

AN INTRIGUING LOGIC WITH TWO IMPLICATIONAL CONNECTIVES

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Abstract Spinks introduces implicative *BCSK*-algebras, expanding implicative *BCK*-algebras with an additional binary operation. Subdirectly irreducible implicative *BCSK*-algebras can be viewed as flat posets with two operations coinciding only in the 1- and 2-element cases, each, in the latter case, giving the two-valued implication truth-function. We introduce the resulting logic (for the general case) in terms of matrix methodology in Section 1, showing how to reformulate the matrix semantics as a Kripke-style possible worlds semantics, thereby displaying the distinction between the two implications in the more familiar language of modal logic. In Sections 2 and 3 we study, from this perspective, the fragments obtained by taking the two implications separately, and—after a digression (in Section 4) on the intuitionistic analogue of the material in Section 3—consider them together in Section 5, closing with a discussion in Section 6 of issues in the theory of logical rules. Some material is treated in three appendices to prevent Sections 1–6 from becoming overly distended.

1. Introduction: Matrices and Axioms from Spinks

Despite its figuring in our main title, we do not define “implicational” here, relying on the reader’s willingness to let the term apply to any connective accorded such logical properties as to result in a resemblance to other connectives which have historically been described as implications (material implication, strict implication, and so on). Typical of the treatment of implicational connectives in matrix semantics is that the matrix elements come equipped with some relation—a partial ordering \leq , perhaps—or with some operations, such as addition and subtraction binary operation on the real numbers. For example, implication is interpreted in the matrices of Gödel [14], in each of which there is a total ordering \leq of the elements, by means of the principle that $a \rightarrow b = 1$ if $a \leq b$ and $a \rightarrow b = b$ otherwise, while in the Łukasiewicz matrices, with $+$ and $-$ as operations on the real numbers and the matrix elements are various

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(depending on the matrix) real numbers between 0 and 1, we have the principle that $a \rightarrow b = \min(1, (1 - a) + b)$. Our notation here employs the same symbol (\rightarrow) for the binary connective and for the algebraic operation, on the understanding that if the matrix element assigned to a formula A is a and that assigned to B is b , then the element to be assigned to $A \rightarrow B$ is $a \rightarrow b$. Each of the principles for interpreting implication is recognizably a generalization of the two-valued truth-table account, if we take the two values T(true) and F(false) to stand in the order $F \leq T$ (for Gödel) and to be the real numbers 1 and 0 (for Łukasiewicz).

On the other hand, suppose we wanted a many-valued generalization of the two-valued truth-table account which employed no antecedently given structure in the form of ordering relations or arithmetical(-like) operations. The only “structure” to be assumed is that we have a designated element (or perhaps several—but here we consider the case of a single designated element) and the relations of identity and difference between the various elements. For convenience let us call a principle for the interpretation of implication *purely qualitative* if it draws on no resources other than those just adumbrated, by contrast with the “quantitative” principles as those extracted from Gödel and Łukasiewicz above. (We might equally well say “absolute” in place of “purely qualitative”, and recommend this substitution to whom the latter phrase has distracting associations.) Spinks [32] has recently called attention to two distinct purely qualitative principles for the interpretation of implication which amount to the same thing—the standard material implication account—in the two-element case but which diverge when matrices are considered with more than two elements. (Some background to Spinks’s discussion will be found below, after Proposition 1.11 and in Appendix B.) Following him we shall use \rightarrow and \Rightarrow for the two implicational connectives (and the corresponding matrix operations), from this point reserving the older symbol ‘ \supset ’ for association with material implication (either the connective itself or the corresponding operation on truth-values).

$$\begin{aligned} a \Rightarrow b &= 1 \text{ if } a = b & a \rightarrow b &= 1 \text{ if } a \neq 1 \\ &= b \text{ if } a \neq b & &= b \text{ if } a = 1 \end{aligned} \tag{1}$$

The (logical) matrix on n elements one of which is the designated element 1, with the two operations \Rightarrow and \rightarrow defined by these principles we call \mathbf{M}_n , with \mathbf{M}_ω for the matrix with such operations on countably many elements. By way of illustration, we consider the five-element Spinks matrix \mathbf{M}_5 , with 1 as the designated element—the only element referred to individually in the above pair of principles—the remaining elements just labeled as 2, 3, 4, 5, below, though the labeling has no “quantitative” (e.g., order-conveying) significance. As we have been emphasizing, all that matters is the distinctness of these elements from each other and from the element 1. A reflection of this aspect of the situation may be seen in the fact that any permutation of the elements (of any one of these matrices) each of which keeps the element 1 fixed extends to an automorphism of the underlying algebra. (Note that we use “matrix” in the sense of “logical matrix”—as explained, for example, in Wójcicki [38] and Wójcicki [37]. In particular, the two tables in Figure 1 form part of a single matrix in this sense, describing the underlying algebra of the matrix; the remaining information constitutive of the matrix’s identity is information as to the set of designated elements—in our case this being $\{1\}$.)

\Rightarrow	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	4	5
3	1	2	1	4	5
4	1	2	3	1	5
5	1	2	3	4	1

\rightarrow	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	1	1
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1

Figure 1

Notice that if we restrict our attention to the subalgebra generated by $\{1,2\}$ (which we may consider as a submatrix since it contains the designated element 1) we have the same operation represented by \Rightarrow and \rightarrow , reflecting the fact that Spinks's two principles ((1) above) for defining implication amount to the same thing in the two-element case, and in fact (identifying 1 with T and 2 with F) to the classical truth-table account for this case. This has an obvious consequence for formulas of the present language, given here as Proposition 1.1, the first of several elementary stage-setting results we give here as preparation for later sections in which we present a “modal commentary” on Spinks's logic. (Readers wanting to pass straight to the explanation of this modal perspective should skip down to the second paragraph following Example 1.8 below.) Before stating it, we should remark that we take the formulas of this language to be those constructed from a denumerable supply of propositional variables (sentence letters), $p_1, p_2, \dots, p_n, \dots$ in the usual recursive manner by means of the two binary connectives \Rightarrow and \rightarrow , and that we call a formula *valid* in a matrix if every assignment of matrix elements to the propositional variables is an assignment on which the whole formula receives a designated value (for the present case: receives *the* designated value).

Proposition 1.1 *If A is a formula of the present language and B is the result of replacing all occurrences of \Rightarrow and of \rightarrow by \supset , then if A is valid in all the Spinks matrices \mathbf{M}_n (for $n \in \omega$) and \mathbf{M}_ω , then B is a two-valued tautology.*

Proof: The hypothesis implies that A is valid in the matrix \mathbf{M}_2 , which gives the result since \mathbf{M}_2 just consists of two copies of the usual two-valued table for \supset . \square

Note that in the future we shall frequently omit the “two-valued” when we say “tautology” (or “tautologous formula”), but we always have the two-valued case in mind with this terminology.

Proposition 1.2 *If a formula A is valid in \mathbf{M}_n and $k < n$, then A is valid in \mathbf{M}_k .*

Proof: Let $f(x_1, \dots, x_m)$ be any term function (i.e., polynomially defined via \Rightarrow and \rightarrow) of the algebra of a Spinks matrix; clearly, for any matrix elements a_1, \dots, a_m , we have $f(a_1, \dots, a_m,) \in \{1, a_1, \dots, a_m, \}$, which implies that \mathbf{M}_k is a submatrix of \mathbf{M}_n , giving the result. \square

Clearly, a similar argument works for \mathbf{M}_ω in place of \mathbf{M}_n . Proposition 1.2, with the observation just made, allows us to rewrite Proposition 1.1 as follows.

Proposition 1.3 *If A is a formula of the present language and B is the result of replacing all occurrences of \Rightarrow and of \rightarrow by \supset , then if A is valid in at least one matrix \mathbf{M}_n (with $n > 2$, including the case also of \mathbf{M}_ω), then B is a two-valued tautology.*

And by the reasoning of the proof of Proposition 1.2, we can set aside \mathbf{M}_ω from further attention.

Proposition 1.4 *If A is a formula constructed from n distinct propositional variables and A is not valid in every Spinks matrix, then A is not valid in the matrix \mathbf{M}_{n+1} .*

Proof: Suppose A is not valid in some Spinks matrix but is valid in \mathbf{M}_{n+1} . By Proposition 1.2, the invalidating matrix contains more than $n + 1$ elements. Let a_1, \dots, a_n be the elements assigned to its distinct propositional variables (taken in some fixed order) by some invalidating assignment in this matrix. The submatrix with elements $1, a_1, \dots, a_n$ is isomorphic to \mathbf{M}_{n+1} or (in case not all the a_i are distinct) to some submatrix thereof, so A is invalid in \mathbf{M}_{n+1} (using Proposition 1.2 again for the submatrix case). \square

Thus the logic comprising the formulas valid in all Spinks matrices has the finite model property (or “is finitely approximable”) and we can equivalently describe it as the logic of those formulas valid in the matrices \mathbf{M}_n for $n \in \omega$, setting aside \mathbf{M}_ω —or indeed any larger matrix on which the two fundamental operations are defined by (1). The question of whether one of these \mathbf{M}_n is a (finite) characteristic matrix in the sense of a matrix validating precisely the formulas valid in all of them has been answered negatively by Spinks (personal communication, using an argument in the general style of [14]). On the other hand, if all we want is some characteristic matrix or other—never mind finiteness—then, by the remark following Proposition 1.2, \mathbf{M}_ω will serve in this capacity. For brevity we call any formula valid in all the Spinks matrices (equivalently, in all the finite Spinks matrices, or again, in the matrix \mathbf{M}_ω) a *Spinks-valid* formula. For a key result (Theorem 1.7) concerning this notion of validity, we need a preliminary observation, for which we introduce the following notation.

Notation 1.5 Let \mathbf{M}_\Rightarrow and \mathbf{M}_\rightarrow be respectively the \Rightarrow -reduct and \rightarrow -reduct of an arbitrary Spinks matrix \mathbf{M}_n ($n \geq 2$). For k with $1 < k \leq n$ define the mappings g_k and h by

$$g_k(i) = 2 \text{ if } i = k, g_k(i) = 1 \text{ if } i \neq k; h(1) = 1, h(i) = 2 \text{ for all } i \neq 1.$$

We also need to recall the following terminology: a *matrix homomorphism* from one matrix to another is a homomorphism of the algebra of the first matrix to that of the second which maps designated elements to designated elements; such a mapping is a *strong matrix homomorphism* if it also maps undesigned elements to undesigned elements.

Lemma 1.6 *With $\mathbf{M}_n, \mathbf{M}_\Rightarrow, \mathbf{M}_\rightarrow, g_k$, and h as defined above, g_k is a matrix homomorphism from \mathbf{M}_\Rightarrow to \mathbf{M}_2 and h is a strong matrix homomorphism from \mathbf{M}_\rightarrow onto \mathbf{M}_2 .*

Proof: By a straightforward consideration of cases using (1), one verifies that $g_k(a \Rightarrow b) = g_k(a) \Rightarrow g_k(b)$ and $h(a \rightarrow b) = h(a) \rightarrow h(b)$, for arbitrary elements a, b , of \mathbf{M}_n . The “strong” (for and “onto” parts of the claim are even more easily checked. \square

Theorem 1.7 *Let $A[\supset]$ be any formula built up from propositional variables by means of the binary connective ‘ \supset ’ and $A[\rightarrow]$ and $A[\Rightarrow]$ be the formulas resulting when all occurrences of ‘ \supset ’ are replaced by ‘ \rightarrow ’ and by ‘ \Rightarrow ’, respectively. Then we have*

- (i) $A[\supset]$ is a two-valued tautology if and only if $A[\rightarrow]$ is Spinks-valid.
- (ii) $A[\supset]$ is a two-valued tautology if and only if $A[\Rightarrow]$ is Spinks-valid.

Proof: For (i): a matrix validates the same formulas as any of its strong homomorphic images ([38], Theorem 3.3), so by Lemma 1.6 all the \rightarrow -reducts of the Spinks matrices validate the same (pure \rightarrow) formulas as each other and in particular, as \mathbf{M}_2 , in which clearly $A[\rightarrow]$ is valid if and only if $A[\supset]$ is a two-valued tautology.

For (ii): the “if” direction is given by Proposition 1.1, and to establish the “only if” direction, suppose $A[\Rightarrow]$ is not Spinks-valid, say because it is invalid in \mathbf{M}_n , and thus in \mathbf{M}_{\Rightarrow} , since \rightarrow is not involved. Thus there is an assignment f of matrix elements to the propositional variables in $A(p_1, \dots, p_m)$ ($= A[\Rightarrow]$): we can assume without loss of generality that the propositional variables from which A is constructed are as listed) for which with $f(A) = k$, say, where $k \neq 1$, in which case, using $A(x_1, \dots, x_m)$ on the right here for the algebraic term corresponding to the formula $A(p_1, \dots, p_m)$, we have

$$k = f(A(p_1, \dots, p_m)) = A(f(p_1), \dots, f(p_m)),$$

and so

$$g_k(k) = g_k(f(A(p_1, \dots, p_m))) = g_k(A(f(p_1), \dots, f(p_m))).$$

Now $g_k(k) = 2$ and by Lemma 1.6, $g_k(A(f(p_1), \dots, f(p_m))) = A(g_k(f(p_1)), \dots, g_k(f(p_m)))$, so $g_k \circ f$ is an invalidating assignment for the formula $A[\Rightarrow] = A(p_1, \dots, p_m)$ in \mathbf{M}_2 , showing that $A[\supset]$ is not tautologous. \square

Theorem 1.7 deals with “unmixed” formulas—those in which only one of our two implicational connectives (\rightarrow, \Rightarrow) appears. What about “mixed” cases? We saw in Proposition 1.1 that the result of replacing all occurrences of ‘ \rightarrow ’ as well as all occurrences of ‘ \Rightarrow ’ by ‘ \supset ’ turns any Spinks-valid formula into a two-valued tautology. On the other hand, from the hypothesis that the result of making such a replacement is a tautology, it does not follow that the formula in which the replacement was made was Spinks-valid. Otherwise, \mathbf{M}_2 would be a finite characteristic matrix for the logic determined by the class of Spinks matrices, contrary to what was reported above. But it is also good to see a concrete counterexample.

Example 1.8 (Spinks) The formula $((p_1 \rightarrow p_2) \rightarrow p_2) \Rightarrow ((p_2 \rightarrow p_1) \rightarrow p_1)$ is not valid in \mathbf{M}_3 though it is valid in \mathbf{M}_2 . (The latter amounts to saying that the result of replacing all occurrences of ‘ \rightarrow ’ and ‘ \Rightarrow ’ by ‘ \supset ’ is tautologous. Note also that the tables for \mathbf{M}_3 can be read off Figure 1 by ignoring the last two rows and columns appearing there.) For consider the assignment (in \mathbf{M}_3) of 2 to p_1 and 3 to p_2 . We calculate:

$$((2 \rightarrow 3) \rightarrow 3) \Rightarrow ((3 \rightarrow 2) \rightarrow 2) = (1 \rightarrow 3) \Rightarrow (1 \rightarrow 2) = 3 \Rightarrow 2 = 2.$$

Thus while the logic consisting of all the Spinks-valid formulas contains, in its unmixed formulas, two copies, one written in terms of ‘ \rightarrow ’ and the other in terms of ‘ \Rightarrow ’, of the \supset -fragment of classical propositional logic, these two arrow connectives

are by no means interchangeable in the logic. (The result of replacing all occurrences of ‘ \rightarrow ’ in the Spinks-invalid formula in Example 1.8, by ‘ \Rightarrow ’, turns it into a Spinks-valid formula, as does the replacement of the main ‘ \Rightarrow ’ by ‘ \rightarrow ’, as also does the interchanging of its occurrences of ‘ \rightarrow ’ and ‘ \Rightarrow ’.) The situation recalls the description in Łukasiewicz [24] of two modal operators with which he was working as being like identical twins, indistinguishable except when appearing together. It is this aspect of the situation that prompts our describing—in the title of the present paper—this logic as *intriguing*. As already intimated, we hope to throw some light on the relationship between the two implicational connectives from a modal perspective, though we do not have Łukasiewicz’s modal logic in mind here so much as the more familiar normal modal logics in general and **S5** in particular.

