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Book Review

V. V. Rybakov. *Admissibility of Inference Rules*. Elsevier Science, Amsterdam, 1997. 617 pages.

A rule is admissible in a logic L if it can be added without increasing the set of tautologies of L. For example, the rule $\varphi/\forall x.\varphi$ is admissible in predicate logic, since if φ is a theorem, so is $\forall x.\varphi$. The notion of an admissible rule is quite central to logic, but it hardly attracts any attention outside a small group of people. Modern textbooks do not teach a student about consequence relations let alone admissible rules, and it is hard to find other books on logic that do. It is one of the aims of this book to fill this lacuna.

It deals specifically with the question of admissibility of inference rules and here mainly in the context of intermediate and modal logic.[1] Nevertheless, the reader will also learn a good deal about algebraic logic, deductive systems, and modal and intuitionistic logic in general. The book contains six chapters, of which the first two present the general theory of algebraic, modal, and intuitionistic logic, while the remaining four chapters deal with the problem of admissibility of rules in modal and intermediate logic. We shall summarize the first two chapters before entering a review of the book in a more chronological fashion.

Even though the results are more general, we shall often take advantage of the fact that we are dealing with extensions of $\mathsf{K4}$ and superintuitionistic logics. This will eliminate certain complications, into which we will not go since they are not relevant for the main results. We assume that the reader is acquainted at least with modal and intuitionistic logic. For a general introduction we refer to [2]. In what is to follow, we shall try to use the most standard terminology, which is not necessarily the author's own. For example, we shall make use of generalized Kripke-frames rather than models. This will help in the formulation of the results. Recall that a *Kripke-frame* is a pair $\langle F, R\rangle$ where F is a set and $R \subseteq F^2$. A *generalized Kripke-frame* (or *frame* henceforth) is a triple $\mathfrak{F} = \langle F, R, U\rangle$ where $\langle F, R\rangle$ is a Kripke-frame and $U \subseteq \wp(F)$ is closed under intersection, complement, and the operation $\tau(A) := \{y : \text{ if } y\ R\ x \text{ then } x \in A\}$. A Kripke-frame $\langle F, R\rangle$ is often tacitly identified with the general frame $\langle F, R, \wp(F)\rangle$. A *modal algebra* is a quintuple $\mathfrak{A} := \langle A, 1, -, \cap, \tau\rangle$ where $\langle A, 1, -, \cap\rangle$ is a Boolean algebra with unit, complement, and intersection, and

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$\tau : A \to A$ a function satisfying $\tau(1) = 1$, $\tau(a \cap b) = \tau(a) \cap \tau(b)$. A frame $\mathfrak{F}$ defines the *wrapping algebra* $\mathfrak{F}_+$ by

$$\mathfrak{F}_+ := \langle U, F, -, \cap, \tau \rangle \ .$$

Conversely, given a modal algebra $\mathfrak{A}$, let $U(\mathfrak{A})$ denote the set of ultrafilters of $\mathfrak{A}$. Put URV if and only if for all $\tau(a) \in U$ we have $a \in V$; finally, put $\widehat{a} := \{U \in U(\mathfrak{A}) : a \in U\}$. Then

$$\mathfrak{A}^+ := \langle U(\mathfrak{A}), R, \{\widehat{a} : a \in A\} \rangle$$

is a generalized frame, called the *dual frame* of $\mathfrak{A}$. Recall that for a modal algebra $\mathfrak{A}$, ${\mathfrak{A}^+}_+ \cong \mathfrak{A}$, but for a generalized frame $\mathfrak{F}$, ${\mathfrak{F}_+}^+ \cong \mathfrak{F}$ only holds if $\mathfrak{F}$ is descriptive. A *(Kripke-)model* is a pair $\langle \mathfrak{F}, \beta \rangle$ where $\mathfrak{F}$ is a generalized frame (Kripke-frame) and β a valuation, that is, a partial function from V into U. $\langle \mathfrak{F}, \beta \rangle \models_x \varphi$ for $x \in F$ is defined by induction on φ as usual. If β is defined only on finitely many variables, we call the model *weak*.

