension with one another. It must, first of all, be a substantive truth, one which implies the basic laws of arithmetic and forms the philosophical basis of our knowledge of them, a point which may be put by saying that it is the principle's account of our application of the numbers in cardinality judgments which underlies our knowledge of their infinity. Secondly, it is supposed to give the sense of the cardinality operator—a function usually reserved for definitions or, in any case, for sentences which are stipulated or laid down.

I believe there is a way of understanding the truth of Hume's principle which addresses both the ambiguity of its status as a stipulation versus a substantive claim and the apparent circularity in Frege's methodology—without, however, having to accept the conventionalist features of Hale and Wright's account.[13] In order to motivate my suggestion, it will be worthwhile to reflect for a moment on a feature of our ordinary model-theoretic analysis of logical truth, according to which a logical truth is one that is true in all models. Even though this analysis appeals to truth-in-a-model, and therefore assumes the notion of interpretation or reference, in the case of the logical truths this dependence turns out to be inessential, since the holding of a logical truth in some particular model does not distinguish that model from any other structure. By contrast, an ordinary mathematical truth, such as the commutative law of group theory, holds precisely in those structures which are commutative groups. Hence, the truth-in-a-model of a mathematical law, as opposed to a logical law, depends essentially on the reference of (at least one of) its constituent expressions. Thus, even though the use of the model-theoretic framework to explain the truth(-in-a-model) of a statement requires the notion of reference, it allows that the situation is importantly different according to whether we are considering logical or nonlogical truths; briefly put, the logical truths make no "special" demands on reference. This is one way of spelling out the topic-neutrality of logical truth in a model-theoretic setting. To put the point another way: the referential demands of the logical truths are reflected in the model-theoretic framework as a whole rather than in any part of it. It is in this sense that the topic-neutrality of the logical truths shows them to have a special status vis-á-vis the existential commitments of the model-theoretic framework: the account of the logical truths requires only the minimal existential commitment that any model would meet.

Logical truths are sometimes said to be "constitutive principles" in the sense that, together with principles of inference, they are constitutive of the meanings of the logical operators. But the topic-neutrality of the logical truths that emerges within our model-theoretic setting affords a slightly different and somewhat more traditional way of viewing their a priori character. Since the logical truths fail to distinguish one model from another—since *every* model is constrained to satisfy them—their truth is independent of the particular model in which the truth or otherwise of sentences of whatever character is being evaluated. In particular, the holding of the logical truths in a structure is independent of whether or not it models the facts of experience. This a prioricity which the topic-neutrality of the truths of logic supports can be plausibly extended to the epistemic status of Hume's principle without, however, requiring that we take a stand on the vexed question of whether Hume's principle is a logical truth.

Although there are, of course, models in which Hume's principle fails, it nevertheless seems plausible to expect of any model which contains concepts of divided reference (sortal concepts), interrelated as our concepts of divided reference are, that it must make some provision for the "unrestricted" application[14] of the cardinality operator, thereby generating the "skeleton" of concepts deployed by Frege in the proof of his theorem. Not only is there associated with such concepts their number, but among the concepts of divided reference the model contains, there are "numerical" concepts—concepts to which the numbers assigned to concepts of divided reference *themselves* belong. Such provision is characteristic of any model containing our concepts of divided reference and numerical identity, and its satisfaction is a feature of any model that might plausibly represent our absolute notion of truth.[15] Since Hume's principle yields the correct analysis of numerical identity, it will be true in any model containing such a family of concepts, and any such model will therefore contain both "the" cardinal numbers and the concepts associated with them in the development of Frege's proof of his theorem. Even if we never achieve a complete specification of the model which the world comprises, it would suffice, so far as this account of our knowledge of arithmetic is concerned, that we should know of any candidate model that it is constrained in this way. Under these circumstances we are justified in supposing Hume's principle true—not just consistent, not just true in some model or other, but true in that model, truth in which coincides with truth. But Hume's principle is not merely true in that model which represents our absolute concept of truth: it would be true in any model containing the relation of numerical identity and a family of concepts of divided reference, interrelated as our concepts of divided reference are. Its generality consists in the fact that it holds in *every* structure capable of representing our concepts of divided reference and notion of numerical identity whether or not such a structure models the facts of experience. The correctness of Frege's analysis of numerical identity in terms of the cardinality operator and the relation of one-one correspondence is a feature of every family of concepts of divided reference belonging to a possible model for our conceptual structure. Hume's principle therefore enjoys a status in the class of models containing such families of concepts which is entirely analogous to the status of the truths of logic in the class of *all* models. Just as a logical truth makes no demand on truth and reference which is not already implicit in every other truth, the referential demands of the truth of Hume's principle are held in common by every truth, and in analogy with the truth of the laws of logic and the model-theoretic framework, its truth is reflected in our conceptual framework as a whole.