In more detail: the purpose of the (remainder of the) present paper is to provide some insight on the logic determined by the class of matrices \mathbf{M}_n by reformulating the matrix semantics as a Kripke-style possible worlds (also called *relational* or *set-theoretic*) semantics. We foreshadow the main idea of this reformulation in somewhat informal terms here. The matrix semantics instructs us to take as the semantic value, in a matrix, of a formula $A \Rightarrow B$ (or $A \rightarrow B$) as the result of applying the like-named operation \Rightarrow (or \rightarrow , respectively) to the semantic values of A and B in that order, where these operations are defined by (1) above. By analogy with the usual relationship between the matrix (or algebraic) semantics for normal modal logic—as explained in van Benthem [35] or Section 7.5 of Chagro and Zakharyashev [6], for example—we will want the designated value 1 to correspond to the set W of worlds (or ‘points’) in a Kripke model, and writing $\|A\|$ for the set of such points at which the formula A is true, we recast (1) in the following form:

$$\begin{aligned} \|A \Rightarrow B\| &= W \text{ if } \|A\| = \|B\| & \|A \rightarrow B\| &= W \text{ if } \|A\| \neq W \\ &= \|B\| \text{ if } \|A\| \neq \|B\| & &= \|B\| \text{ if } \|A\| = W \end{aligned} \quad (2)$$

We can achieve this effect in a fragment of the usual language of propositional modal logic with material implication (\supset), material equivalence (\equiv), and necessity (\Box) as primitive if we restrict attention to the class of models in which the accessibility relation is universal, implying that ‘ $\Box A$ ’ is true at a point in a model just in case $\|A\| = W$. The modal translation we require is given by τ in (3).

$$\begin{aligned} \tau(p_i) &= p_i; \\ \tau(A \Rightarrow B) &= \Box(\tau(A) \equiv \tau(B)) \vee \tau(B); \\ \tau(A \rightarrow B) &= \Box\tau(A) \supset \tau(B). \end{aligned} \quad (3)$$

Note that the scope of \Box in the scheme for \Rightarrow is just the biconditional which follows it, while its scope in the scheme for \rightarrow is just the formula $\tau(A)$. Thus the upshot of (3) may be put by saying that to obtain $\tau(C)$ from C , one successively replaces subformulas $A \Rightarrow B$ and $A \rightarrow B$ by, respectively, the formulas $\Box(A \equiv B) \vee B$ and $\Box A \supset B$. To check that we obtain the desired effect—namely (2)—from this translation, we consider the case of \Rightarrow , thinking of $A \Rightarrow B$ as simply another notation for $\Box(A \equiv B) \vee B$. Suppose that $\|A\| = \|B\|$ which implies that $\|\Box(A \equiv B)\| = W$; thus $\|A \Rightarrow B\| = \|\Box(A \equiv B) \vee B\| = \|\Box(A \equiv B)\| \cup \|B\| = W \cup \|B\| = W$, as required by (2). On the other hand, suppose that $\|A\| \neq \|B\|$. In this case we have $\|\Box(A \equiv B)\| = \emptyset$; and again $\|A \Rightarrow B\| = \|\Box(A \equiv B) \vee B\| = \|\Box(A \equiv B) \vee B\| = W \cup \|B\| = W$.

$B) \cup \|B\| = \emptyset \cup \|B\| = \|B\|$, as (2) demands. We leave the reader to check that the demands placed on \rightarrow formulas by (2) are also satisfied when the identification suggested by (3) of $A \rightarrow B$ with $\Box A \supset B$ is made.

These modal translations of \rightarrow and \Rightarrow are of some independent interest. In \Box -based modal logic, there has been much confusion as to what form a suitable (Detachment) Deduction Theorem should take, with the question of the existence of a compound $A\#B$ such that there is a derivation of $A\#B$ from a set of formulas if and only if there is a derivation of B from that set together with A . Apparent disagreements are due to working with different conceptions of derivation, and principally over the question of whether the rule of Necessitation (prefixing a \Box to an earlier formula) may be applied. (See note 3 and the text to which it is appended in Smiley [31] for citations and diagnostic remarks on a related rule, and p. 123 for Necessitation itself.) All is further clarified in Porte [25] where it is noted that with Necessitation allowed to apply in deriving a formula from others, the appropriate form (for **S4** and **S5**) for the desired $A\#B$ to take is none other $\Box A \supset B$: our $A \rightarrow B$. We return to this matter in Section 6. For \Rightarrow , we are not aware of any great interest having been taken in compounds of A and B of the form $\Box(A \equiv B) \vee B$ in modal logic; though it is interesting to note that the conjunction of this formula, $A \Rightarrow B$, call it (as a mere abbreviation in the present context), with its converse, $B \Rightarrow A$, is equivalent even in the weakest normal modal logic **K** to the formula $\Box(A \equiv B)$ itself. The corresponding compounds in intuitionistic logic will receive our attention in Section 4. Finally, on the independent interest of the \Box -translations of \rightarrow and \Rightarrow provided by τ , we can use them separately to define faithful translations of the implicational fragment of classical logic into **S5**. To reduce interruptions, we relegate the discussion of this topic to Appendix A.

We can use the idea of the above translation to turn modal algebras into (the algebras of) Spinks matrices. In particular, we recall that an algebra $\mathbf{A} = (A, \cup, \cap, ', \Box, 0, 1)$ of type $\langle 2, 2, 1, 1, 0, 0 \rangle$ is a *Henle algebra* if $(A, \cup, \cap, ', 0, 1)$ is a Boolean algebra and the 1-ary operation \Box satisfies, for all $a \in A$: $\Box a = 1$ if $a = 1$, and $\Box a = 0$ if $a \neq 1$. For terminological parity, we call a matrix based on a Henle algebra with the 1 of that algebra as its sole designated element a *Henle matrix*. (It is customary to ignore the distinction between an algebra having a distinguished element 1 with the matrix based on that algebra in which 1 is the sole designated value. The traditional use of the term—as in Scroggs [29]—allowed for Henle matrices with several designated values. Cf. the closing paragraph of Section 6 below.) The following observations are stated without (their routine) proofs; [29] explains the “and hence” part of Proposition 1.10.

Proposition 1.9 *Let \mathbf{A} be a Henle algebra and \mathbf{A}^τ be the algebra $(A, \Rightarrow, \rightarrow, 1)$ of type $\langle 2, 2, 0 \rangle$ in which for $a, b \in A$: $a \Rightarrow b = \Box((a \cap b) \cup (a' \cap b')) \cup b$ and $a \rightarrow b = (\Box a)' \cup b$. Then \mathbf{A}^τ is a Spinks matrix. Further, for any formula B we have: B is valid in \mathbf{A}^τ if and only if $\tau(B)$ is valid in \mathbf{A} .*

Proposition 1.10 *For any formula B , B is valid in every Spinks matrix if and only if $\tau(B)$ is valid in every Henle matrix (and hence if and only if $\tau(B)$ is provable in **S5**).*

The following axiomatization was proposed by Spinks with the conjecture (eventually proved in [32]—see Corollary 1.13 of these nine schemata by means of the rule given comprise exactly the Spinks-valid formulas.

- (A1) $A \Rightarrow (B \Rightarrow A)$
- (A2) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
- (A3) $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$
- (A4) $A \Rightarrow (B \rightarrow A)$
- (A5) $(A \rightarrow (B \rightarrow C)) \Rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
- (A6) $(A \rightarrow (B \rightarrow C)) \Rightarrow (B \rightarrow (A \rightarrow C))$
- (A7) $((A \rightarrow B) \rightarrow A) \Rightarrow A$
- (A8) $(A \Rightarrow B) \rightarrow (A \rightarrow B)$
- (A9) $((A \Rightarrow B) \rightarrow B) \Rightarrow ((B \Rightarrow A) \rightarrow A)$

Rule: Modus ponens for \rightarrow : From $A \rightarrow B$ and A to B .

Although it is modus ponens for \rightarrow that is listed here as a primitive rule, it is not hard to see that the analogous rule for \Rightarrow is a derived rule: to obtain B from $A \Rightarrow B$ and A , use (A8) and modus ponens for \rightarrow twice. Since (A1)–(A3) are a well-known basis, together with modus ponens for \Rightarrow , for the set of two-valued truth-functional tautologies for implication (i.e., interpreting \Rightarrow as \supset), and the corresponding formulas in \rightarrow are also provable—being given by (A4), (A5), and (A7) when the main \Rightarrow is replaced by \rightarrow (as (A8) and modus ponens for \rightarrow allow), we can see that all the Spinks-valid unmixed formulas are provable from this basis and, moreover, the following “syntactic” analogue of Theorem 1.7 whose notation we use here.

Proposition 1.11 *For any unmixed formula $A = A[\rightarrow]$ or $A[\Rightarrow]$, A is provable in the above axiomatic system if and only if $A[\supset]$ is a two-valued tautology and thus if and only if A is valid in \mathbf{M}_2 .*

When we turn our attention to arbitrary formulas, including the mixed cases, we can see by a laborious induction on the length of proofs that all formulas provable from the axioms are Spinks-valid—soundness result—and the question of the corresponding completeness result arises. Spinks’s own discussion (in [32]) is in the tradition of algebraic semantics à la Blok and Pigozzi [3], where the class of algebras providing an algebraic semantics comprises what he calls implicative *BCSK*-algebras whose precise definition may be found in Appendix B to this paper, except to note that they are of the form $(A, \Rightarrow, \rightarrow, 1)$ of type $\langle 2, 2, 0 \rangle$, with reducts $(A, \Rightarrow, 1)$ being implicative *BCK*-algebras (sometimes called Abbott or Tarski algebras: algebraic analogues of the implicational fragment of classical logic), that they form a variety, and—an observation Spinks credits to (his thesis supervisor) R. J. Bignall—that the subdirectly irreducible such algebras are what we have been calling Spinks matrices. Similarly, the $(A, \rightarrow, 1)$ reducts of these algebras are what (Spinks [32], [33], and in a 1998 unpublished manuscript) calls implicative *BCS*-algebras whose equational definition along with those for the *BCK* and *BCSK* cases, may be found in Appendix B. (Again, the “implicative” is not just a modifier indicating that some kind of implication is at issue, but specifically classical implication, as opposed to “positive implicative,” which is used in the literature Spinks draws on for the analogous algebraic correlates for intuitionistic implication.)

By a *consequence relation* on a language \mathcal{L} we understand a relation \vdash between finite subsets of \mathcal{L} and elements (formulas) of \mathcal{L} satisfying the usual conditions:

- (i) $A \vdash A$;
- (ii) $\Gamma \vdash A$ implies $\Gamma, B \vdash A$;
- (iii) $\Gamma \vdash A$ and $\Gamma, A \vdash B$ together imply $\Gamma \vdash B$.

Some familiar notational abuses have been employed here: thus (i) means $\{A\} \vdash A$, and the consequent of (ii) means $\Gamma \cup \{B\} \vdash A$. (Γ is an arbitrary finite set of formulas and A and B individual formulas.) We allow ourselves to use the ‘ $\Gamma \vdash$ ’ notation even when Γ is infinite, in which case ‘ $\Gamma \vdash A$ ’ means: for some finite $\Gamma_0 \subseteq \Gamma$, we have $\Gamma_0 \vdash A$. In subsequent sections, the language \mathcal{L} on which we consider various consequence relations is a propositional language, and in particular on the language of the axiomatic system considered above (in Section 5) as well as its \rightarrow - and \Rightarrow -fragments (in Sections 2 and 3, respectively). But the above definition subsumes also the case—a case we need to mention here—in which we are dealing with an equational language over some similarity type in which the formulas are equations $t \approx u$ with t and u terms constructed from a countable supply of individual variables (x_1, x_2, \dots) by application of operation symbols provided by the type. (We here follow by convention of equational logic according to which when mentioning rather than using the equality symbol one writes ‘ \approx ’ rather than ‘ $=$ ’.) For the present case these operation symbols are ‘ \Rightarrow ’ and ‘ \rightarrow ’. The term t_A corresponding to a propositional formula A is defined in the standard way. Thus, for example, for A as $p_1 \rightarrow (p_2 \Rightarrow p_1)$ then t_A is the term $x_1 \rightarrow (x_2 \Rightarrow x_1)$, while the equation corresponding to the formula A is $t_A \approx 1$. For any class \mathbf{K} of algebras of a given similarity type we can define the relation $\models_{\mathbf{K}}$ holding between finite sets $\{e_1, \dots, e_n\}$ of equations (for that type) and individual equations e just in case every algebra in \mathbf{K} satisfies the quasi identity “ e_1 and \dots and e_n , imply e ”, and so defined, $\models_{\mathbf{K}}$ is a consequence relation. Taking \mathbf{K} as the class of implicative *BCK*-algebras, this provides the algebraic characterization of a certain consequence relation defined on the basis of the above axiomatization. We write simply “ $\Gamma \vdash_{\text{MP}} A$ ” to mean that there is a sequence of formulas each of which is either an element of Γ or an instance of one of the axiom-schemata (A1)–(A9) or follows from earlier members of the sequence by an application of the rule modus ponens (for \rightarrow). Spinks [32] then gives the following result in which **BCSK** denotes the class of all implicative *BCK*-algebras:

Theorem 1.12 (Spinks) *For all formulas A of the propositional language with binary connectives \Rightarrow and \rightarrow and all sets Γ of such formulas: $\Gamma \vdash_{\text{MP}} A$ if and only if $\{t_C \approx 1 \mid C \in \Gamma\} \models_{\text{BCSK}} t_A \approx 1$.*

Note that the class of algebras **BCSK** is not the only class with respect to which such an equivalence (with \vdash_{MP}) obtains, as indeed one would expect since the only aspect of the class used here is the set of quasi identities involving equations with “1” on one side. (For an example of this multiplicity of determining classes of algebras—this time for *BCK*-logic—see Kabziński [21].) However, we note that Spinks (unpublished) has shown that there is nonetheless *something* special about **BCSK** in this connection: this class of algebras serves as what Blok and Pigozzi [3] call the “equivalent quasi-variety semantics” for the consequence relation \vdash_{MP} .

Corollary 1.13 *For any formula A , A is provable in the above axiomatic system just in case A is valid in every Spinks matrix.*

Proof: Putting $\Gamma = \emptyset$, we get that A is provable if and only if the identity $t_A \approx 1$ holds in every implicative $BCSK$ -algebra, and since the equations satisfied by a variety are the same as those holding in all subdirectly irreducible algebras of the variety, this amounts to saying that A is provable if and only if $t_A \approx 1$ holds in every Spinks matrix, which means that A is Spinks-valid. \square

Our own version of a result along the lines of Corollary 1.13 appears as Corollary 5.3, and it is also obtained by specializing a semantic characterization of a syntactically defined consequence relation to the special case of $\Gamma = \emptyset$. Although agreeing on this case, the consequence relation of our later discussion (to be called \vdash_* in Section 5) is very different from \vdash_{MP} , as we shall note in Section 6, which gives (Theorem 6.1) a comparative account of the two relations from the model-theoretic perspective. As already remarked, our semantic characterization of the consequence relation in question is in terms of Kripke models rather than of algebras, and we shall not be concerned to relate the classes of models to classes either of modal algebras or of suitable generalizations of Spinks matrices. We have postponed the precise definition of the notion of an implicative $BCSK$ -algebra to Appendix B rather than giving it here, since it is only the (logically representative) Spinks matrices we are concerned with. We remark only that just as the Spinks matrices are the subdirectly irreducible implicative $BCSK$ -algebras, so Henle matrices are the subdirectly irreducible $\mathbf{S5}$ -algebras. (See [29], and, for the not quite invariable—though here certainly attested—link between subdirect irreducibility of modal algebras and the property of being a generated Kripke frame, Sambin [26], where the point-generated Kripke frames are referred to as “initial” frames.)

Since not only the identities but also the quasi identities holding in all subdirectly irreducible members of a variety of algebras are exactly those holding in the whole class, we can replace the reference to **BCSK** in Theorem 1.12 by a reference to the Spinks matrices, which means that we can reformulate the result of making this replacement as in Corollary 1.14, in which the consequence relation determined by a class of matrices is to be understood in the usual (designation-preserving) way (see, e.g., Wójcicki [37], Definition 3.1.4, p. 195).

Corollary 1.14 \vdash_{MP} is the consequence relation determined by the class of all Spinks matrices.

This is the last we shall hear of \vdash_{MP} until Section 5. We close this introduction with a comment on the partial orders definable on implicative BCS -algebras and implicative BCK -algebras with which Spinks makes considerable play. In the BCK case this ordering is defined by $a \leq b$ if and only if $a \Rightarrow b = 1$. A similar definition for implicative BCS -algebras gives only a preorder, and to obtain “the natural” partial order there we must instead set $a \leq b$ if and only if $(a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a = b$. For implicative $BCSK$ -algebras the two partial orders must coincide (see Appendix B), and in the subdirectly irreducible (Spinks matrix) case what we have are just “flat” posets (the element 1 covering an antichain comprising the remaining elements).

We note that, to take the case of \Rightarrow , the ‘=’ in the left-hand half of (1) could have been replaced by ‘ \leq ’—the latter introduced as the flat partial ordering—giving the kind of principle cited from [14] in our opening paragraph. This would be completely equivalent to the formulation given for \Rightarrow in (1). Does this mean we have to retract the motivation (for the study of the Spinks matrices) in terms of purely quantitative or

absolute conditions? No. The relation of identity used in (1) itself is after all already a partial ordering. And the flat ‘ \leq ’ can be defined entirely in terms of identity and the top element. The corresponding strict (irreflexive) partial order relates a to b just in case $a \neq 1$ and $b = 1$ which makes this relation “monadically representable” in the sense of Humberstone ([16], [18]): its holding between elements is a matter of those elements satisfying independently stateable monadic conditions. (See the cited discussions for a precise formulation.) Such relations are intuitively not genuinely relational. Famously, some philosophers have even cast aspersions on the status of identity as a binary relation. (See Chapter 2 of Williams [36] and references there cited.) With this philosophical background in mind, it is perhaps of interest to observe that the flat \leq ordering is the union of a monadically representable relation with the identity relation.