We shall briefly mention a few results on the connection between modal and intermediate logics. The so-called Gödel-McKinsey-Tarski translation T from intuitionistic formulas to modal formulas is defined as follows.

$$\begin{array}{lcl} T(p) & := & \Box p \\ T(\varphi \wedge \psi) & := & T(\varphi) \wedge T(\psi) \\ T(\varphi \vee \psi) & := & T(\varphi) \vee T(\psi) \\ T(\varphi \to \psi) & := & \Box(T(\varphi) \to T(\psi)) \\ T(\neg\varphi) & := & \Box\neg T(\varphi) \end{array}$$

Given an intermediate logic L, we define $\rho(L) := \mathsf{S4} \oplus T[L]$ and $\rho(L) := \mathsf{Grz} \oplus T[L]$. Here, the notation $L \oplus X$ is used to denote the (normal) extension of L by X where X is a set of formulas. Given a modal logic L' containing $\mathsf{S4}$ we put $\tau(L') := \{\varphi : T(\varphi) \in L'\}$. We call L' a *modal companion* of $\tau(L')$. For each intermediate logic the set of modal companions is exactly the interval $[\rho(L), \sigma(L)]$. The mapping σ is an isomorphism between the lattice of superintuitionistic logics and the lattice of normal extensions of Grz. The translation is faithful with respect to a number of properties, such as tabularity, and the finite model property (fmp).

Take a language $\mathcal{L}$. A *consequence relation* over $\mathcal{L}$ is a relation $\vdash \subseteq \wp(\mathcal{L}) \times \mathcal{L}$ such that

1. if $\varphi \in \Delta$ then $\Delta \vdash \varphi$,
2. if $\Delta \vdash \varphi$ then $\Delta \cup \Gamma \vdash \varphi$,
3. if $\Delta \vdash \psi$ for every $\psi \in \Gamma$ and if $\Gamma \vdash \varphi$ then $\Delta \vdash \varphi$.

$\vdash$ is *structural* if from $\Delta \vdash \varphi$ follows $\Delta^\sigma \vdash \varphi^\sigma$ where σ is a substitution, and $\vdash$ is *finitary* if $\Delta \vdash \varphi$ implies that there exists a finite $\Delta_0 \subseteq \Delta$ such that $\Delta_0 \vdash \varphi$. We will consider in sequel only structural and finitary consequence relations.[2] Given $\vdash$, we put $\mathsf{Taut}(\vdash) := \{\varphi : \varnothing \vdash \varphi\}$ and call it the *set of tautologies* of $\vdash$.

A *rule* is a pair $\rho = \langle \Delta, \varphi \rangle$ where $\Delta \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$. For example, $\mathrm{MP} = \langle \{p, p \to q\}, q \rangle$ is the well-known rule of Modus Ponens. Alternative notations for rules are $\delta_0, \ldots, \delta_{n-1}/\varphi$. So, the rule MP is also written like this: $p, p \to q/q$ or even, more visually, like this

$$\frac{p, p \to q}{q} \ .$$

ρ is a *derived rule* of $\vdash$ if $\rho \in \vdash$. Given a set R of finitary rules we let $\vdash^R$ denote the least finitary structural consequence relation in which all rules from R are derived rules. This is uniquely defined. We call R an *axiomatization* of $\vdash$ if $\vdash = \vdash^R$.

Definition 1 Let $\rho = \langle \Delta, \varphi \rangle$ be a rule and $\vdash$ a consequence relation. ρ is *admissible in* (or *for*) $\vdash$ if for every substitution σ: if $\Delta^\sigma \subseteq \mathsf{Taut}(\vdash)$ then $\varphi^\sigma \in \mathsf{Taut}(\vdash)$. $\vdash$ is called *structurally complete* if every admissible rule of $\vdash$ is also derivable.

Clearly, every derived rule is also admissible. The converse is not true as we shall see, and this of course makes the notion of admissibility all the more interesting.