It might be thought that if Hume's principle is true, we are justified, by the context principle, in holding to a limited realism concerning the cardinal numbers. But the realism to which we are entitled fails to be a complete vindication of Frege's program because the basic laws of arithmetic derivable from Hume's principle allow us to capture the natural numbers at most only up to isomorphism; in particular, using only the resources of the context principle and the contextual definition, we are unable to characterize "the" cardinal numbers. There is, therefore, a sense in which the realism concerning the numbers which these considerations vindicate is attenuated relative to that which attaches to realism regarding the physical world: we do not think of the furniture of the world as specifiable only up to isomorphism but regard it as something we will have captured uniquely should we succeed in characterizing the model which the physical world comprises. Frege seems also to have partially conceded this qualification when he sought an account of the numbers that would single out one natural sequence of numbers[16] from all the rest by characterizing the numbers as extensions which comprehend all their applications. But if the numbers are captured only "structurally," there is a certain conventionalism which attaches to the assertion of their existence: as far as the basic laws of arithmetic derived from Hume's principle are concerned, *any* ω sequence will serve as the sequence of natural numbers; but this is a feature without parallel in our conception of the constituents of the physical world. A complete vindication of Fregean realism would require being able to distinguish "the" natural numbers, something demanding resources going beyond the contextual definition.[17] Nevertheless, we have achieved something more than the conclusion that the contextual definition is merely true-in-a-structure. We have explained how it might be seen as *true*; as such, it would have a distinguished status among mathematical principles, as would the referential commitments of its constituent expressions.

Leaving aside the historical question of Frege's view in *Grundlagen*, I am claiming that a novel and plausible philosophy of arithmetic is at least suggested by the work, one which, moreover, is free of the conventionalism that has characterized later developments.[18] On this view, the explanation of the significance of Hume's principle and Frege's theorem is this: the contextual definition is a constraint on the classes of models that are capable of adequately representing the structure of our concepts of divided reference, the relation of numerical identity and our absolute notion of truth, a constraint which bears a direct analogy to that imposed by the logical truths on the class of all structures. It is not, however, a constraint which is arbitrarily laid down, like a stipulation governing the introduction of a new concept. Rather, the sense in which the contextual definition is "definitional" is that it advances an analysis of a concept in use, namely, our preanalytic notion of numerical identity. The argument for the correctness of this analysis depends on several important and nontrivial intermediate conclusions which are argued for throughout the course of *Grundlagen*. The most significant of these are: (i) the fundamental thought of Section 46 according to which a statement of number involves the predication of something of a concept; (ii) the assumption that the numbers are arguments to concepts of first level and that the cardinality operator is, therefore, properly interpreted by a type-lowering function; and (iii) the contention that *as many as* and, therefore, *sameness of number* are to be understood in terms of one-one correspondence. By Frege's theorem, in order for any such structure to satisfy Hume's principle, it must contain "the" numbers. The existence of the numbers—the "limited realism" of this view—is justified to the degree that we are justified in assuming the correctness of this analysis and of the supposition that the world is modeled by a structure in which it is reflected.