2. The Single-Shafted Arrow

In this section we consider only those formulas of the language described above which are constructed without the aid of \Rightarrow . We study consequence relations on this restricted language with the following semantic interpretation in mind. A model is a triple $\langle W, R, V \rangle$ with W a nonempty set on which R is a binary relation (henceforth the *accessibility* relation of the model), with V a function assigning to each p_i a subset of W . If $\mathcal{M} = \langle W, R, V \rangle$ is a model, the truth of a formula A at a point $u \in W$ —notated ‘ $\mathcal{M} \models_u A$ ’—is given by the following inductive definition:

$$\begin{aligned} \mathcal{M} \models_u p_i & \text{ iff } u \in V(p_i); \\ \mathcal{M} \models_u B \rightarrow C & \text{ iff whenever } \mathcal{M} \models_v B \text{ for all } v \text{ such that } Ruv, \\ & \text{ we have } \mathcal{M} \models_u C \end{aligned}$$

The rationale for the second clause here may be gleaned from Section 1, with the ‘ \Box, \supset ’ representation of $B \rightarrow C$ given by the translation τ there. Our notation $\|A\|$ from that section stands for what in the present notation would be $\{u \in W \mid \langle W, R, V \rangle \models_u A\}$ though in that discussion we were considering the special case—to which we shall return in due course after working our way up to it—of R as the universal relation $W \times W$ on W .

For a model \mathcal{M} as above, we define the \mathcal{M} -relative semantic consequence relation $\models_{\mathcal{M}}$ thus: for any set of formulas Γ and formula A , we define

$$\Gamma \models_{\mathcal{M}} A \text{ iff } u \in W, \text{ if } \mathcal{M} \models_u C \text{ for all } C \in \Gamma, \text{ then } \mathcal{M} \models_u A.$$

An arbitrary consequence relation \vdash is *sound* with respect to a class \mathbb{C} of models when for all Γ, A we have: if $\Gamma \vdash A$ then for each $\mathcal{M} \in \mathbb{C}$, $\Gamma \models_{\mathcal{M}} A$; \vdash is *complete* with respect to \mathbb{C} when for all Γ, A we have: if $\Gamma \models_{\mathcal{M}} A$ for each $\mathcal{M} \in \mathbb{C}$, then $\Gamma \vdash A$; and \vdash is *determined* by \mathbb{C} when \vdash is both sound and complete with respect to \mathbb{C} (i.e., when \vdash is the intersection of the relations $\models_{\mathcal{M}}$ for $\mathcal{M} \in \mathbb{C}$). In practice, we use this terminology when \vdash has been (perhaps partially) specified in syntactic or proof-theoretic terms. The first piece of syntactic terminology to be used in this connection is that of \rightarrow -normality which is intended to be analogous to the familiar notion of normality for modal logics with a primitive necessity operator (\Box). We say a consequence relation \vdash is \rightarrow -normal if it satisfies the following three conditions (in which all variables for formulas and sets of formulas are to be understood as universally quantified).

- (\rightarrow N1) $A_1, \dots, A_n \vdash B$ implies $B \rightarrow C \vdash A_1 \rightarrow (A_2 \rightarrow \dots \rightarrow (A_n \rightarrow C) \dots)$
 (\rightarrow N2) $\Gamma, A \rightarrow C \vdash D$ and $\Gamma, B \vdash D$ imply $\Gamma, A \rightarrow B \vdash D$
 (\rightarrow N3) $B \vdash A \rightarrow B$

For (\rightarrow N1), we understand the schematically indicated conclusion on the right to be the formula C in the case of $n = 0$. (This case of (\rightarrow N1) corresponds to the rule of Necessitation from the familiar \Box -based approach to modal logic; there is also a connection with a version of modus ponens we call (ii)* in our discussion in Section 6.)

Since the above conditions are somewhat unfamiliar, we pause here to tease out some of their consequences.¹ As a special case of (\rightarrow N2) we have the following, derived by setting $D = B$ (thus securing the “ $\Gamma, B \vdash D$ ” automatically):

$$(\rightarrow\text{N2})_0 \quad \Gamma, A \rightarrow C \vdash B \text{ implies } \Gamma, A \rightarrow B \vdash B.$$

Thus in particular, taking $\Gamma = \emptyset$ and putting $B = A \rightarrow C$, we get the familiar contraction principle:

$$(\rightarrow\text{Contraction}) \quad A \rightarrow (A \rightarrow C) \vdash A \rightarrow C.$$

Returning to (\rightarrow N2) itself, we can use it to derive a “prefixing” condition for \rightarrow -normal consequence relations:

$$(\rightarrow\text{Prefixing}) \quad \text{If } \Gamma, B \vdash C \text{ then } \Gamma, A \rightarrow B \vdash A \rightarrow C.$$

To derive this condition, assume $\Gamma, B \vdash C$. By (\rightarrow N3), we get $\Gamma, B \vdash A \rightarrow C$. The desired conclusion (that $\Gamma, A \rightarrow B \vdash A \rightarrow C$) then follows by (\rightarrow N2), once we put $D = A \rightarrow C$ in our formulation of the latter condition above.

The corresponding “suffixing” principle (to the effect that if $\Gamma, A \vdash B$ then $\Gamma, B \rightarrow C \vdash A \rightarrow C$) is not derivable except in special cases—for example, when $\Gamma = \emptyset$, in which case we are dealing with an instance of (\rightarrow N1). For commuting of antecedents,

$$(\rightarrow\text{Permutation}) \quad A \rightarrow (B \rightarrow C) \vdash B \rightarrow (A \rightarrow C),$$

one may begin by prefixing a ‘B’ to both sides of an instance of (\rightarrow N3), getting

$$B \rightarrow C \vdash B \rightarrow (A \rightarrow C)$$

and by a second appeal to (\rightarrow Prefixing):

$$A \rightarrow (B \rightarrow C) \vdash A \rightarrow (B \rightarrow (A \rightarrow C))$$

from which we get the desired condition by the ‘Cut’ properties of \vdash and the contraction-like condition

$$A \rightarrow (B \rightarrow (A \rightarrow C)) \vdash B \rightarrow (A \rightarrow C)$$

which itself results from another appeal to (N3)

$$B \rightarrow C \vdash B \rightarrow (A \rightarrow C)$$

from which (\rightarrow N2)₀ delivers the contraction-like condition above. So much for deriving some familiar implicational principles from our basic conditions. Note that we have not derived Contraction and Permutation, for example, in the “theorematic” form as conditions:

$$\vdash (A \rightarrow (A \rightarrow C)) \rightarrow (A \rightarrow C)$$

$$\vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$$

(where we write “ $\vdash D$ ” for “ $\emptyset \vdash D$ ”). Indeed no such “ \emptyset on the left” \vdash -conditions are satisfied by all \rightarrow -normal consequence relations, as one may see by an induction on the

length of any derivation of such a condition from $(\rightarrow N1) - (\rightarrow N3)$, or alternatively, given the semantic analysis to follow, by an appeal to Proposition 2.8 below.

Lemma 2.1 *If a consequence relation \vdash is determined by some class \mathbb{C} of models, then \vdash is \rightarrow -normal.*

Proof: It suffices to show that for each $\mathcal{M} \in \mathbb{C}$, the consequence relation $\vdash_{\mathcal{M}}$ is \rightarrow -normal. \square

Lemma 2.1 is useful for purposes of showing that a particular consequence relation is sound with respect to a class of models. For showing completeness, we need a canonical model construction. (The canonical model for a given \vdash will be a *characteristic* model, that is, a model \mathcal{M} such that $\Gamma \vdash A$ if and only if $\Gamma \models_{\mathcal{M}} A$, for all A, Γ .) The elements of such models will be certain sets of formulas. A set of formulas Γ is a \vdash -theory when $\Gamma \vdash A$ implies $A \in \Gamma$, for all A ; a \vdash -theory Γ *maximally avoids* a formula C when $\Gamma \not\vdash C$ while $\Delta \vdash C$ for every $\Delta \supseteq \Gamma$. Finally, Γ is an *ma-set* (relative to \vdash , a qualification often omitted when the \vdash in question is clear) just in case for some formula C , Γ is a \vdash -theory which maximally avoids C . It is these ma-sets which will comprise the elements of our canonical models. For \rightarrow -normal \vdash , such sets Γ possess what we shall call the *special* ($\rightarrow N2$)-*secured property*, for a reason the proof of Lemma 2.2(ii) makes clear:

For all formulas A, B, C , if $A \rightarrow C \in \Gamma$ then either $A \rightarrow B \in \Gamma$ or $C \in \Gamma$.

The ma-sets are all “consistent” in the sense of not containing every formula, since any formula they maximally avoid fails to belong to them. (On the other hand, they are not guaranteed to be “maximally consistent” in the sense of maximally avoiding all their nonmembers.) We have no need for further reference to consistency (of \vdash -theories) in this sense, but we shall need a notion of consistency for consequence relations themselves, and call \vdash *consistent* when for some set Δ and formula A , we have $\Delta \not\vdash A$ (equivalently, by the conditions on consequence relations, when this holds for $\Delta = \emptyset$).

Lemma 2.2 *For any \rightarrow -normal consequence relation \vdash :*

- (i) *if $\Gamma \not\vdash D$ then there exists an ma-set (relative to \vdash) $\Delta \supseteq \Gamma$ for which $\Delta \not\vdash D$.*
- (ii) *every ma-set (relative to \vdash) possesses the special ($\rightarrow N2$)-secured property.*

Proof: (i) By a Lindenbaum’s Lemma argument. (ii) Say Δ maximally avoids D . Then if $A \rightarrow B \notin \Delta$ and $C \notin \Delta$, we have $\Delta, A \rightarrow B \vdash D$ and $\Delta, C \vdash D$, so by ($\rightarrow N2$) $\Delta, A \rightarrow C \vdash D$. Thus $A \rightarrow C \notin \Delta$. (Note that this appeal to ($\rightarrow N2$) swaps around the roles played by the schematic letters ‘B’ and ‘C’ in the above formulation of ($\rightarrow N2$)). \square

We are now in a position to introduce the *canonical model* for a consistent \rightarrow -normal consequence relation \vdash , denoted $\mathcal{M}_{\vdash} = \langle W_{\vdash}, R_{\vdash}, V_{\vdash} \rangle$. Here W_{\vdash} is the set of all ma-sets (relative to \vdash), and $V_{\vdash}(p_i) = \{u \in W_{\vdash} \mid p_i \in u\}$.² The definition of R_{\vdash} requires an auxiliary notion. For $u \in W_{\vdash}$, we put $nec(u) = \{A \mid \text{for all } C \text{ for which } A \rightarrow C \in u, \text{ we have } C \in u\}$. Then we set $R_{\vdash}uv$ if and only if $nec(u) \subseteq v$, for $u, v \in W_{\vdash}$.

Lemma 2.3 *Where $u \in W_{\vdash}$ for a \rightarrow -normal consequence relation \vdash , $nec(u)$ is a \vdash -theory.*

Proof: We must show that if $nec(u) \vdash A$, then $A \in nec(u)$. Suppose then, that $nec(u) \vdash A$, while $A \notin nec(u)$. Since $A \notin nec(u)$, there is some formula C with $A \rightarrow C \in u$ and $C \notin u$. Since $nec(u) \vdash A$, there are $B_1, \dots, B_n \in nec(u)$ with $B_1, \dots, B_n \vdash A$. By (\rightarrow N1),

$$A \rightarrow C \vdash B_1 \rightarrow (B_2 \rightarrow \dots \rightarrow (B_n \rightarrow C) \dots).$$

As $A \rightarrow C \in u$, $B_1 \rightarrow (B_2 \rightarrow \dots \rightarrow (B_n \rightarrow C) \dots) \in u$. But $B_1 \in nec(u)$, so

$$(B_2 \rightarrow \dots \rightarrow (B_n \rightarrow C) \dots) \in u,$$

and similarly proceeding in turn through B_2, \dots, B_n , each of them belonging to $nec(u)$, we conclude that $C \in u$, contradicting our earlier assumption concerning C . \square

Theorem 2.4 *Let \vdash be a consistent \rightarrow -normal consequence relation and $\mathcal{M}_\vdash = \langle W_\vdash, R_\vdash, V_\vdash \rangle$ be its canonical model. Then for all formulas C we have*

$$\text{for all } u \in W_\vdash, \mathcal{M}_\vdash \models_u C \text{ iff } C \in u.$$

Proof: By induction on the complexity of (= number of occurrences of “ \rightarrow ” in) C . The basis case (complexity 0) is secured by the way V_\vdash was defined. Induction step: assuming the result holds (for all $u \in W_\vdash$) for formulas of lower complexity than $C = A \rightarrow B$, show that it holds for C itself. Availing ourselves of this inductive hypothesis (invoked for A and B), we need only show that (for all $u \in W_\vdash$),

$$A \rightarrow B \in u \text{ iff, if } A \in v \text{ for each } v \text{ such that } R_\vdash uv, \text{ then } B \in u.$$

We do the “only if” direction first. Suppose that $A \rightarrow B \in u$ and that $A \in v$ for each v such that $R_\vdash uv$, with a view to showing that in that case $B \in u$. The supposition that $A \in v$ for all v for which $R_\vdash uv$ means that for all $v \in W_\vdash$ with $nec(u) \subseteq v$, $A \in v$. This implies that $A \in nec(u)$. For if $nec(u) \not\vdash A$ then there exists $v \in W_\vdash$ with $nec(u) \subseteq v$ and $A \notin v$, by Lemma 2.2; thus $nec(u) \vdash A$ and so by Lemma 2.3, $A \in nec(u)$, as claimed. Since $A \rightarrow B \in u$ and $A \in nec(u)$, we have the desired conclusion that $B \in u$.

For the “if” direction, suppose that $A \rightarrow B \notin u$, with a view to showing (1) $A \in v$ for each v such that $R_\vdash uv$, and (2) $B \notin u$. As to (1), suppose, for a contradiction, that (1) is false. Thus for some $v \in W_\vdash$ with $nec(u) \subseteq v$, we have $A \notin nec(u)$, which means that for some formula C , $A \rightarrow C \in u$, while $C \notin u$. But from the fact that $A \rightarrow C \in u$, Lemma 2.2(ii) gives us—the special (\rightarrow N2)-secured property—that either $A \rightarrow B \in u$ or $C \in u$, contradicting what we already have and establishing (1). For (2), note that if $B \in u$ then $A \rightarrow B \in u$, by appeal to (\rightarrow N3). \square

Corollary 2.5 *The smallest \rightarrow -normal consequence relation is determined by the class of all models.*

Proof: Soundness: Lemma 2.1. Completeness: Suppose $\Gamma \not\vdash A$, where \vdash is the smallest \rightarrow -normal consequence relation. Then by Lemma 2.2(i) there is an element $x \in W_\vdash$ with $\Gamma \subseteq x$ and $A \notin x$, for which x we have, for $\mathcal{M}_\vdash = \langle W_\vdash, R_\vdash, V_\vdash \rangle$, by Theorem 2.4, $\mathcal{M}_\vdash \models_x B$ for each $B \in \Gamma$, while also $\mathcal{M}_\vdash \not\models_x A$. \square

The accessibility relations R_\vdash of the canonical models we have been working with were defined in terms of the function nec which was itself defined by taking $nec(u)$, for a suitable set of formulas u , to consist of those formulas A with the property

that whenever $A \rightarrow C \in u$, we have $C \in u$. This is intuitively intelligible as a way of simulating the more direct definition available in a modal language with \Box , for which—though the notion would hardly be worth separately encapsulating in a definition—we could define $nec(u) = \{A \mid \Box A \in u\}$. We cannot define \Box (with the usual semantic properties—see Proposition 2.7) in the present language, so this direct route is not available to us. But if \Box along with material implication (\supset) were available, we recall from Section 1 that $A \rightarrow C$ would be expressible as $\Box A \supset C$. So what our definition puts into $nec(u)$ are all those formulas with the property that anything their necessitations materially imply is true: the closest we can come without an explicit ‘ \Box ’ to saying that their necessitations are true. (Here for expository simplicity, we rely on Theorem 2.4’s allowing us to conflate talk of truth and membership.) For certain purposes, a somewhat less obvious—though shorter—way of characterizing the function nec is convenient.