The notion of an admissible rule is defined on the basis of the set of tautologies alone, and this means that there are several consequence relations with the same set of admissible rules. Some examples may illustrate this. Let L be a modal logic. Then two special consequence relations are generally associated with L. These are called the *local consequence relation*, $\vdash_L$, and the *global consequence relation*, $\Vdash_L$. They are defined as follows. $\Delta \vdash_L \varphi$ if φ is provable from $\Delta \cup L$ by means of MP alone; $\Delta \Vdash_L \varphi$ if φ is provable from $\Delta \cup L$ by means of MP and MN $:= p/\Box p$. The rules MN and DN $:= \Box p/p$ are both admissible rules of $\vdash_{\mathsf{K}}$, but neither is derivable. MN is a derived rule of $\Vdash_{\mathsf{K}}$, while DN is admissible but not derivable. In the book, the symbol $\vdash_L$ is used for what we have called the global consequence relation of L. We shall not follow this usage, since most people would find that irritating. There is a particular reason for preferring $\Vdash_L$ over $\vdash_L$. The latter is in general not finitely axiomatizable even when L is. This is so because we cannot ensure the closure of L under MN other than by assuming infinitely many axioms. For an intermediate logic L, the consequence relation $\vdash_L$ is defined as the local consequence relation, but it matches the global consequence relation of $\sigma(L)$ under the translation T. The Theorem 3.2.2 (called the Translation Theorem) says that a rule ρ is admissible in L if and only if $T(\rho)$ is admissible in $\sigma(L)$. Given this, it suffices to develop the theory of admissibility for modal logics. The results for intermediate logics can be derived from them. This is how we shall present the results here.

The consequence relations over a language form a complete lattice, and for each set L of formulas, the set of consequence relations whose set of tautologies is L form an interval whose maximal element is structurally complete. We denote this consequence relation by $\vdash^m_L$. We say that in L the *admissibility of inference rules is decidable* if for any finitary ρ it is decidable whether or not ρ is admissible, that is, whether or not $\rho \in \vdash^m_L$. Given a modal logic, the following questions naturally arise:

1. Is $\vdash^m_L$ finitely axiomatizable?
2. Is $\vdash^m_L$ decidable?
3. How many consequence relations exist in the interval $[\Vdash_L, \vdash^m_L]$?
4. Is $\Vdash_L$ structurally complete?

Clearly, the last problem is a special case of the third one, but it is this one which is treated in this book. To answer these questions, some more machinery needs to be developed. Recall Birkhoff's theory of equationally definable classes. From this theory it follows that there is an anti-isomorphism between the lattice of normal modal logics and the lattice of varieties of modal algebras where both are ordered by class inclusion. Namely, in the present context, an equation of the form $\varphi \doteq \psi$ can be replaced by the equation $\varphi \longleftrightarrow \psi \doteq \top$, which in turn is equivalent to the axiom $\varphi \longleftrightarrow \psi$. Conversely, the axiom φ is equivalent to the equation $\varphi \doteq \top$. This opens the way for an algebraic model theory of modal logic, aided by Stone's representation theorems for Boolean algebras. Now, a similar correspondence holds between quasi identities and quasi varieties. Quasi identities are equivalent in this context to what is known as Horn-clauses. A *Horn-clause* is a sentence of the form $(\forall \vec{x})(\bigwedge_{i<n} \delta_i \to \varphi)$ where all the δ_i $(i < n)$ and φ are atomic formulas. Quasi varieties are those classes that are axiomatizable by means of quasi identities. They are therefore elementary, and closed under products and subalgebras. We say that a rule $\rho = \delta_0, \ldots, \delta_{n-1}/\varphi$ is *valid* in a modal algebra $\mathfrak{A}$ if the corresponding Horn-clause $(\forall \vec{x})(\bigwedge_{i<n} \delta_i \doteq \top \to \varphi \doteq \top)$ is valid in $\mathfrak{A}$. (Here, $\vec{x}$ contains all variables occurring free in ρ.)

Now, given a modal logic L and a cardinal κ, denote by $\mathfrak{F}_L(\kappa)$ the freely κ-generated L-algebra. Of particular interest in the study of admissible rules is $\mathfrak{F}_L(\omega)$. For the following holds.

Theorem 2 *ρ is admissible for L if and only if ρ is valid in $\mathfrak{F}_L(\omega)$.*

Hence, the quasi variety axiomatized by $\vdash_L^m$ is exactly $\mathfrak{F}_L(\omega)^Q$, where $\mathcal{K}^Q$ denotes the smallest quasi variety containing $\mathcal{K}$.