4. A Foundation or an Analysis?

At this point it might be appropriate to address a concern which Dummett has expressed, if only to further clarify what is and is not achieved by the view presented here. According to Dummett, Frege's approach to the numbers is fundamentally misguided. Russell's discovery of the contradiction in Frege's theory of classes served only to bring the difficulty into sharper focus. But Frege's basic approach would have been problematic even if *no* inconsistency had been discovered since there is an unacceptable circularity in Frege's procedure: the abstraction principle which introduces the numbers contains an implicit first-order quantifier, so the numbers introduced on the left occur within the range of the variables bounded on the right in the explicit definition of one-one correspondence. Moreover, this feature is essential if Hume's principle is to support the proof of the infinity of the number sequence since that proof turns on the possibility of forming, for each n, a concept under which the numbers up to (and including) n all fall. But this is possible only if the numbers fall within the range of the first-order variables, otherwise we will not have allowed for the formation of the concepts on which the success of the argument depends. Hence, if it is to support the proof of Frege's theorem, Hume's principle would seem to require that the numbers are "given to us" independently of the abstraction principle or contextual definition which introduces them—contrary to the promise of the neo-Fregean's "abstractionist" methodology that it would replace the metaphor of the numbers as something "ostended in intuition" with an altogether different and nonmetaphorical account of our knowledge of them. Dummett concludes[19] that not only Frege, but also neo-Fregeans who have succeeded him, are unable to solve the two problems to which logicism originally directed its attention: to secure the existence of the objects of number theory and to show what our conception of a countably infinite domain rests upon without in either case relying on an appeal to intuition or the facts of experience. Does our account fare any better? To see in what sense it does, it is necessary to distinguish two rather different ways of viewing the significance of Frege's theorem. If we think of the proof as exhibiting a "construction"[20] of the numbers, then in order to avoid circularity, it would be illegitimate to assume at any stage of the construction the existence of numbers which arise at some later stage. If, however, we view the proof less ambitiously, as a verification that Hume's principle implies the Dedekind infinity of the numbers, so that we are therefore justified in claiming of it both that it captures a central feature of our notion of number and that it reveals the assumptions on which our conception of their infinity may be based, then there can be no circle. In fact, we may take the practice of recovering a central feature of a concept in use by revealing the assumptions on which our use of the concept depends as a characterization of what traditionally passes for a conceptual analysis. Thus understood, Frege's theorem confirms that his analysis of numerical identity in terms of Hume's principle is a compelling solution to the *second* of the two problems that Dummett poses, that of explaining how we attain the concept of a countably infinite domain.

Of course, the Dedekind infinity of the numbers can be obtained directly from Dedekind's and Peano's well-known axiomatizations. Indeed, if the concepts over which the variables of Hume's principle range are restricted to (the Kuratowski-) finite concepts, Frege Arithmetic is equivalent to the Dedekind-Peano axiomatization. What is the advantage of Frege's development of arithmetic from Hume's principle? The answer, I think, is that by contrast with Dedekind and Peano, Frege derives the number-theoretic or pure properties of the numbers from an analysis of our applications of them in judgments of cardinality. By being based on the fundamental thought, Frege's account shows how a mathematical analysis such as Dedekind's or Peano's arises out of the most common everyday applications we make of the numbers. Such an account ties our conception of the numbers to our conception of families of concepts of divided reference and the interconnections of which they are capable and in so doing, locates the origin of our conception of number in the character of our conceptual framework.

But what of Dummett's *first* problem? Have we given up on securing the existence of the numbers? If Hume's principle is regarded as a conceptual analysis of our arithmetical knowledge, it presupposes rather than vindicates the existential commitments of what we take ourselves to know. Such an analysis can clarify the nature and extent of those commitments, and it can clarify the assumptions from which they derive, but it may still fall short of persuading someone who, for whatever reason, denies the existence of the numbers. We have simply not addressed the question whether it is coherent to deny the existential assumptions which, on the analysis on offer, the analysandum presupposes. Our account does not yield what Dummett has called a "suasive argument"[21] for the existence of the numbers. At best, it affords an explanation of how it is possible to arrive at a conception of the natural number sequence, an explanation that is based on our applications of numbers to concepts. It is only in this sense that our account affords a foundation for arithmetic, a sense which falls short of one that would secure our arithmetical knowledge against someone who would question the existential commitments it presupposes. But although such an explanation falls short of the traditional justification Dummett is demanding, it may be said in its defense that it is not at all clear how anything stronger could be achieved.