Proposition 2.6 *Where $\mathcal{M}_\vdash = \langle W_\vdash, R_\vdash, V_\vdash \rangle$ is the canonical model for a consistent \rightarrow -normal consequence relation, for the function $nec: W_\vdash \rightarrow W_\vdash$ as defined above we have*

$$nec(u) = \{A \mid \exists B. A \rightarrow B \notin u\}.$$

Proof: To show $nec(u) \subseteq \{A \mid \exists B. A \rightarrow B \notin u\}$, suppose $A \in nec(u)$, but—for a contradiction—that for all formulas B , $A \rightarrow B \in u$. Since $A \in nec(u)$, for every B with $A \rightarrow B \in u$, we have $B \in u$. This means then, that for every formula B , $B \in u$, contradicting the consistency of u . For the converse inclusion, suppose that for some B , $A \rightarrow B \notin u$. We must show that for all formulas C , if $A \rightarrow C \in u$, then $C \in u$. But this is an immediate consequence of Lemma 2.2. \square

We remarked in passing above on the undefinability of \Box (as usually interpreted) in the present language; by way of substantiation, let us show that no formula of the present language is equivalent to the formula (standardly interpreted) $\Box p_1$.

Proposition 2.7 *There is no formula A of the present language with the property that for all models $\mathcal{M} = \langle W, R, V \rangle$ and all $u \in W$: $\mathcal{M} \models_u A$ if and only if for all $v \in W$ with Ruv , we have $\mathcal{M} \models_v p_1$. Further, no such A can be found even when attention is restricted to the class of all models with accessibility relation an equivalence relation.*

Proof: First, note that if a formula A can be found which is true, in any model, at precisely those points all of whose R -successors verify p_1 , then such a formula can be found in which the only variable to occur is p_1 itself. (For if a formula B involving other variables can be found meeting this condition, there being no conditions on the V of the models concerned, we may substitute p_1 for each of those extraneous variables, obtaining A .) Thus we treat only the case in which A is constructed out of p_1 by means of \rightarrow , and an easy induction on the complexity of such A shows that for all of them $p_1 \vdash A$, where \vdash is the consequence relation determined by the class of all models. From this it follows that no such A can be true at precisely those points all of whose R -successors verify p_1 . The argument requires only that the determining class of models does not require a point to have no R -successors other than itself, whence the assertion about equivalence-relational models. \square

Proposition 2.7 provides something of a contrast with the usual run of ventures into expressively impoverished modal languages. There has been considerable interest

in what becomes of familiar completeness results in the absence of various *Boolean* connectives—for example, in Humberstone [17], Dunn [10], Celani and Jansana [5]—but even if \Box is not taken as primitive, as for example in pure strict implication systems (e.g., Hacking [15], Lemmon et al. [23]), it is at least definable in terms of the primitives. A case relevantly similar in this respect to the present case arises with the modal logic of noncontingency, in which for weaker normal modal logics (determined by classes of frames which need not be reflexive) where even with the aid of a functionally complete set of Boolean connectives, necessity is not definable. (See Humberstone [19] and Kuhn [22].)

While we are showing the nonexistence of formulas satisfying certain semantic conditions, we should also note another significant case—though this time (by contrast with Proposition 2.8), not one that survives to the extensions of the basic logic (cf. the schema $(\rightarrow\mathbf{T})$ below).

Proposition 2.8 *There is no formula A of the present language with the property that for all models $\mathcal{M} = \langle W, R, V \rangle$ and all $u \in W : \mathcal{M} \models_u A$.*

Proof: Let $W = \{u\}$ for some u , and $R = \emptyset$, $V(p_i) = \emptyset$ for all i , and show by induction on the complexity of A that for the model $\mathcal{M} = \langle W, R, V \rangle$ we have in the case of no formula A , $\mathcal{M} \models_u A$. \square

We turn to some special classes of models. We name the following conditions after schemata \mathbf{T} ($= \Box A \supset A$) and \mathbf{B} ($= A \supset \Box \diamond A$) achieving a similar effect in the more familiar \Box -based approach to modal logic, though they are independently familiar as pure implicational principles (Identity and Peirce, respectively):

$$\begin{aligned} (\rightarrow\mathbf{T}) \quad & \vdash A \rightarrow A \\ (\rightarrow\mathbf{B}) \quad & (A \rightarrow B) \rightarrow A \vdash A. \end{aligned}$$

Theorem 2.9 *If \vdash is a consistent \rightarrow -normal consequence relation satisfying $(\rightarrow\mathbf{T})$ or $(\rightarrow\mathbf{B})$, then the relation R_{\vdash} in the canonical model $\mathcal{M}_{\vdash} = \langle W_{\vdash}, R_{\vdash}, V_{\vdash} \rangle$ is reflexive or symmetric, respectively.*

Proof: For the case of $(\rightarrow\mathbf{T})$, suppose we have $u \in W_{\vdash}$. To show that R_{\vdash} is reflexive, that is, that $nec(u) \subseteq u$, take $A \in nec(u)$ with a view to showing that $A \in u$. But this conclusion is immediate, since we always have $A \rightarrow A \in u$, by $(\rightarrow\mathbf{T})$. Turning to the case of $(\rightarrow\mathbf{B})$, suppose for $u, v \in W_{\vdash}$ we have $R_{\vdash}uv$ but not $R_{\vdash}vu$, that is, $nec(u) \subseteq v$, while for some formula A , $A \in nec(v)$ but $A \notin u$. By Proposition 2.6, since $A \in nec(v)$, there is some formula B with $A \rightarrow B \notin v$. Since $A \notin u$, $(\rightarrow\mathbf{B})$ gives us that $(A \rightarrow B) \rightarrow A \notin u$. So now appealing again to Proposition 2.6 and taking the current $A \rightarrow B$ and A as the “ A ” and “ B ” of our formulation of that Proposition, we get $A \rightarrow B \in nec(u)$; since $nec(u) \subseteq v$, this contradicts the choice of B as a formula for which $A \rightarrow B \notin v$. \square

Corollary 2.10 *The smallest \rightarrow -normal consequence relations satisfying (i) $(\rightarrow\mathbf{T})$, (ii) $(\rightarrow\mathbf{B})$, (iii) both $(\rightarrow\mathbf{T})$ and $(\rightarrow\mathbf{B})$, are determined by the classes of models whose accessibility relations are respectively (i) reflexive, (ii) symmetric, (iii) both reflexive and symmetric.*

Proof: Soundness (in each case): left to the reader. Completeness: by the canonical model method (as in Theorem 2.4 and Corollary 2.5), appealing to Theorem 2.9. \square

The condition ($\rightarrow\mathbf{T}$) is equivalent, as a condition imposed on \rightarrow -normal consequence relations, to ($\rightarrow\mathbf{CP}$) below. The latter implicational principle is familiar from natural deduction and sequent calculus approaches to logic as the rule of \rightarrow -introduction (Conditional Proof) or \rightarrow -insertion on the right, respectively. (More accurately: as the claim that such a rule is admissible.) It is also familiar from the axiomatic approach to logic (with an ancillary notion of the derivability of a formula from a set of formulas) under the rubric of the *Deduction Theorem*:

($\rightarrow\mathbf{CP}$) $\Gamma, A \vdash B$ implies $\Gamma A \rightarrow B$.

Clearly, if a consequence relation \vdash satisfies ($\rightarrow\mathbf{CP}$), then it satisfies ($\rightarrow\mathbf{T}$), since we may take $\Gamma = \emptyset$ and $B = A$. To see that the converse applies in the case of \rightarrow -normal consequence relations at least, we recall that all such \vdash satisfy the condition called (\rightarrow Prefixing) above, so if we have $\Gamma, A \vdash B$, we may prefix an A to obtain $\Gamma, A \rightarrow A \vdash A \rightarrow B$. The consequent of ($\rightarrow\mathbf{CP}$) then follows by appeal to ($\rightarrow\mathbf{T}$) (and the fact that \vdash is a consequence relation).

In view of (CP), the \rightarrow -normal consequence relations satisfying ($\rightarrow\mathbf{T}$) also satisfy the “theorematic” versions of such conditions as (\rightarrow Contraction) and (\rightarrow Permutation) noted above not to be forthcoming for the basic \vdash of Corollary 2.5:

$$\begin{aligned} &\vdash (A \rightarrow (A \rightarrow C)) \rightarrow (A \rightarrow C) \\ &\vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)). \end{aligned}$$

The formulas schematically represented here are what in the combinator-derived terminology of such phrases as “*BCK*-logic” (originated by Meredith in the 1950s) and the like would be called *W* and *C*, respectively, while ($\rightarrow\mathbf{T}$) itself provides *I*, and one application of (CP) to ($\rightarrow\mathbf{N3}$) gives *K*. For *B* itself, the theorematic condition on \vdash would be

$$\vdash (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$$

and this is not satisfied by any of the consequence relations mentioned in Corollary 2.10, as one may verify model-theoretically. (For further explanation of the combinator-derived labels for implicational logics and—as evidenced in Section 1—algebras, see Bunder [4]; further information on the relation between the logics and the classes of algebras may be found in Section 5.2.3 of [3], as well as in references there cited.) The same goes for the condition which this one is a single application of (CP) away from, namely, what we might—for a reason explained below—call ($\rightarrow\mathbf{4}$):

($\rightarrow\mathbf{4}$) $B \rightarrow C \vdash (A \rightarrow B) \rightarrow (A \rightarrow C)$

The three consequence relations isolated in Corollary 2.10 correspond to the familiar normal modal logics called **KT**, **KB**, and **KTB** in the nomenclature of Chellas [7] (except that we have converted Chellas’s italics to roman boldface), whereas, for reasons indicated in Section 1, what we are especially interested in would be the analogue of **S5**, for which that nomenclature provides numerous labels, suggestive of various axiomatizations: **KT5**, **KTB4**, and so on. Since we have been unable to provide a result along the lines of Theorem 2.9 and Corollary 2.10 where the condition on accessibility relations is transitivity (i.e., to provide a syntactic condition on consequence relations on the present language which has the effect of the modal principle $\Box A \vdash \Box\Box A$ —a version of the schema known as **4**), we work with the first of these suggested axiomatizations, isolating the analogue of the **5** schema; to fill out the discussion we include also a further condition closely related to the defining equation for what have been called quasi-commutative *BCK*-algebras—as a reminder

of which we shall use the letter ‘Q’. (This use of the term “quasi-commutative” is from Abbott [1]; see, further, Appendix B, for our dissatisfaction with the more widespread terminology of ‘commutative BCK-algebras’.)

$$\begin{aligned} (\rightarrow\mathbf{5}) \quad & A \rightarrow B, (A \rightarrow C) \rightarrow B \vdash B \\ (\rightarrow\mathbf{Q}) \quad & \vdash ((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A) \end{aligned}$$

To consider $(\rightarrow\mathbf{5})$ and $(\rightarrow\mathbf{Q})$ from a model-theoretic perspective, we need to recall that a binary relation R on a set W is said to be *euclidean* when for all $u, v, w \in W$, Ruv and Ruw together imply Rvw , and to be *cyclic* when Ruv and Rvw together imply Rwu . One readily observes that the equivalence relations on a set W are precisely the reflexive euclidean relations on W , and that these are, again, precisely the reflexive cyclic relations on W . As already indicated, we wish to sidestep the property of transitivity and propose to do so by considering $(\rightarrow\mathbf{5})$, which amounts to concentrating on euclideaness; we mention $(\rightarrow\mathbf{Q})$ here only for its familiarity as, *inter alia*, a homogenized version of the “mixed” schema (A9) in Section 1—and invite the reader to check that for all A, B ,

$$\models_{\mathcal{M}} ((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)$$

where \mathcal{M} is any model whose accessibility relation is reflexive and cyclic. (From this and the content of Theorem 2.11(ii) below, it follows that the consequence relation mentioned there satisfies $(\rightarrow\mathbf{Q})$.)

Theorem 2.11 (i) *If \vdash is a consistent \rightarrow -normal consequence relation satisfying $(\rightarrow\mathbf{5})$, then the relation R_{\vdash} of the canonical model $\mathcal{M}_{\vdash} = \langle W_{\vdash}, R_{\vdash}, V_{\vdash} \rangle$ is euclidean.*
(ii) *The smallest \rightarrow -normal consequence relation satisfying $(\rightarrow\mathbf{T})$ and $(\rightarrow\mathbf{5})$ is determined by the classes of models with equivalence relations as their accessibility relations.*

Proof: (i) Suppose for $u, v, w \in W_{\vdash}$, we have Ruv and Ruw but not Rvw . Then there is some $A \in \text{ nec}(v)$ with $A \notin w$. Since $A \in \text{ nec}(v)$ there is (by Proposition 2.6) some formula C for which we have $A \rightarrow C \notin v$. Since Ruw and $A \notin w$, $A \notin \text{ nec}(u)$, that is, for all formulas B we have $A \rightarrow B \in u$. But also since Ruv and $A \rightarrow C \notin v$, $A \rightarrow C \notin \text{ nec}(u)$, so for all formulas B we have $(A \rightarrow C) \rightarrow B \in u$. Finally, as u is consistent, we may choose B as some formula not belonging to u , giving a contradiction by $(\rightarrow\mathbf{5})$ and the fact that we now have $\{A \rightarrow B, (A \rightarrow C) \rightarrow B\} \subseteq u$.

(ii) Soundness being clear, we do only the completeness half of the claim. This follows from part (i) and Theorem 2.9, given that reflexive euclidean relations are equivalence relations. \square

Corollary 2.12 *The consequence relation (on the language of the present section) determined by the class of all models with universal accessibility relations is that mentioned under Theorem 2.11(ii).*

Proof: By a routine adaptation of the corresponding argument for \square -based modal logic using a lemma on generated submodels (as in [7], Theorems 3.12, 3.13, or [6], Propositions 3.2, 3.76) and the fact that for equivalence-relational models such submodels have a universal accessibility relation and our Theorem 2.11(ii). \square

As already noted, we have not been able to provide a completeness result for a syntactically characterized class of consequence relations, and in particular for its least element, corresponding to the \square -based logic **K4**, in which (for the least element case)

the semantic characterization is in terms of determination by the class of all models whose accessibility relations are transitive. A pertinent tool for such a syntactic characterization would seem to be the condition we accordingly called $(\rightarrow 4)$ which we now observe (without proof) to be equivalent, as a condition on \rightarrow -normal consequence relations to a condition for whose formulation we introduce the following abbreviative convention (which we shall employ extensively in later sections, especially a propos of conditions involving \Rightarrow). We write “ $\Gamma \vdash A_1, \dots, A_n$ ” to mean: “for all formulas D and all $\Delta \supseteq \Gamma$, if $\Delta, A_i \vdash D$ (for each $i \in \{1, \dots, n\}$), we have $\Delta \vdash D$ ”. (See Scott [27], where, however, this is not introduced as an abbreviation.)

$(\rightarrow 4)'$ $B \rightarrow C \vdash A \rightarrow D, (A \rightarrow B) \rightarrow E, C.$

Using the characterization of nec given in Proposition 2.6, one can see the condition $(\rightarrow 4)'$ as securing the following property of the canonical model for any \rightarrow -normal consequence relation satisfying the condition: if $A \rightarrow B \in nec(u)$ and $A \in nec(u)$, then $B \in nec(u)$, for all $u \in W_{\vdash}$. But the author has been unable to use this fact concerning $(\rightarrow 4)'$, or any more direct manipulations with $(\rightarrow 4)$ itself, to show that the relation R_{\vdash} is transitive (or equivalently, that when $nec(u) \subseteq v$, we always have $nec(u) \subseteq nec(v)$). The reason for concentrating on $(\rightarrow 4)$ despite these difficulties will emerge in Theorem 2.14(iii). We pause to notice, for use in later sections, a feature of such “multiple-succedent” formulations as $(\rightarrow 4)'$ here.

Proposition 2.13 *With the multiple right-hand side notation understood as above, if u is an ma-set with respect to a consequence relation \vdash for which we have $\Gamma \vdash A_1, \dots, A_n$, then if $\Gamma \subseteq u$, for at least one of the i ($i = 1, \dots, n$) we have $A_i \in u$.*

Proof: Suppose u maximally avoids the formula C and no $A_i \in u$. Then $u \cup \{A_i\} \vdash C$ for each i . By our explanation of the notation above, then, on the assumption that $\Gamma \subseteq u$, we get $u \vdash C$, contradicting the choice of C as maximally avoided by u . \square

Many of our observations about the logic of \rightarrow have been formulated in terms of models where a more convenient formulation is available in terms of frames, the *frame* of a model $\mathcal{M} = \langle W, R, V \rangle$ being the pair $\mathcal{F} = \langle W, R \rangle$; we also put this by calling \mathcal{M} a model *on* the frame \mathcal{F} . To adapt our earlier terminology and notation we write $\Gamma \Vdash_{\mathcal{F}} A$ to mean: $\Gamma \Vdash_{\mathcal{M}} A$ for every model \mathcal{M} on \mathcal{F} . A consequence relation \vdash is *sound with respect to*, *complete with respect to*, or *determined by* a class of frames just in case, respectively, \vdash is included in, includes, or is equal to the intersection of the collection of consequence relations $\Vdash_{\mathcal{F}}$ for \mathcal{F} in the class. Thus we can reformulate part (iii) of Corollary 2.10, by way of example, as saying that the consequence relation there characterized syntactically is the consequence relation determined by the class of reflexive symmetric frames (where we apply the terminology for binary relations to frames when it applies to their accessibility relations); for the sake of the casual reader, however, perhaps not keen to absorb the frame-based terminology, we continue to state completeness results in terms of classes of model, using the frame apparatus only for the modal definability discussion which follows and for the corresponding discussion at the end of the following section.

As is well known, abstracting away from the V component of our models and considering frames in their own right suggests new questions, such as, in particular, the modal definability of classes of frames, where a condition on \rightarrow -normal consequence relations—and here we have in mind simple conditions of the “ $\Gamma \vdash A$ ” form—*modally defines* a class \mathbb{F} of frames just in case $\Gamma \Vdash_{\mathcal{F}} A$ if and only if $\mathcal{F} \in \mathbb{F}$. (See

van Benthem [34] or [35].) We close the present section with some observations on definability for the main conditions we have been discussing. Since these have been stated schematically, rather than for particular formulas, we take the names of the schemata to refer here to representative instances, in which distinct schematic letters for formulas are replaced by distinct propositional variables. Thus part (ii), for example, of Theorem 2.14 below means that

$$(p_1 \rightarrow p_2) \rightarrow p_1 \vdash p_1$$

modally defines the class of symmetric frames. (This explanation makes sense only for schematically formulated conditions in which all schematic letters are—like our ‘A’, ‘B’, . . .—metalinguistic variables for formulas—as opposed to the more general case in which there appear schematic letters—our ‘ Γ ’, ‘ Δ ’, . . .—for sets of formulas. Since the object language contains nothing which would play the role played by propositional variables in the latter case—a point noted in a related connection by Scott [28]—the conditions mentioned in the following result are all formulated without such set-variables.)

Theorem 2.14 (i) *The condition $(\rightarrow \mathbf{T})$ modally defines the class of reflexive frames; (ii) $(\rightarrow \mathbf{B})$ modally defines the class of symmetric frames; (iii) $(\rightarrow \mathbf{4})$ modally defines the class of transitive frames; (iv) $(\rightarrow \mathbf{5})$ modally defines the class of euclidean frames.*

Proof: For (i): we have to show that $\Vdash_{\mathcal{F}} p_1 \rightarrow p_1$ if and only if \mathcal{F} is reflexive. Since we have taken the reader already to have verified what amounts to the “if” direction here for the soundness half of Corollary 2.10(i), we have only the “only if” direction to check. So suppose that $\mathcal{F} = \langle W, R \rangle$ is a frame with R not reflexive, say because for $u \in W$, not Ruu . To show, as required, that we do not have $\Vdash_{\mathcal{F}} p_1 \rightarrow p_1$, we must show how to construct a model $\mathcal{M} = \langle W, R, V \rangle$ on \mathcal{F} for which not $\Vdash_{\mathcal{M}} p_1 \rightarrow p_1$, which we do by specifying V in such a way that $\mathcal{M} \not\models_u p_1 \rightarrow p_1$. As long as we choose $V(p_1) = \{v \in W \mid Ruv\}$ the resulting model is easily seen to satisfy this demand. In future, we denote $\{v \in W \mid Ruv\}$ by $R(u)$.

For (ii):—and we work only the “only if” direction, as in (i)—suppose we have $\langle W, R \rangle$ with $u, v \in W$ for which Ruv and not Rvu . Any V satisfying $V(p_1) = R(u)$, $V(p_2) = \emptyset$, yields a model, as required, with $(p_1 \rightarrow p_2) \rightarrow p_1$ true at u and p_1 false at u .

For (iii): given $\langle W, R \rangle$ and $u, v, w \in W$ with Ruv, Rvw , and not Ruw , let V satisfy $V(p_1) = R(u)$, $V(p_2) = \{x \in W \mid R(x) \subseteq R(u)\}$, $V(p_3) = \emptyset$, and we have $\langle W, R, V \rangle \models_u p_2 \rightarrow p_3$ while $\langle W, R, V \rangle \not\models_u (p_1 \rightarrow p_2) \rightarrow (p_1 \rightarrow p_3)$, showing that, *contra* (4), for $\mathcal{F} = \langle W, R \rangle$, we do not have $p_2 \rightarrow p_3 \Vdash_{\mathcal{F}} (p_1 \rightarrow p_2) \rightarrow (p_1 \rightarrow p_3)$.

For (iv): given $\langle W, R \rangle$ and $u, v, w \in W$ with Ruv, Ruw , and not Rvw , let V satisfy $V(p_1) = W \setminus \{w\}$, $V(p_2) = V(p_3) = \emptyset$, and we have $\langle W, R, V \rangle \models_u p_1 \rightarrow p_2$, $\langle W, R, V \rangle \models_u (p_1 \rightarrow p_3) \rightarrow p_2$, while, *contra* (5), $\langle W, R, V \rangle \not\models_u p_2$. \square

It might be of interest to know which classes of frames are modally definable in the customary language with \square are not modally definable in the present language with \rightarrow as sole connective, but we do not pursue this question here as it is somewhat off the main track for our concern with questions of completeness for logics in the

combined \rightarrow, \Rightarrow language and its \rightarrow and \Rightarrow fragments. Having considered such questions for the former fragment, we now turn our attention to the latter.

3. The Double-Shafted Arrow

Our language here will be as in Section 2 but with the binary connective \Rightarrow in place of \rightarrow . Again by a model \mathcal{M} we understand a triple $\langle W, R, V \rangle$ whose components are as before, though this time the truth-definition takes the following form (inspired by the ' \Box, \equiv ' treatment of \Rightarrow provided by the translation τ in Section 1):

$$\begin{aligned} \mathcal{M} &\models_u p_i \text{ iff } u \in V(p_i), \\ \mathcal{M} &\models_u B \Rightarrow C \text{ iff either for all } v \text{ such that } Ruv, \\ &\text{either (i) } \mathcal{M} \models_v B \text{ iff } \mathcal{M} \models_v C \text{ or else (ii) } \mathcal{M} \models_u C. \end{aligned}$$

Notions of soundness with respect to a class \mathbb{C} of models, completeness with respect to \mathbb{C} , and determination by \mathbb{C} are to be understood as before, with the current truth-definition in force in place of that in Section 2. As there, we begin by asking about the consequence relation determined by the class of all models, and syntactically characterizing a notion of normality for which the consequence relation in question is the weakest to fall under that notion. Accordingly, call a consequence relation $\vdash \Rightarrow$ -normal provided it satisfies the following conditions, for whose formulation we have (for $(\Rightarrow N3)$) used the abbreviative convention introduced a propos of $(\rightarrow 4)'$ in the preceding section. $(\Rightarrow N3)$ may be regarded as the imposition of all the conditions listed here as $(\Rightarrow N3)_n$ for $n = 0, 1, \dots$, each of which is rather cumbersome to formulate even with the aid of this convention:

$$\begin{aligned} (\Rightarrow N1) \quad & A \Rightarrow B \vdash B \Rightarrow A, B; \\ (\Rightarrow N2) \quad & B \vdash A \Rightarrow B; \\ (\Rightarrow N3)_n \quad & \text{For } N = \{1, \dots, n\} \text{ if, for all } I, J, \text{ such that } I \cup J = N \text{ and } I \cap J = \\ & \emptyset, \bigcup_{i \in I} \{C_i, D_i\}, A \vdash B, \bigcup_{j \in J} \{C_j, D_j\} \text{ and } \bigcup_{i \in I} \{C_i, D_i\}, B \vdash \\ & A, \bigcup_{j \in J} \{C_j, D_j\}, \text{ then } C_1 \Rightarrow D_1, \dots, C_n \Rightarrow D_n \vdash A \Rightarrow \\ & B, D_1, \dots, D_n. \end{aligned}$$

The condition $(\Rightarrow N1)$ is in fact redundant here, following from $(\Rightarrow N3)_1$ when A and B are taken as D_1 and C_1 , respectively. For this reason it does not figure explicitly in the completeness proof below; we list and label it separately because we shall need it for some later applications in which the cumbersome $(\Rightarrow N3)$ condition is not present (being no longer needed).

Lemma 3.1 *If a consequence relation \vdash (on the present language) is determined by some class \mathbb{C} of models then \vdash is \Rightarrow -normal.*

For a consistent \Rightarrow -normal consequence relation \vdash , we introduce the canonical model \mathcal{M}_\vdash for \vdash , using the same notation as in Section 2, as the structure $\langle W_\vdash, R_\vdash, V_\vdash \rangle$, where W_\vdash is the set of all ma-sets of formulas (relative to \vdash), and $V_\vdash(p_i) = \{u \in W_\vdash \mid p_i \in u\}$. The definition of R_\vdash is given by: $R_\vdash uv$ if and only if for all formulas C , if $C \Rightarrow D \in u$ and $D \notin u$, then $C \in v$ if and only if $D \in v$. We are now in a position to pass to the analogue of Theorem 2.4.

Theorem 3.2 *Let \vdash be a consistent \Rightarrow -normal consequence relation and $\mathcal{M}_\vdash = \langle W_\vdash, R_\vdash, V_\vdash \rangle$ be its canonical model. Then for all formulas C we have: for all $u \in W_\vdash$, $\mathcal{M}_\vdash \models_u C$ if and only if $C \in u$.*

Proof: The strategy is as in the proof of Theorem 2.4, so we give only the inductive part of the proof, where C is $A \Rightarrow B$ and what has to be shown is that

$$A \Rightarrow B \in u \text{ iff either for all } v \text{ such that } R_{\perp}uv, \text{ we have } A \in v \text{ iff } B \in v, \\ \text{or else } B \in u.$$

The “only if” direction is immediate from the definition of R_{\perp} . The “if” direction requires us to show that $A \Rightarrow B \in u$ if (i) $B \in u$, which follows by (\Rightarrow N2), and (ii) if for all v such that $R_{\perp}uv$, $A \in v$ if and only if $B \in v$, to show which argue contrapositively that given $A \Rightarrow B \notin u$ we can find $v \in W_{\perp}$ with either $A \in v$ and $B \notin v$, or else $A \notin v$ and $B \in v$. So suppose for u with $A \Rightarrow B \notin u$ that we have an enumeration $C_1 \Rightarrow D_1, \dots, C_n \Rightarrow D_n, \dots$ of all \Rightarrow -formulas in u whose consequents (the D_i) do not belong to u . We want to construct the desired v by dividing the set of pairs $\{C_i, D_i\}$ into two classes, comprising what we shall call the *in*-pairs and the *out*-pairs. The idea will be that both members of an *in*-pair are to belong to v while neither member of an *out*-pair belongs to v , while at the same time exactly one of A, B , is in v . If such a division of the set of pairs $\{C_i, D_i\}$ is impossible this must be because it is impossible for some finite subset, $\{C_1, D_1\}, \dots, \{C_n, D_n\}$, which means that we have

$$\bigcup_{i \in I} \{C_i, D_i\}, A \vdash B, \bigcup_{j \in J} \{C_j, D_j\} \text{ and } \bigcup_{i \in I} \{C_i, D_i\}, B \vdash A, \bigcup_{j \in J} \{C_j, D_j\}$$

for all I, J , for which $I \cup J = N$ and $I \cap J = \emptyset$, in which case (\Rightarrow N3) _{n} tells us that

$$C_1 \Rightarrow D_1, \dots, C_n \Rightarrow D_n \vdash A \Rightarrow B, D_1, \dots, D_n.$$

But this cannot be, by Proposition 2.13, since all of the formulas on the left of the ‘ \vdash ’ belong to u while *ex hypothesi* none of those on the right do. \square

Before proceeding to consider some more restricted classes of models, we pause to give the analogue of Proposition 2.7, showing that \Rightarrow fares no better than \rightarrow in providing a formula with the same truth-conditions as $\Box p_1$.

Proposition 3.3 *There is no formula A of the present language with the property that for all models $\mathcal{M} = \langle W, R, V \rangle$ and all $u \in W : \mathcal{M} \models_u A$ if and only if for all $v \in W$ with Ruv , we have $\mathcal{M} \models_v p_1$. Moreover, this continues to be the case as we restrict the class of models under consideration to those in which accessibility is an equivalence relation.*

Proof: Exactly the same argument as was given for Proposition 2.7 works here. \square

We now turn our attention to extensions of the basic logic. Consider the condition

$$(\Rightarrow \mathbf{T}) \quad A, A \Rightarrow B \vdash B$$

which could, in honor of its form, equally well be called (\Rightarrow **MP**). We have named it after its function rather than its form, since this is a condition playing the role of the familiar **T** axiom of (\Box -based) modal logic, as we shall note in Corollary 3.5(i). The resemblance to modus ponens is of interest, however, because the corresponding function was played in Section 2 by the condition we accordingly there called (\rightarrow **T**), and whereas modus ponens is generally thought of as an elimination principle for an implicational connective, Conditional Proof—a variant of (\rightarrow **T**), as we noted with the discussion of (\rightarrow **CP**) following Corollary 2.10—has instead the status of an introduction principle. (As to the various things that might be meant by “modus

ponens”—the current meaning being evidently somewhat different from the sense in which the phrase was used in Section 1—some discussion will be found in Section 6 below.) For part (ii) of the following result, we need to isolate a symmetry-related condition, analogous to the modal schema **B**, which may be compared with the functionally similar though (as with the **T** conditions) formally very different condition ($\Rightarrow\mathbf{B}$) of the preceding section:

$$(\Rightarrow\mathbf{B}) \quad A \vdash (A \Rightarrow B) \Rightarrow B.$$

Theorem 3.4 (i) *If \vdash is a consistent \rightarrow -normal consequence relation satisfying $(\Rightarrow\mathbf{T})$, then the relation R_{\vdash} in the canonical model $\mathcal{M}_{\vdash} = \langle W_{\vdash}, R_{\vdash}, V_{\vdash} \rangle$ is reflexive.*
(ii) *If \vdash is a consistent \rightarrow -normal consequence relation satisfying $(\Rightarrow\mathbf{T})$ and $(\Rightarrow\mathbf{B})$, then the relation R_{\vdash} in the canonical model $\mathcal{M}_{\vdash} = \langle W_{\vdash}, R_{\vdash}, V_{\vdash} \rangle$ is both reflexive and symmetric.*

Proof: (i) Take $u \in W_{\vdash}$, to show that $R_{\vdash}uu$, that is, for all A, B with $A \Rightarrow B \in u$ and $B \notin u$, we have $A \in u$ if and only if $B \in u$. On the hypothesis that $B \notin u$, the latter reduces to showing that $A \notin u$, which follows by $(\Rightarrow\mathbf{T})$.

(ii) Take $u, v \in W_{\vdash}$, with $R_{\vdash}uv$, with a view to showing that $R_{\vdash}vu$. Suppose otherwise: that is, that for some A, B , we have $A \Rightarrow B \in v$ and $B \notin v$, while A and B differ as to their membership in u . There are two cases: (1) $A \in u$ and $B \notin u$, (2) $A \notin u$ and $B \in u$.

Case 1: By $(\Rightarrow\mathbf{B})$ since $A \in u$, we have $(A \Rightarrow B) \Rightarrow B \in u$. Since $B \notin u$ and $R_{\vdash}uv$, $A \Rightarrow B$ and B must in that case agree as to their membership in v : but this is impossible as we already have $A \Rightarrow B \in v$ while $B \notin v$.

Case 1: This time we instantiate the condition $(\Rightarrow\mathbf{B})$ with the roles of the schematic letters A, B , interchanged, getting $(B \Rightarrow A) \Rightarrow A \in u$, since $B \in u$. As $A \notin u$ and $R_{\vdash}uv$, $(B \Rightarrow A)$ and A must agree in respect of membership in v . Now $A \Rightarrow B \in v$ and $B \notin v$, so (by $(\Rightarrow\mathbf{N1})$) $B \Rightarrow A \in v$, so we must have $A \in v$. But since $A \Rightarrow B \in v$ and $B \notin v$, this contradicts $(\Rightarrow\mathbf{T})$. \square

Corollary 3.5 (i) *The smallest \rightarrow -normal consequence relations satisfying $(\Rightarrow\mathbf{T})$ is determined by the classes of models whose accessibility relations are reflexive;*
(ii) *The smallest \rightarrow -normal consequence relations satisfying $(\Rightarrow\mathbf{T})$ and $(\Rightarrow\mathbf{B})$ is determined by the classes of models whose accessibility relations are reflexive and symmetric.*

Proof: Soundness: in each case left to the reader. Completeness: by Theorem 3.4(i) and (ii) for the respective cases here. \square

To parallel the development in Section 2 for ' \rightarrow ', we should make our way to a version of **S5** in the current language by articulating a condition whose effect on the canonical accessibility relation is, when taken in conjunction with reflexivity, perhaps together with symmetry, to make it an equivalence relation. Several properties of binary relations were mentioned in Section 2 as playing this role: transitivity, cyclicity, and euclideaness, the last being that concentrated on in Theorem 2.11. Indeed, as we remark below (Theorem 3.6(iii)) the following simple condition—named in the light of this fact—modally defines the class of euclidean frames:

$$(\Rightarrow\mathbf{5}) \quad \vdash B \Rightarrow (A \Rightarrow B).$$

It is far from clear how a canonical model (or indeed any other) completeness argument for this condition might go, so our exploration of the route to the \Rightarrow -language incarnation of **S5** comes to an abrupt terminus at this point. This does not mean that the eventual goal of our exploration, which is a version of **S5** in the combined language with both \Rightarrow and \rightarrow , is unattainable. We shall see in Section 5 that the presence of \rightarrow alongside \Rightarrow enables us to use the definition from Section 2 of the canonical accessibility relation to obtain the desired result. We close this section with an elaboration of the difficulty of using (\Rightarrow 5) to show that the relation R_{\vdash} (for \vdash satisfying this condition) is euclidean and then some observations on modal definability in the present language.