The foregoing proposal has some affinity with earlier views, but I have not adequately explored the extent to which this is true to comment in detail on precedents except to say that from Frege and Russell I have taken the idea that logic and arithmetic are concerned with truths of extreme generality, while from Carnap I have appropriated the thesis that the truths of arithmetic are partly constitutive of our conceptual framework. I differ from Carnap over the matter of the basis for the truth of a constitutive principle like Hume's: for Carnap, as with Wright, its truth rests upon a decision of ours. (Carnap, more plausibly I think than Wright, takes this to undermine the notion that a constitutive principle can possess a "factual" content.) I also differ from Carnap by preserving the notion that constitutive truths can incorporate significant existential presuppositions, that they can "have a content." (Carnap would dismiss this contention as a confusion of internal and external questions.) Where I diverge from Frege and Russell is in my conception of the generality that constitutive principles enjoy: Hume's principle, for example, is not merely a general truth, characteristic of everything, but a principle which holds in any possible model of our conceptual structure. It is from this generality that its constitutive and a priori status derive.

By way of conclusion, I would like to remark on the role of analyticity in the preceding account of the truth of Hume's principle. Although I have appealed to the notion *analytic of*, arguing that Hume's principle is analytic of the notion of numerical identity, I have said nothing of its status as an *analytic truth*. Indeed, I have conceded that Hume's principle need not be regarded as a truth of logic and have thereby renounced one means of securing its analyticity. I have also argued that it is not a mere convention because it fails to possess the triviality of a stipulation—something which the traditional use of the doctrine of analyticity was supposed to support. On the present account, Hume's principle is a general truth, not just in the sense that it holds universally of everything in the relevant domain of quantification, but in the stronger sense that it is true in every "world," that is, in every model which is capable of representing our conceptual structure. While it is possible to introduce a notion of analyticity according to which a principle like Hume's would come out analytic insofar as it satisfies this notion of generality, it is unclear what would be gained from doing so. Certainly there is nothing in such a notion of analyticity that would justify the usual conclusions that have been thought to follow from such a characterization: for example, it would not follow that the principle is trivial and it would not follow that it is without existential commitment or "factual content." The plausibility of the idea that Hume's principle is analytic of the concept of numerical identity depends on the plausibility of a conceptual analysis; but the truth of the principle that expresses this analysis depends on the presuppositions of the framework of which the analysis *is* an analysis. These presuppositions are rather strong, and it is on their satisfaction, rather than its analyticity of the relation of numerical identity, that the truth of Hume's principle depends. How, then, are we to answer the question, "Is Hume's principle analytic?" We should answer "Yes," if our goal is to emphasize that the principle expresses the result of a conceptual analysis. The significance of such a positive answer is that it vindicates an important methodology, the method of analysis exhibited in *Grundlagen*, the first great work of the analytic tradition. If, however, the point of the question is to suggest that the principle is distinguished in some manner other than what is implied by its generality—that it is a convention of language or without significant existential presupposition—our answer must be "No."

Notes

1. This had been observed by Parsons about twenty years earlier in his paper "Frege's theory of number" in Parsons [16] and in my collection Demopoulos [6]. So far as I am aware, the terminology "partial contextual definition" originates with Parsons.

2. See Wright [19], [20], and [21].

3. The first bad company objection was urged by Dummett in [10]; it is elaborated in Dummett [8], pp. 369–88. The second (as well as the first) was presented by Boolos in "The standard of equality of numbers," [2] (reprinted in Boolos [4] and Demopoulos [6], pp. 234–54).

4. As shown by Boolos in [1], also reprinted in Boolos [4] and in Demopoulos [6], pp. 211–33.

5. Whether this is the theory of pure second-order logic or the theory of second-order logic with a single function symbol N of mixed type, one whose intended interpretation is that of a map from concepts to objects. It is Hume's principle that makes N a cardinality function capable of supporting definitions of immediately precedes and zero which satisfy the basic laws of arithmetic.

6. That is, $n(F) = n(G)$ if and only if $[x : x$ is F but not G or G but not $F]$ is finite.

7. See Wright [19], pp. 222–25. The restriction to the full power set is necessary since there is a Henkin model with an infinite domain in which NP is true. The proof of this claim is outlined in the Appendix to my paper, Demopoulos [7].

8. Essentially the same point has been made by Boolos—although in a different context—in [3]; see especially, p. 253.

9. See, for example, Wright [22], p. 9.

10. In this connection, compare Wright's penultimate paragraph to "Is Hume's principle analytic?" ([22], p. 19).

11. See Frege [12], p. 29.xii.99, draft or excerpt by Hilbert. For a discussion of the issues raised by this correspondence, see Demopoulos [5], pp. 211–25.