Suppose then that $\langle W_{\vdash}, R_{\vdash}, V_{\vdash} \rangle$ is the canonical model for a consistent \Rightarrow -normal consequence relation \vdash satisfying (\Rightarrow 5), and, hoping for a contradiction, that R_{\vdash} is not euclidean; thus there are $u, v, w \in W_{\vdash}$ with $R_{\vdash}uv, R_{\vdash}uw$ and not $R_{\vdash}vw$. Since not $R_{\vdash}vw$, there are formulas A, B , for which $A \Rightarrow B \in v$ and $B \notin v$, while A and B differ as to their membership in w . In view of (\Rightarrow 5), we have $B \Rightarrow (A \Rightarrow B) \in u$, so since $R_{\vdash}uv$ and B and $A \Rightarrow B$ differ, as we have just seen, in respect of membership in v , we cannot also have $A \Rightarrow B \notin u$. Thus $A \Rightarrow B \in u$. Now we also know that $R_{\vdash}uw$ and A and B differ as to their membership in w . If we had $B \notin u$, the fact that $A \Rightarrow B \in u$ would be inconsistent with this, and so we conclude that $B \in u$. But from this point on, there seems no further need to travel in the direction of a contradiction from our assumptions. To use the fact that u bears the relation R_{\vdash} to various points—here v and w —we need to exploit \Rightarrow -implications whose consequents do not belong to u , and while we know that there are formulas not in u , since u is consistent, it is not clear—though, of course, it may after all be possible with the aid of greater insight than the present author is able to muster—how to get such nonmembers of u to engage with the A and B delivered to us by the hypothesis that $R_{\vdash}vw$.

On the other hand, there would have been no difficulty if we had been working with a ternary connective $\#$ instead of our binary \Rightarrow , with $\#(A, B, C)$ being interpreted as $\Box(A \equiv B) \vee C$ in the same way that $A \Rightarrow B$ is interpreted as $\Box(A \equiv B) \vee B$, that is, with the following clause in the definition of truth at a point in a model $\mathcal{M} = \langle W, R, V \rangle$:

$$\begin{aligned} \mathcal{M} \models_u \#(A, B, C) \text{ iff either for all } v \text{ such that } Ruv, \text{ either } \mathcal{M} \models_v A \\ \text{iff } \mathcal{M} \models_v B, \text{ or else } \mathcal{M} \models_u C. \end{aligned}$$

We leave the reader to see how to modify the notion of a \Rightarrow -normal consequence relation to get a corresponding notion for this ternary connective which allows for a proof that truth and membership coincide for the canonical model of any consistent consequence relation of the kind isolated, when the accessibility relation R_{\vdash} is defined in the obvious way: $R_{\vdash}uv$ just in case for all formulas A, B, C , with $C \notin u, A \in v$ if and only if $B \in v$. We content ourselves with considering a suitable variation on the theme of (\Rightarrow 5) for this more expressive language:

$$(\#5) \quad \vdash \#(\#(A, B, C), C, \#(A, B, D)).$$

(We could equally well replace the schema after the ‘ \vdash ’ here with ‘ $\#(C, \#(A, B, C), \#(A, B, D))$ ’.) It is easy to see (especially if one considers the translation given above using \Box, \equiv , and \vee) that for all formulas A, B, C, D and all models \mathcal{M} with euclidean accessibility relations

$$\models_{\mathcal{M}} \#(\#(A, B, C), C, \#(A, B, D))$$

which gives the required soundness result, and for completeness, by contrast with the \Rightarrow case, it is easy to show the canonical accessibility relation to be euclidean. As with the unsuccessful argument of the previous paragraph, we aim for a contradiction from the supposition that there are $u, v, w \in W_{\vdash}$ with $R_{\vdash}uv$, $R_{\vdash}uw$, and not $R_{\vdash}vw$. Since not $R_{\vdash}vw$, there are formulas A, B, C for which $\#(A, B, C) \in v$ and $C \notin v$, while A and B differ as to their membership in w . By (#5), for this choice of A, B, C , and any choice of D , we have $\#(\#(A, B, C), C, \#(A, B, D)) \in u$. But $R_{\vdash}uv$, so since the first two components of this compound differ as to their membership in v , we must have $\#(A, B, D) \in v$, for all D , and in particular therefore for a D (and by consistency we know there is one such) such that $D \notin u$. Finally, since also we were given that $R_{\vdash}uw$ and A and B differ over membership in w , we get our contradiction (since $\#(A, B, D) \in v$). The earlier argument (for \Rightarrow) was blocked at the stage corresponding to that at which here we concluded that $\#(A, B, D) \in v$, for all D , since in the \Rightarrow language we can only express the special case in which D is B itself. In Appendix C to this paper, we give a simpler example of how this “special case” type of restriction can affect completeness proofs in the hope that it may contain lessons for the present example.

The above difficulties with completeness notwithstanding, we conclude this section as we concluded Section 2, with some observations on modal definability of classes of frames. We take the definitions of the key concepts (introduced for Theorem 2.14) to be transferred to the present language and semantics *mutatis mutandis*.

Theorem 3.6 (i) *The condition $(\Rightarrow \mathbf{T})$ modally defines the class of reflexive frames;* (ii) *$(\Rightarrow \mathbf{B})$ modally defines the class of symmetric frames;* (iii) *$(\Rightarrow \mathbf{5})$ modally defines the class of euclidean frames.*

Proof: In each case we content ourselves with showing how to construct, on any frame outside the class, a suitable countermodel (as in the proof of Theorem 2.14). The resulting models \mathcal{M} refute respectively the claims that (i) $p_1, p_1 \Rightarrow p_2 \models_{\mathcal{M}} p_2$; (ii) $p_1 \models_{\mathcal{M}} (p_1 \Rightarrow p_2) \Rightarrow p_2$; and (iii) $\models_{\mathcal{M}} p_1 \Rightarrow (p_2 \Rightarrow p_1)$. For (i), with $\langle W, R \rangle$ and $u \in W$ for which not Ruu , put $V(p_1) = W$, $V(p_2) = W \setminus \{u\}$; for (ii) take $\langle W, R \rangle$ with $u, v \in W$ such that Ruv and not Rvu , and put $V(p_1) = \{u\}$, $V(p_2) = W \setminus \{u\}$; for (iii), given $\langle W, R \rangle$ and $u, v, w \in W$ with Ruv , Ruw , and not Rvw , put $V(p_1) = \emptyset$, $V(p_2) = \{w\}$. \square

4. Intermission: An Intuitionistic Version of the Double-Shafted Arrow

We interrupt the investigation from a modal perspective of the issues raised in Section 1, to report on the intuitionistic analogue of the ‘ \Rightarrow ’ of the preceding section. The language is as there, but is now to be interpreted by Kripke models for intuitionistic logic; that is, we take a model \mathcal{M} to be a triple $\langle W, R, V \rangle$ but with R specifically a partial ordering of the nonempty set W and with V satisfying the special (“Persistence”) condition that we have $v \in V(p_i)$ whenever Ruv and $u \in V(p_i)$. The definition of truth at a point in such a model is exactly as in Section 3; it follows from this definition that whenever any formula is true at a point in a model $\langle W, R, V \rangle$ (“All formulas are persistent”). With this semantics in force, the logic determined by the class of all models is accordingly the fragment of intuitionistic logic we get by considering $A \Rightarrow B$ to be the formula $(A \longleftrightarrow B) \vee B$. (Here ‘ \longleftrightarrow ’ is the usual intuitionistic biconditional.) Classically, such a formula (taking ‘ \longleftrightarrow ’ as the ‘ \equiv ’ of our customary notation) is just another way of writing the implication we have

been notating as $A \supset B$. (There would be nothing to be gained, by contrast, from considering an intuitionistic version of the implicational connective considered in Section 2: since truth at a point is equivalent to truth at all accessible points, the ‘ \Box ’ on the antecedent of the translation of a compound is redundant and we should be simply considering intuitionistic implication again, rather than, as with the above ‘ \Rightarrow ’, something new.)

Let \vdash be the least consequence relation on the present language to satisfy, for all formulas A, B, C , and all sets of formulas Γ , the following four conditions:

- (IL \Rightarrow 1) $A, A \Rightarrow B \vdash B$;
- (IL \Rightarrow 2) $\Gamma, B \Rightarrow A \vdash C$ and $\Gamma, B \vdash C$ imply $\Gamma, A \Rightarrow B \vdash C$;
- (IL \Rightarrow 3) $\Gamma, A \vdash B$ and $\Gamma, B \vdash A$ imply $\Gamma \vdash A \Rightarrow B$;
- (IL \Rightarrow 4) $B \vdash A \Rightarrow B$.

(Note that with the abbreviative conventions of the preceding section in force, we could write (IL \Rightarrow 2) in the form: $A \Rightarrow B \vdash B, B \Rightarrow A$.) The result below reveals \vdash as the basic (i.e., weakest) consequence relation susceptible of a completeness proof in terms of models with the present semantic stipulations in force. The interesting point is that we are spared the complexities of a condition like (\Rightarrow N3) —essentially because of the phenomenon of persistence.

Theorem 4.1 *The consequence relation \vdash here defined is determined by the class of all models.*

Proof: The soundness part of the claim is left to the reader to verify. The completeness part requires a variation on the canonical model method used for minimal, intuitionistic, and intermediate logics in Segerberg [30]. We specify the canonical model $\mathcal{M}_\vdash = \langle W_\vdash, R_\vdash, V_\vdash \rangle$ by putting into W_\vdash , all those consistent \vdash -theories with the special property that for all formulas A, B , whenever $A \Rightarrow B$ is in the theory, then so is either $B \Rightarrow A$ or else the formula B itself. (Whenever $\Gamma \not\vdash C$, the set Γ can be extended to a consistent \vdash -theory not containing C but possessing this special property, in virtue of \vdash 's satisfying the condition (IL \Rightarrow 2).) R_\vdash is the relation \subseteq on these sets of formulas and V_\vdash is defined as usual in terms of the presence or absence of the p_i in the sets concerned. As usual, we have to show that truth at a point in this canonical model coincides with membership in that point (considered as a set of formulas), and the crucial case is the inductive case for \Rightarrow :

$A \Rightarrow B \in u$ iff either for all v such that $R_\vdash uv$, we have $A \in v$ iff $B \in v$,
or else $B \in u$.

We show the “if” direction first. If $B \in u$, we have $A \Rightarrow B \in u$, by (IL \Rightarrow 4). So suppose that for all $v \in W_\vdash$, with $v \supseteq u$, we have $A \in v$ if and only if $B \in v$. This means that some formulas in u taken together with A have B as a consequence (by the relation \vdash) and that some formulas in u taken together with B have A as a consequence, since otherwise we could find the least $v \in W_\vdash$ extending $u \cup \{A\}$ to get an element of the canonical model containing A but not B , or else the least $v \in W_\vdash$ extending $u \cup \{B\}$ to arrive at an element containing B but not A . Taking Γ as the union of the supplementary sets of formulas from u for these consequences, (IL \Rightarrow 3) gives $A \Rightarrow B \in u$.

For the “only if” direction, suppose that $A \Rightarrow B \in u$ while $B \notin u$, with a view to showing that for all $v \in W_\vdash$ with $v \supseteq u$, $A \in v$ if and only if $B \in v$. By the above

“special property” of the \vdash -theories making up W_{\vdash} , the supposition gives $B \Rightarrow A \in u$. Thus for any $v \supseteq u$ we have both $A \Rightarrow B \in v$ and $B \Rightarrow A \in v$, so (IL \Rightarrow 1) gives $A \in v$ if and only if $B \in v$, completing the proof. \square

5. The Two Implications Together

Returning to our modal investigation of Spinks’s connectives \rightarrow and \Rightarrow , whose separate behavior was considered in Section 2 and Section 3, respectively, we come now to the question of their interaction. We need to isolate the consequence relation on the language with these two binary connectives, determined by the class of all equivalence-relational models when truth in a model is treated for \rightarrow -compounds as in Section 2 and for \Rightarrow -compounds as in Section 3, with a single equivalence relation R for both cases. The most elegant and informative route to such a result would no doubt begin by examining the basic logic on this mixed language—the consequence relation determined by the class of all models (with a single accessibility relation, concerning which no further assumptions are made), showing completeness by a canonical model argument in which the separate \rightarrow -based and \Rightarrow -based definitions of R_{\vdash} are shown to define the same relation. It would then proceed to monitor the effect of further conditions on the accessibility relation until the case in which we are ultimately interested is reached—the case of accessibility as an equivalence relation. However, in view of the difficulties over \Rightarrow in Section 3, it is not clear how to implement this ideal strategy, and we accordingly settle for something less: we pass straight to the case of equivalence-relational models and use a canonical model argument in which the accessibility relation is given the \rightarrow -based definition (using the function *nec* introduced in Section 2) showing that this still allows the fundamental equation of truth and membership to be derivable for \Rightarrow -compounds.

We list the conditions we shall need to consider, most of them having already been encountered in earlier sections, though now they are to be taken as conditions on consequence relations \vdash on the language with both \rightarrow and \Rightarrow as connectives. As usual, we understand these as conditions to be satisfied for all formulas and sets of formulas as indicated by the schematic letters; in addition, we have abbreviated (\rightarrow N2) from Section 2 in accordance with the multiple-succedent convention of Section 3:

- (\rightarrow N1) $A_1, \dots, A_n \vdash B$ implies $B \rightarrow C \vdash A_1 \rightarrow (A_2 \rightarrow \dots \rightarrow (A_n \rightarrow C) \dots)$
- (\rightarrow N2) $A \rightarrow B \vdash B, A \rightarrow C$
- (\rightarrow N3) $B \vdash A \rightarrow B$
- (\rightarrow T) $\vdash A \rightarrow A$
- (\rightarrow 5) $A \rightarrow B, (A \rightarrow C) \rightarrow B \vdash B$
- (\Rightarrow N1) $A \Rightarrow B \vdash B, B \Rightarrow A$
- (\Rightarrow N2) $B \vdash A \Rightarrow B$
- (\Rightarrow T) $A, A \Rightarrow B \vdash B$
- (\Rightarrow \rightarrow 1) $A \Rightarrow B, (A \Rightarrow B) \rightarrow C \vdash B, C$
- (\Rightarrow \rightarrow 2) $C_1, \dots, C_n, A \vdash B$ and $C_1, \dots, C_n, B \vdash A$ imply $\vdash A \Rightarrow B, C_1 \rightarrow D_1, \dots, C_n \rightarrow D_n$

Let \vdash_* be the least consequence relation satisfying the above ten conditions. Then we have the following theorem.

Theorem 5.1 \vdash_* is determined by the class of all models $\langle W, R, V \rangle$ in which R is an equivalence relation, when truth is defined by the semantical clause for \rightarrow given in Section 2 and by the clause for \Rightarrow given in Section 3.

Proof: The soundness half is left to the reader. For completeness we construct the canonical model, which we shall call—to avoid a proliferation of subscripts— $\langle W_*, R_*, V_* \rangle$, in which W_* comprises all ma-sets of formulas (relative to \vdash_*), R_* is defined as in Section 2: R_*uv if and only if $nec(u) \subseteq v$, where nec is defined as in that section in terms of \rightarrow -formulas, and $V_*(p_i) = \{u \in W_* \mid p_i \in u\}$. As usual we want to check that for any formula C , we have, for all $u \in W_*$, C is true at u in this model if and only if $C \in u$, and the inductive part of this verification required that we check, on the assumption that this holds for A and B , that we have (1) and (2):

- (1) $A \rightarrow B \in u$ iff, if $A \in v$ for each v such that R_*uv , then $B \in u$;
- (2) $A \Rightarrow B \in u$ iff either for all v such that R_*uv , we have $A \in v$ iff $B \in v$, or else $B \in u$.

Now, since \vdash_* satisfies conditions $(\rightarrow N1)–(\rightarrow N3)$, we have (1) by the proof of Theorem 2.4, so this leaves only (2) to check. For the “only if” direction of (2), suppose that $A \Rightarrow B \in u$ and $B \notin u$. We must show that for any $v \in W_*$, R_*uv implies $A \in v$ if and only if $B \in v$. So take such a v , that is, a v for which $nec(u) \subseteq v$. We first show that $A \Rightarrow B \in nec(u)$. This means that for any C with $(A \Rightarrow B) \rightarrow C \in u$, we have $C \in u$. But this follows by $(\Rightarrow \rightarrow 1)$ and the fact that $A \Rightarrow B \in u$ while $B \notin u$. (Here we have used Proposition 2.13.) Since $A \Rightarrow B \in nec(u)$ and $nec(u) \subseteq v$, we have $A \Rightarrow B \in v$. That $A \in v$ implies $B \in v$ follows by $(\Rightarrow T)$. To see that we also have the converse implication, recall that $A \Rightarrow B \in u$ and $B \notin u$, so by $(\Rightarrow N1)$ we have $B \Rightarrow A \in u$, and thus by the previous argument, interchanging A and B , we have $B \Rightarrow A \in nec(u)$, and thus $B \Rightarrow A \in v$; therefore appealing again to $(\Rightarrow T)$, if $B \in v$ then $A \in v$.

For the “if” direction of (2), we get from $B \in u$ to $A \Rightarrow B \in u$ by $(\Rightarrow N2)$; to show that if for all v such that R_*uv , we have $A \in v$ if and only if $B \in v$, then $A \Rightarrow B \in u$, suppose $A \Rightarrow B \notin u$ in the hope of finding $v \in W_*$, with $nec(u) \subseteq v$ and A, B , differing in respect of membership in v . If $nec(u) \cup \{A\} \not\subseteq B$ or $nec(u) \cup \{B\} \not\subseteq A$ we get the desired v as a superset of $nec(u) \cup \{A\}$ maximally avoiding B or as a superset of $nec(u) \cup \{B\}$ maximally avoiding A , since in either case A and B differ in membership of the ma-set in question. So we are only in trouble if $nec(u) \cup \{A\} \vdash B$ and $nec(u) \cup \{B\} \vdash A$, in which case there are C_1, \dots, C_n , with each $C_i \in nec(u)$ and

$$C_1, \dots, C_n, A \vdash B \text{ as well as } C_1, \dots, C_n, B \vdash A.$$

For each of these C_i , let D_i be such that $C_i \rightarrow D_i \notin u$. (These D_i can be found, by Proposition 2.6, since each $C_i \in nec(u)$.) From the inset \vdash -statements here it follows by $(\Rightarrow \rightarrow 2)$ that

$$\vdash A \Rightarrow B, C_1 \rightarrow D_1, \dots, C_n \rightarrow D_n,$$

so by Proposition 2.13 one of the formulas on the right belongs to u . Since none of the $C_i \rightarrow D_i$ is in u this leaves only $A \Rightarrow B$. But our starting assumption was that $A \Rightarrow B \notin u$. This contradiction shows that the anticipated trouble cannot after all arise, completing the proof. \square

By considering generated submodels in the usual way (cf. 2.12), we obtain Corollary 5.2 below; to tie matters up with our discussion in Section 1, we include Corollary 5.3.

Corollary 5.2 \vdash_* is determined by the class of all models $\langle W, R, V \rangle$ in which $R = W \times W$.

Corollary 5.3 For any formula A of the present language, we have $\vdash_* A$ if and only if A is Spinks-valid.

Proof: By Corollary 5.2, $\vdash_* A$ just in case A is true throughout every universal model, which is equivalent—cf. our discussion of Henle matrices in Section 1—to the claim that the \Box -translation $\tau(A)$ is valid in every Henle matrix and hence by Proposition 1.10 to the claim that A is Spinks-valid. \square

6. Rules and Consequence Relations

Scott ([28], p. 148) lists in a table labeled “Four forms of modus ponens” the following conditions on a consequence relation \vdash and a binary connective (for which he writes ‘ \Rightarrow ’, but which we change to ‘ \gg ’ to avoid confusion with our own—that is, Spinks’s—‘ \Rightarrow ’ notation):

$A, A \gg B \vdash B$	$\frac{\vdash A \gg B}{A \vdash B}$	$\frac{\vdash A}{\vdash A \gg B}$	$\frac{\vdash A}{\underline{A \vdash B}}$
(i)	(ii)	(iii)	(iv)

Although he uses the “rule notation” of a line separating premises from conclusion, Scott’s discussion makes it clear that he is simply considering four metatheoretical statements to the effect that, in the case of (ii), for instance, whenever we have $\vdash A \gg B$ for formulas A, B , we also have $A \vdash B$. Each statement implies that to its right (for arbitrary but fixed \vdash and \gg , of course). As he points out, modus ponens is traditionally taken as a principle concerning an implicational connective (schematically indicated here by ‘ \gg ’) so (iv), which is just a general feature of consequence relations having nothing to do with any particular connectives, is hardly a good thing to mean by modus ponens. Scott has a preferred candidate, from among the survivors, (i)–(iii), for this role, answering the question as to which of them deserves the name modus ponens:

The correct answer seems to be (iii), for this is the metatheoretic statement that the validities of the system are closed under the rule allowing for the detachment of the conclusion of an implication (provided it and the antecedent are valid). ([28], p. 148)

The merits of this proposal are of less interest to us here than Scott’s subsequent demonstration of the nonequivalence of candidates (ii) and (iii), since to the latter end if one took \vdash as the consequence relation on the usual \Box -based language of modal logic obtained by putting $\Gamma \vdash A$ just in case for all models \mathcal{M} with a transitive and reflexive accessibility relations we have $\Gamma \Vdash_{\mathcal{M}} A$. (The reference to transitivity is not actually relevant, but we are here mirroring Scott’s discussion, in which it is clear that in alluding, as he does on p. 149, to the modal logic **S4**, it is the consequence relation just specified he has in mind as an illustration.) To see that (iii) does not

imply (ii), Scott draws our attention to the connective \gg (our notation) defined by: $A \gg B = \Box A \supset B$. This is, of course, none other than the ‘ \rightarrow ’ of our own discussion. Scott has overlooked a “fifth form of modus ponens” and for this particular choice of \gg , (iii) is satisfied precisely because this missing form—here given as (ii)*—is satisfied:

$$\frac{\vdash A}{A \gg B \vdash B}$$

(ii)*

Note that (ii)*, with \gg as \rightarrow , is the $n = 0$ case of the condition (\rightarrow N1) from Section 2, and that in general, while neither (ii) nor (ii)* implies the other, (ii)* is, like (ii) itself, weaker than (i) and stronger than (iii). Further, taking \vdash as our consequence relation \vdash_* , whereas (ii)* and so (iii) (and of course (iv)) holds for \rightarrow , for \Rightarrow we have the strongest form of all, namely, (i)—alias (\Rightarrow T).

The range of conditions we have been considering here show, incidentally, that it is an oversimplification to think that transitions from a set of formulas (here comprising A and $A \gg B$ for a given A and B) to a formula (here B) can be exhaustively divided—taking this terminology from [28]—into the “horizontal” transitions such as (i) and the “vertical” transitions such as (iii), since this plainly leaves out the mixed intermediate cases (ii) and (ii)*. The same holds for other versions of this binary distinction, such as that of Smiley ([31], p. 114) between rules of *inference*—licensing the horizontal transitions, and rules of *proof*—licensing the vertical transitions. (In the terminology of Gabbay (Gabbay [13], p. 9) this is the distinction between “consequence rules” and “provability rules”.) For one attempt at providing what amounts to (when applied in the case of **S5**) a unified framework in which transitions in which some (formula-)premises behave like rule-of-inference premises and others like rule-of-proof premises, see Blamey and Humberstone [2].

Returning to modus ponens, we recall that for a given \vdash and \gg , (i) above is equivalent to the “only if” (Detachment) half of the Detachment Deduction Theorem—that is, the claim that for all sets Γ of formulas and all formulas A, B , we have

$$\text{(DDT)} \quad \Gamma, A \vdash B \text{ if and only if } \Gamma \vdash A \gg B.$$

Taking \vdash as \vdash_* , we have the forward (only if) direction of (DDT) for \gg as \Rightarrow but not for \gg as \rightarrow , while for this same consequence relation, the backward (if) direction of (DDT) holds for \gg as \rightarrow , but not for \gg as \Rightarrow . (See the discussion of (\rightarrow CP) and (\Rightarrow T) preceding Theorem 3.4.) It can be shown in fact that for no binary \gg definable in terms of \rightarrow and \Rightarrow does (DDT) hold, still taking \vdash as \vdash_* ; the argument we know of for this conclusion relies on Spinks’s characterization of the free implicative *BCSK*-algebra on two free generators and will not be given here. (This algebra has fourteen elements, which, viewed as 2-ary polynomials, correspond to the various definable connectives to serve as \gg , from which an easy examination of cases rules out any candidate satisfying (DDT) for \vdash_* .)

The standard proof of the Deduction Theorem is available for the consequence relation \vdash_{MP} introduced for Theorem 1.12 in terms of Spinks’s axioms and the rule of modus ponens; in the current terminology \vdash_{MP} is the least \vdash satisfying (i) above for \gg as \rightarrow and such that $\vdash A$ for any formula A instantiating one of Spinks’s schemata (A1)–(A9) given in Section 1. Thus we have (DDT) for \gg as \rightarrow and \vdash as \vdash_{MP} . This syntactic characterization of \vdash_{MP} is cumbersome to work with, however. We

do have a characterization in terms of Spinks matrices (Corollary 1.14) and we shall close by viewing this from the model-theoretic perspective of our discussion. We could translate (1.14) directly into these terms, *via* the connections between Spinks matrices and Henle matrices and between the latter and universal Kripke models, as in the proof of Corollary 5.3, but prefer a slightly different route emphasizing some aspects of the situation—in particular in respect to the differing preservation-characteristics of rules—touched on above.

Let us consider the failure of modus ponens in the form (i) for \vdash as \vdash_* and \gg as \rightarrow , replacing the exhibited occurrence of \rightarrow by its \Box -translation (cf. the use of the example by Scott, cited above). We do not in general have, as truth-preserving at a point in a Kripke model, the transition from A and $\Box A \supset B$ to B , because although we can pass in such a “locally” truth-preserving manner from $\Box A$ and $\Box A \supset B$ to B , to make this transition we should first have to pass from A to $\Box A$ (Necessitation), a transition which preserves truth-throughout-a-model but not truth-at-an arbitrarily-selected-point-in-a-model. So the problem arises because of smuggling in applications of a rule of proof as what should be rules of inference, to use the terminology of Smiley mentioned above. Accordingly let us distinguish the following two notions of consequence, defined in terms of a class \mathbb{C} of Kripke models:

1. The formula A is a *local \mathbb{C} -consequence* of the set of formulas Γ just in case for each $\mathcal{M} \in \mathbb{C}$, where $\mathcal{M} = \langle W, R, V \rangle$, whenever $u \in W$ and $\mathcal{M} \models_u C$ for all $C \in \Gamma$, we have $\mathcal{M} \models_u A$.
2. A is a *global \mathbb{C} -consequence* of Γ just in case for each $\mathcal{M} \in \mathbb{C}$, where $\mathcal{M} = \langle W, R, V \rangle$, whenever $\mathcal{M} \models_u C$ for all $C \in \Gamma$ and all $u \in W$, we have $\mathcal{M} \models_u A$ for all $u \in W$.

This use of the local/global terminology is far from new: it can be found in Fitting [12] or van Benthem [34], for instance. (Compare also Humberstone [20].) We would certainly not want to claim it as the all-purpose semantic embodiment of Smiley’s distinction between rules of proof and rules of inference, since rules of proof sometimes used which are not rules of inference—one thinks principally of the rule of Uniform Substitution here—do not deliver as conclusions global $\{\mathcal{M}\}$ -consequences of their premises. (A fuller discussion would require consideration of the analogous “frame-based” local/global contrast—as in [34], Definition 2.32—but this is not needed for the present application.) The above definitions are particularly useful for making manifest the following two facts: (1) the local \mathbb{C} -consequences (for any \mathbb{C}) of a set of formulas are always among its global \mathbb{C} -consequences, but not in general conversely, though (2) for a fixed \mathbb{C} , the local \mathbb{C} -consequences and the global \mathbb{C} -consequences of the empty set coincide. Given Theorem 6.1 below, (1) explains why $\vdash_* \subseteq \vdash_{\text{MP}}$, and (2) why $\vdash_* A$ if and only if $\vdash_{\text{MP}} A$.

The consequence relation *locally determined by \mathbb{C}* is that relation \vdash for which (for all Γ, A) $\Gamma \vdash A$ if and only if A is a local \mathbb{C} -consequence of Γ . This, of course, is simply what in Section 2 (and beyond) we called the consequence relation determined by \mathbb{C} , since local \mathbb{C} -consequence is just the intersection, for $\mathcal{M} \in \mathbb{C}$, of the relations $\Vdash_{\mathcal{M}}$ there defined. We rebaptize it here to emphasize the contrast with the following. The consequence relation *globally determined by \mathbb{C}* is that relation \vdash for which $\Gamma \vdash A$ if and only if A is a global \mathbb{C} -consequence of Γ . We are now in a position to compare \vdash_* with \vdash_{MP} :

Theorem 6.1 (i) \vdash_* is the consequence relation locally determined by the class of all universal models. (ii) \vdash_{MP} is the consequence relation globally determined by the class of all universal models.

Proof: (i) is just a restatement in the present terminology of Corollary 5.2. For (ii), we have to show that $A_1, \dots, A_n, \vdash_{MP} B$ if and only if B is a global \mathbb{C} -consequence of $\{A_1, \dots, A_n\}$ where \mathbb{C} is the class of models with universal accessibility relations. But, appropriately instantiating the (DDT) schema above, the former amounts to $\vdash_{MP} A_1 \rightarrow (A_2 \rightarrow \dots \rightarrow (A_n \rightarrow B) \dots)$, which by (1.13) and (5.3) is equivalent to the same with the ‘MP’ subscript replaced by ‘*’. This, by Corollary 5.2, means that for any point u in a universal model throughout which model all of the A are true, we have B true at u , which is equivalent to saying that B is true throughout any universal model throughout which all the A_i are true: that is, to B ’s being a global \mathbb{C} -consequence of $\{A_1, \dots, A_n\}$ for the current choice of \mathbb{C} . \square

As has already been mentioned, an alternative route to Theorem 6.1 could be provided with the aid of Henle matrices, though we should have to allow (generalizing our official definition in Section 1—though cf. the reference there to [29]—such matrices to have designated values other than simply the Boolean top element 1, since the consequence relation determined by these restricted matrices amounts to that *globally* determined by the class of universal Kripke models (1 corresponding to the set W in such a model). It has not seemed worthwhile for our limited purposes here to introduce such variations—with designated elements as arbitrary principal filters of Henle algebras (thus encoding truth at the point whose unit set generates the filter)—or analogous variations on the concept of a Spinks matrix.

Appendix A. Two Translational Embeddings

We can separate out the treatment of the connectives \Rightarrow and \rightarrow provided by the translation τ of Section 1, to give two translational embeddings of the implicational fragment of classical propositional logic into (propositional) **S5**. Such embeddings may not seem to have the informativeness of, for example, the well-known translations of intuitionistic propositional logic into **S4** ([6], §3.9), since here already the identity translation (the inclusion map) is faithful—where faithfulness is a matter of the “if” half of the claim that a formula is provable in the source logic if and only if its translation is provable in the target logic. However, some interest must attach to the existence within the more inclusive system of an exact replica (assuming faithfulness) of the smaller system constructed out of formulas of a specified form provided by the translation. At any rate, embeddings with this property have attracted a certain amount of attention: witness Fitting [11], Czermak [8], [9], in which papers the target logic is (quantified) **S4** and the source logic is (the whole of) classical predicate logic. Since the translations we consider are related to τ , we shall call them ρ and σ . In each case the source (domain) is the language whose only connective is \supset and the target (codomain) is the full language of (propositional) modal logic with \square as the non-Boolean primitive. We define ρ and σ by imitating the behavior of τ (as in (3) from Section 1) on \Rightarrow -formulas and \rightarrow -formulas respectively, but applying them to \supset -formulas instead:

$$\begin{aligned}
\rho(p_i) &= \sigma(p_i) = p_i \\
\rho(A \supset B) &= \Box(\rho(A) \equiv \rho(B)) \vee \rho(B) \\
\sigma(A \supset B) &= (\Box\sigma(A)) \supset \sigma(B)
\end{aligned}$$

Observation A.1 For any formula A built up using only the connective \supset :

- (i) A is a classical tautology if and only if $\rho(A)$ is a theorem of **S5**.
- (ii) A is a classical tautology if and only if $\sigma(A)$ is a theorem of **S5**.