12. This point is made by Dummett in [10], p. 654.

13. This idea was sketched in Demopoulos [7]. What follows is an elaboration of this earlier discussion.

14. That is, it is restricted in its application only by the demand that the concepts to which it applies are concepts of divided reference.

15. By such a "skeleton of concepts" we mean any family $\Im$ of concepts of divided reference such that
    (i) $\Im$ contains the empty concept and is closed under Boolean combinations of the concepts it contains;
    (ii) If F is a finite concept belonging to $\Im$, then the concept of F's number—$[x : x =$ the number of Fs$]$—belongs to $\Im$ where the notion of finiteness may be any one of the several weak notions of finiteness (e.g., Kuratowski-finiteness) properly adapted to the case of concepts;
    (iii) For any concepts F and G belonging to $\Im$, if F is not in one-one correspondence with G, then the number of Fs is distinct from the number of Gs.

16. For Frege, the natural sequence of numbers is the sequence, $0, 1, \ldots, \aleph_0$, which is contrasted with the sequence of natural numbers, $0, 1, \ldots$.

17. As is well known, this is what led to Frege's explicit definition of the numbers in terms of classes of equinumerous concepts.

18. Thus I am not claiming the unequivocal authority of Frege for the view advanced here. Indeed, there is a basic difference between the approach to abstraction principles I have been urging and the account one finds in *Grundgesetze*'s discussion of Basic Law V—a principle which we may express by the universal closure of the biconditional:

The class of Fs *is identical with the class of* Gs *if and only if all and only* Fs *are* Gs.

This difference emerges from the recent discussion in Heck's "Grundgesetze der Arithmetik I §10" [15]. Heck argues that Frege, like Wright, is explicitly committed to the idea that the truth of Basic Law V should be secured by nothing more than a series of stipulations, the central one of which Heck calls "The Initial Stipulation", namely, that the truth value of the left-hand side of the above biconditional is the truth value of the right-hand side. In Heck's view, Frege's dissatisfaction with the account of *Grundgesetze* was not centered on the conventionalist character of this strategy, but on its failure—on the failure of the series of stipulations on which it relies—to provide us with a conception of classes as objects, and consequently, on its failure to provide a defense of the thesis that class expressions refer to objects. Although the methodology Heck describes may be central to Frege's strategy in *Grundgesetze* in connection with Basic Law V and classes, *Grundlagen*'s account of the truth of abstraction principles is susceptible of a greater variety of interpretations, and to the extent that this is true, *Grundlagen* is the philosophically richer work.

19. See his paper, *Neo-Fregeans: In bad company?* [8], pp. 369 and 381.

20. Perhaps after the manner of the method of "domain extension" which Wilson has argued was suggested to Frege by the work of von Staudt and others. See Wilson [17].

21. See "The justification of deduction" in Dummett [9], pp. 290–318, cf. esp. 295 f.

References

[1] Boolos, G., "The consistency of Frege's *Foundations of Arithmetic*," pp. 3–20 in *On Being and Saying. Essays in Honor of Richard Cartwright*, edited by J. Thomson, The MIT Press, Cambridge, 1987. MR 1 376 397. 223

[2] Boolos, G., "The standard of equality of numbers," pp. 261–77 in *Meaning and Method. Essays in Honor of Hilary Putnam*, Cambridge University Press, Cambridge, 1990. MR 1 376 398. 215, 222

[3] Boolos, G., "Is Hume's Principle analytic?" pp. 245–61 in *Language, Thought, and Logic. Essays in honour of Michael Dummett*, edited by R. G. Heck, Jnr., Oxford University Press, New York, 1997. Zbl 0938.03506. MR 2000h:03005. 223

[4] Boolos, G., *Logic, Logic, and Logic*, Harvard University Press, Cambridge, 1998. Zbl 0955.03008. MR 2000b:03005. 222, 223

[5] Demopoulos, W., "Frege, Hilbert, and the conceptual structure of model theory," *History and Philosophy of Logic*, vol. 15 (1994), pp. 211–25. Zbl 0808.01002. MR 95i:03003. 223

[6] Demopoulos, W., editor, *Frege's Philosophy of Mathematics*, Harvard University Press, Cambridge, 1995. MR 96h:03015. 222, 223

[7] Demopoulos, W., "The philosophical basis of our knowledge of number," *Noûs*, vol. 32 (1998), pp. 481–503. MR 2000b:03014. 223