Proof: (i) A formula $A = A[\supset]$ is a classical tautology if and only if $A[\Rightarrow]$ is Spinks-valid, by Theorem 1.7(ii), which by Proposition 1.10 holds if and only if $\tau(A[\Rightarrow])$ is **S5**-provable. But $\tau(A[\Rightarrow])$ is the formula $\rho(A)$. The case of part (ii) is similar. \square

The author has not seen anything resembling part (i) in the literature, but something close to the “only if” direction of part (ii) may be found in Section 3, written by Meredith of Lemmon et al. [23]. Meredith considers a variant on σ above, which we shall call σ' , defined like σ except that we add an initial \Box : $\sigma'(A \supset B) = \Box((\Box\sigma'(A)) \supset \sigma'(B))$, and he shows that if a purely implicational formula A is a classical tautology then $\sigma'(A)$ is **S5**-provable. A consideration of the one-element Kripke models—in which any formula and its σ' -translation (or indeed its ρ - or σ -translation) are equivalent—reveals that conversely, $\sigma'(A)$ is only **S5**-provable if A is a classical tautology, so we have here a faithful embedding and the above point about the fragment of the target logic applies. In the case of σ' the interest lies in the fact that formulas of the form $\sigma'(A)$ all lie (to within equivalence) in the “strict implication” fragment of **S5**: so there is an exact replica, within this fragment, of the \supset -fragment of classical propositional logic. Our interest in σ is due to a similar replication being revealed when instead of the strict implicational fragment, the less familiar “ \rightarrow fragment”, as we might call it, is considered. (It should be added that Meredith also shows that his σ' embeds the implicational fragment of intuitionistic logic—faithfully, we again add—in the strict implication fragment of **S4**.)

Observation A.1 above rather crudely conceives of logics as nothing other than certain sets of formulas, and we leave the interested reader to consider what becomes of our translations in the context of logics as consequence relations. A translation would be a faithful embedding in this context when A is a source-consequence of Γ just in case the translation of A is a target-consequence of the translation of Γ (i.e., the set of translations of elements of Γ). While it is clear what the source consequence relation should be for such a generalization of the above result, the reader is advised to consult Section 6 on the choice between two candidates for the role of target consequence relation—the local and the global consequence relation determined by the class of models for **S5**. Certainly, with the local consequence relation in mind, σ does not fare very well, since although (for example) p_2 is a tautological consequence of p_1 and $p_1 \supset p_2$, it is not a local **S5**-consequence of p_1 and $\Box p_1 \supset p_2$.

Appendix B. The Varieties Considered by Spinks

The following definitions, from Spinks [32], of some classes of algebras which figured in our discussion in Section 1 are included here to keep the present paper self-contained. (For background on *BCK*-algebras *simpliciter*, see Section 5.2.3 of [3]. Spinks does not work with a notion of *BCS*-algebra *simpliciter*, introducing only the

terminology of implicative *BCS*-algebras; we use his terminology here for convenience. Note that while the class of *BCK*-algebras is a proper quasi variety, the three classes of algebras introduced all comprise varieties.)

An *implicative BCK-algebra* is an algebra $(A, \Rightarrow, 1)$ of similarity type $\langle 2, 0 \rangle$ satisfying the following identities (in which we write the variables x_1, x_2, x_3 of Section 1 as x, y, z):

$$\begin{aligned} x \Rightarrow x &\approx 1 \\ (x \Rightarrow y) \Rightarrow x &\approx x \\ (x \Rightarrow y) \Rightarrow y &\approx (y \Rightarrow x) \Rightarrow x \\ x \Rightarrow (y \Rightarrow z) &\approx y \Rightarrow (x \Rightarrow z) \end{aligned}$$

An *implicative BCS-algebra* is an algebra $(A, \rightarrow, 1)$ of the same similarity type, satisfying

$$\begin{aligned} x \rightarrow x &\approx 1 \\ (x \rightarrow y) \rightarrow x &\approx x \\ x \rightarrow (y \rightarrow z) &\approx (x \rightarrow y) \rightarrow (x \rightarrow z) \\ x \rightarrow (y \rightarrow z) &\approx y \rightarrow (x \rightarrow z) \end{aligned}$$

The third of these conditions is a two-way analogue of the propositional schema called in the combinator-derived parlance of *B, C, K*, and so on (on which we have already recommended [4]), *S*:

$$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$$

which explains Spinks's choice of the terminology "*BCS*-algebra".

Of course, the mere notational difference between ' \Rightarrow ' and ' \rightarrow ' marks no contrast in itself, and we use the two notations only to facilitate the definition of implicative *BCSK*-algebras (see below); setting it to one side we can say (again following Spinks) that every implicative *BCK*-algebra is an implicative *BCS*-algebra, though not conversely; the converse does, however, hold if we replace the reference to every implicative *BCS*-algebra by one to every implicative *BCS*-algebra satisfying the identity

$$(x \rightarrow y) \rightarrow y \approx (y \rightarrow x) \rightarrow x.$$

(Compare in this connection the formula of Example 1.8.) This is sometimes referred to as the commutative identity, since it makes the operation, \vee (say) defined by letting $a \vee b$ be $(a \rightarrow b) \rightarrow b$, commutative. *BCK*-algebras satisfying it are usually called commutative, though we prefer to avoid such terminology for an algebra whose sole fundamental binary operation is not itself commutative, preferring to say instead "quasi-commutative"—following [1], as noted in Section 2. Unfortunately this term has itself acquired a different meaning in the *BCK*-algebraic literature, following its introduction by Yutani [39]. As an alternative, one might consider saying "join-commutative", since the above identity makes the above \vee a least upper bound operation for the *BCK* partial order mentioned in the discussion after Corollary 1.14. There is even a complication with this proposal, though, since much of the *BCK*-algebraic literature is conducted in a notation dual to that employed here (and in most discussions on *BCK* logic and its extensions), in which what we write as $x \rightarrow y$ is written as $y * x$ or yx , and 1 is replaced by 0, the effect of which is to turn the \leq

defined earlier into \geq and joins into meets! (This is because the fundamental operation of a *BCK*-algebra is thought of as a generalization of subtraction rather than of implication.)

As remarked in Section 1, the word “implicative” in these cases is standard *BCK*-algebraic terminology intended to suggest the algebraic analogue not just of any implicational connective but specifically of *classical* implication. More precisely, it is the second identity—an equational form of Peirce’s Law—in each of the above two lists that is signaled by use of the term “implicative” in this context.

An *implicative BCSK-algebra* is an algebra $(A, \Rightarrow, \rightarrow, 1)$ of type $(2, 2, 0)$ whose reducts $(A, \Rightarrow, 1)$ and $(A, \rightarrow, 1)$ are, respectively, an implicative *BCK*-algebra and an implicative *BCS*-algebra, and satisfying the further identities

$$x \Rightarrow (y \rightarrow x) \approx 1 \quad \text{and} \quad ((y \Rightarrow x) \rightarrow x) \rightarrow x \approx y \Rightarrow x.$$

Satisfaction of these two additional equations is equivalent to saying that the natural *BCK* and *BCS* partial orders (as explained in Section 1, following Corollary 1.14) coincide.

Appendix C. Anomalies à la Section 3 in a simpler setting

Letting \top be the formula $p_1 \Rightarrow p_1$, then for any A the formula $\top \Rightarrow A$ is true at a point in a model (using the semantics of Section 3) just in case either A is true at all points accessible to that point or else A is true at that point itself, amounting to what in the familiar \Box -based language would be written as the disjunction $\Box A \vee A$. We could equally well have used the formula $(A \Rightarrow A) \Rightarrow A$ for this purpose. But now let us take the singularly connective which delivers from A a formula with these truth-conditions as a new primitive connective in its own right and consider the language in which there are no other connectives. We write this connective as Ω ; thus for a model $\mathcal{M} = \langle W, R, V \rangle$, we stipulate that

$$\begin{aligned} \mathcal{M} \models_u p_i & \text{ iff } u \in V(p_i); \\ \mathcal{M} \models_u \Omega B & \text{ iff either for all } v \text{ such that } Ruv, \text{ we have } \mathcal{M} \models_v B, \text{ or else} \\ & \mathcal{M} \models_u B. \end{aligned}$$

It is easy to see that a consequence relation determined by the class of all models may be characterized syntactically as the least consequence relation on this language satisfying for all formulas

$$\begin{aligned} (\Omega N1) \quad & B_1, \dots, B_n, \vdash A \text{ implies } \Omega B_1, \dots, \Omega B_n, \vdash \Omega A, B_1, \dots, B_n; \\ (\Omega N2) \quad & A \vdash \Omega A. \end{aligned}$$

The soundness part of this claim of determination is left to the reader while for completeness we use the canonical model $\mathcal{M}_\vdash = \langle W_\vdash, R_\vdash, V_\vdash \rangle$ with W_\vdash as the set of all \vdash -theories and $R_\vdash uv$ if and only if for all A such that $\Omega A \in u$ and $A \notin u$, we have $A \in v$. (V_\vdash is defined as usual.) In the inductive proof that truth and membership coincide, this leaves us with the following to show.

$$\Omega A \in u \text{ iff either for all } v \text{ such that } R_\vdash uv, \text{ we have } A \in v, \text{ or else } A \in u.$$

The “only if” is an immediate consequence of the way R_\vdash was defined. For the “if” part, we have $\Omega A \in u$ on the supposition that $A \in u$, by $(\Omega N2)$, while, finally, to show that $\Omega A \in u$ if for all v such that $R_\vdash uv$, we have $A \in v$, suppose $\Omega A \notin u$ with a view to finding $v \in W_\vdash$ with $R_\vdash uv$ and $A \notin v$. We get the desired v from the fact that $\{B \mid \Omega B \in u \ \& \ B \notin u\} \not\vdash A$ by appeal to $(\Omega N1)$ and the fact that $\Omega A \notin u$.

Some of the key features of the situation with ‘ \Rightarrow ’ in Section 3 are reproduced here, though the above completeness proof was considerably simpler than Theorem 3.2. We have the hidden disjunctive form (here $\Box A \vee A$, there $\Box(A \equiv B) \vee A$) wrapped up in our sole primitive, along with the attendant lack of generality: no way of expressing $\Box A \vee B$ (as there we had no way of expressing $\Box(A \equiv B) \vee C$). As to whether the feature we are about to draw attention to in the case of the Ω -language is exactly parallel to what caused our difficulties—for example, over the completeness proof for the $(\Rightarrow 5)$ case, in which we were unable to show the canonical accessibility relation to be euclidean with the aid of $(\Rightarrow 5)$ —we are not entirely clear. But we certainly have here a striking and unusual feature which is caused by the “special case” nature of the Ω -compounds and which we can bring out by making two observations.

Observation C.1 The smallest consequence relation on the present language satisfying $(\Omega N1)$, $(\Omega N2)$, and the converse of $(\Omega N2)$ is determined by the class of all models whose accessibility relations are reflexive.

We should note before proceeding further that the characterization of the consequence relation just given involved redundancy, since $(\Omega N1)$ follows from $(\Omega N2)$ its converse, taken together. Indeed, the logic we are now considering is a renotating, though without the usual supply of Boolean connectives, of what is sometimes called the *Trivial* system in \Box -based modal logic, determined by the class of models in which each point is accessible to itself and only itself (alternatively: by the class of one-point models with a reflexive accessibility relation). Since we have a different truth-definition in mind, as given above, it would be confusing to use the \Box notation here, however. (The present line of exploration is, however, very much in the spirit of the remark of van Benthem [35], p. 179: “The Kripke truth definition is not sacrosanct.” Van Benthem goes on to consider a semantical clause for \Box which has $\Box A$ true at u if and only if for all $v \in W$, if either Ruv or Rvu , then A is true at v , and he remarks that the basic logic for this semantics—that determined by the class of all models, with this truth-definition in force—is **KB**.)

The soundness claim built into Observation C.1 is easily verified, and for completeness we check that the canonical accessibility relation, defined as above— $R_{\vdash}uv$ if and only if for all A such that $\Omega A \in u$ and $A \notin u$, we have $A \in v$ —is reflexive, that is, that for all $u \in W_{\vdash}$ (where, of course, we take \vdash as the consequence relation mentioned in Observation C.1) and all formulas A , if $\Omega A \in u$ and $A \notin u$, we have $A \in u$. But this is evidently equivalent to the result of deleting the ‘ $A \notin u$ ’ antecedent since this is the negation of the consequent, leaving as all that has to be shown for reflexivity that $\Omega A \in u$ implies $A \in u$, an immediate consequence of the converse of $(\Omega N2)$.

Observation C.1 may not seem very surprising. But here is a different completeness result for the same logic.

Observation C.2 The smallest consequence relation on the present language satisfying $(\Omega N1)$, $(\Omega N2)$, and the converse of $(\Omega N2)$ is determined by the class of all models whose accessibility relations are universal.

Soundness is clear from Observation C.1, since universality implies reflexivity. For completeness, we just have to look more closely at the previous argument that $R_{\vdash}uv$ for $u, v \in W_{\vdash}$, when $u = v$, to notice that the latter identity is not actually exploited. For suppose that not $R_{\vdash}uv$; thus for some formula A such that $\Omega A \in u$ and $A \notin u$,

we have $A \notin v$. But this is impossible, since by the converse of ($\Omega N2$), we cannot have $\Omega A \in u$ and $A \notin u$ to begin with—never mind what is happening in respect of v . Note that the canonical model method by itself for \Box -based modal logic (with the standard truth-definition) does not yield this result for **S5**, since the canonical model is an equivalence relational model with many equivalence classes, and we have to pass to generated submodels to get the accessibility relations to be universal (see the proof of Corollary 2.12). By contrast, here, as the above argument shows the canonical accessibility relation is already universal in its own right.

Whether the example discussed here throws any light on the case of ‘ \Rightarrow ’ in Section 3 remains to be seen. Either way, it seems not without interest in its own right.

Notes

1. A referee has asked for some light to be shed on the somewhat surprising appearance of the ‘C’ in ($\rightarrow N2$). Let us recall that a standard Gentzen-style sequent-calculus for inserting an implication, \supset , say, on the left would take us from premise-sequents (which can here allow ourselves to confuse with \vdash -statements) (1) $\Gamma \vdash A$, and (2) $\Gamma, B \vdash D$, to the conclusion (3) $\Gamma, A \supset B \vdash D$. We want to alter this to give a similar rule for \rightarrow , with $A \rightarrow B$ amounting to $\Box A \supset B$. This suggests working with something along the lines of (1') $\Gamma \vdash \Box A$ and (2) as above, to the conclusion (3') $\Gamma, A \rightarrow B \vdash D$. Unfortunately no formula in the pure \rightarrow -language is available (as we show in Proposition 2.7) to do the work of $\Box A$ for an arbitrary formula A . If we had negation available we could negate this ‘ $\Box A$ ’ and place it on the left, filling the gap remaining by the succedent formula of (2) and (3'), represented by D . Again, we cannot write anything in the present language which is equivalent to $\neg \Box A$. (Proposition 2.7 does not itself yield this result, but the proof we give of Proposition 2.7 does deliver the result.) But we can write something which *follows from* $\neg \Box A$, namely, $\neg \Box A \vee C$ (for any formula C), which we are writing as $A \rightarrow C$. Thus replacing $\neg \Box A$ on the left by $A \rightarrow C$ gives us a premise, (1'') $\Gamma, A \rightarrow C \vdash D$ which is stronger than the unavailable $\Gamma, \neg \Box A \vdash D$. Thus the rule we end up with—whose admissibility is what condition ($\rightarrow N2$) requires—having this stronger premise (1'') alongside (2), and with conclusion (3'), is *weaker* than the (unavailable) rule which has, in place of $A \rightarrow C$ in (1''), simply $\neg \Box A$. Thus there should be no surprise that the mysterious appearance of the otherwise absent schematic letter ‘C’ does not cause trouble by being too strong. Is it strong enough (alongside ($\rightarrow N1$) and ($\rightarrow N3$)), however? Here we simply point ahead to what we shall be calling the special ($\rightarrow N2$)-secured property and the role this plays in the completeness proof below (see the appeal to Lemma 2.2(ii) in the proof of Theorem 2.4).
2. We could equally well let the elements of the canonical model be just the \vdash -theories which have the special ($\rightarrow N2$)-secured property, but choosing to take all ma-sets makes for greater continuity with corresponding constructions in later sections. The special ($\rightarrow N2$)-secured property is analogous to the condition of *primeness* placed on theories for the usual canonical model completeness proofs for minimal and intuitionistic logics—for example, in Segerberg [30]. (A theory is *prime* if it contains at least one disjunct of any disjunction it contains.) Instead of just requiring that the theories involved as elements of the canonical model be prime, the stronger demand that they be what we are calling ma-sets (relative to the consequence relations involved) could be imposed and the completeness arguments would still go through—though it is not clear to me whether

this replacement would yield interesting further results. The analogy between primeness and the special (\rightarrow N2)-secured property is closer than this structural similarity might suggest extending to the content of the two properties. Recall that the property secured by (\rightarrow N2) for a set Γ is that $A \rightarrow C \in \Gamma$ implies that either $A \rightarrow B \in \Gamma$ or $C \in \Gamma$, for arbitrary A, B, C . Rewriting $A \rightarrow C$ in modal terms so as to reveal a disjunction, we get the hypothesis that $\neg \Box A \vee C \in \Gamma$, from which primeness would deliver: either $\neg \Box A \in \Gamma$ or $C \in \Gamma$. But the first of these alternatives does not make sense as it stands for the \rightarrow -language (cf. the preceding note), motivating its transformation into $\neg \Box A \vee B \in \Gamma$ (for arbitrary B), which translates into $A \rightarrow B \in \Gamma$, and turning this particular appeal to primeness precisely an appeal to the special (\rightarrow N2)-secured property.

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