[8] Dummett, M., "Neo-Fregeans: In bad company?" pp. 369–405 in *The Philosophy of Mathematics Today (Munich, 1993)*, edited by M. Schirn, Clarendon Press, New York, 1998. With a reply by C. Wright. Zbl 0937.03004. MR 2001f:03025. 222, 224

[9] Dummett, M. A. E., *Truth and Other Enigmas*, Harvard University Press, Cambridge, 1978. 224

[10] Dummett, M. A. E., *Frege: Philosophy of Mathematics*, Harvard University Press, Cambridge, 1991. 222, 223

[11] Frege, G., *Die Grundlagen der Arithmetik. Eine Logisch Mathematische Untersuchung über den Begriff der Zahl*, Basil Blackwell, Oxford, 1950. Zbl 0041.14701. MR 11,487a. 210

[12] Frege, G., *Philosophical and Mathematical Correspondence*, University of Chicago Press, Chicago, 1980. Abridged from the German edition and with a preface by B. McGuinness. Translated from the German by H. Kaal with an appendix by P. E. B. Jourdain. Zbl 0502.03002. MR 81d:03001. 223

[13] Frege, G., *Collected Papers on Mathematics, Logic, and Philosophy*, Basil Blackwell, Oxford, 1984. Translated by M. Black, V. Dudman, P. Geach, H. Kaal, E. Kluge, B. McGuinness, and R. Stoothoff. Zbl 0652.01036. MR 86k:01050. 216

[14] Hale, B., *Abstract Objects*, Oxford University Press, Oxford, 1987. 210

[15] Heck, R. G., Jr., "Grundgesetze der Arithmetik I §10," *Philosophia Mathematica*, vol. 7 (1999), pp. 258–92. The George Boolos Memorial Symposium, Notre Dame IN, 1998. Zbl 0976.03006. MR 2001j:03007. 224

[16] Parsons, C., *Mathematics in Philosophy*, Cornell University Press, Ithaca, 1983. Selected essays. MR 86a:01050. 222

[17] Wilson, M., "To err is Humean," *Philosophia Mathematica*, vol. 7 (1999), pp. 247–57. The George Boolos Memorial Symposium, Notre Dame IN, 1998. MR 2000h:03017. 224

[18] Wright, C., *Frege's Conception of Numbers as Objects*, Aberdeen University Press, Aberdeen, 1983. Zbl 0524.03005. MR 85g:00035. 210

[19] Wright, C., "On the philosophical significance of Frege's Theorem," pp. 201–44 in *Language, Thought, and Logic. Essays in Honour of Michael Dummett*, edited by R. G. Heck, Jnr., Oxford University Press, New York, 1997. Zbl 0938.03508. MR 2000h:03005. 212, 213, 215, 222, 223

[20] Wright, C., "On the harmless impredicativity of N= ('Hume's Principle')," pp. 339–68 in *The Philosophy of Mathematics Today*, edited by M. Schirn, Clarendon Press, New York, 1998. Including papers from the conference held in Munich, 1993. Zbl 0925.03022. MR 2000c:00036. 222

[21] Wright, C., "Response to Dummett," pp. 389–406 in *The Philosophy of Mathematics Today*, edited by M. Schirn, Clarendon Press, New York, 1998. Including papers from the conference held in Munich, 1993. Zbl 0937.03005. MR 2000c:00036. 216, 222

[22] Wright, C., "Is Hume's Principle analytic?" *Notre Dame Journal of Formal Logic*, vol. 40 (1999), pp. 6–30. Special issue in honor and memory of George S. Boolos, Notre Dame IN, 1998. Zbl 0968.03009. MR 1 811 201. 211, 223

[23] Wright, C., and B. Hale, "Implicit definition and the a priori," pp. 286–319 in *New Essays on the a Priori*, edited by P. Boghossian and C. Peacocke, Oxford University Press, Oxford, 2000. 213

Acknowledgments

This paper has benefited enormously from the comments and criticisms of many friends and colleagues, to all of whom I am deeply indebted. My research was partially supported by the Social Sciences and Humanities Research Council of Canada and the Erskine Foundation of the University of Canterbury.

Department of Philosophy
Talbot College
The University of Western Ontario
London ONT N6A 3K7
CANADA
wgdemo@uwo.ca