We now enter the book at Chapter 3. This is really the heart of the whole book. It contains the most difficult and powerful theorems. Throughout we shall assume that L is a modal logic containing $\mathsf{K4}$. An important notion is that of an *n-characterizing model*. It can be described in standard terms as follows. Take the free L-algebra $\mathfrak{F}_L(n)$ on n generators. Denote the dual frame of $\mathfrak{F}_L(n)$ by $\mathfrak{Can}_L(n)$. This frame is infinite but it contains as a generated subframe the frame of all points of finite depth which we denote by $\mathfrak{Ch}_L(n)$. Together with the natural valuation this constitutes the n-characterizing model which we also denote by $\mathfrak{Ch}_L(n)$. It is known that each point of infinite depth sees a point of arbitrary finite depth (see Fine [6]). Theorem 3.3.6 asserts that if L has the finite model property every nontheorem of L is refutable in one characterizing model.[3] The n-characterizing models can be used to determine the admissibility of a rule as follows. Let L have fmp. Then the wrapping algebra of $\mathfrak{Ch}_L(n)$ is actually isomorphic to $\mathfrak{F}_L(n)$. Since $\mathfrak{F}_L(n)$ is a subalgebra of $\mathfrak{F}_L(\omega)$, one can show that $\{\mathfrak{F}_L(n) : n \in \omega\}^Q = \mathfrak{F}_L(\omega)^Q$. Now, suppose we are given a rule ρ. Then ρ is admissible in L if and only if it is admissible in every n-characterizing model. Finally, Lemma 3.4.2 says that if ρ contains k variables, ρ is admissible in $\mathfrak{Ch}_L(n)$ if and only if it is admissible in $\mathfrak{Ch}_L(k)$. This gives the first general result.

Theorem 3 *Let L be a finitely axiomatizable logic containing* $\mathsf{K4}$. *If the variety of L-algebras is locally finite, the admissibility of rules for L is decidable.*

This includes all tabular logics, since they generate a locally finite variety and are finitely axiomatizable. There is a criterion on local finiteness, which runs as follows. Call a logic L of *depth d* if no refined L-frame contains a sequence of points x_i, $i < d+1$, such that $x_i \; R \; x_{i+1}$ but not $x_{i+1} \; R \; x_i$. L generates a locally finite variety if and only if it is of depth d for some $d \in \omega$.

Theorems 3.5.1 and 3.5.2 are still more general. We will state them as one:

Theorem 4 *Suppose that L is a logic containing* $\mathsf{K4}$. *Suppose further that*

1. L has fmp,

2. *L has branching below m for some* $m \in \omega$,

3. *L has the effective m-drop point property for some* $m \in \omega$.

Let ρ *be a rule with k variables. Then* ρ *is admissible in L if and only if it is valid in the wrapping algebra of the Kripke-frame underlying the k-characterizing frame. Furthermore, suppose that there is an algorithm which decides for a finite frame whether it is an L-frame. Then there exists an algorithm deciding whether a given inference rule is admissible for L.*

Here, a logic has branching below m if whenever in some frame for L there is a cluster with d immediate successor clusters, then whenever we find d incomparable clusters in $\mathfrak{Ch}_L(n)$, there is a cluster C having these clusters as its immediate successor clusters. The effective m-drop point property is still more cumbersome to define. To understand it, recall the selection procedure of Fine and Zakharyaschev (see [6] and [9]). This procedure extracts a finite model out of a given model $\mathfrak{M}$ on the basis of a set Y of formulas closed under subformulas. Denote this frame by $X(\mathfrak{M}, Y)$ and by $X_m(\mathfrak{M}, Y)$ the model containing both $X(\mathfrak{M}, Y)$ and the points of depth at most m. (We are assuming that the model is weak.) Crucially, this procedure does not preserve the truth of all formulas (since we are taking subframes which are in general not generated) but it does preserve the truth of all formulas from Y. For cofinal subframe logics this shows that they have the finite model property. The m-drop point property says the following. Suppose that we have a finite n-generated L-model $\mathfrak{M}$ and that it is large. Then it contains a submodel $\mathfrak{W} \supseteq X_m(\mathfrak{M}, Y)$, which is contractible onto an L-frame of no more than $g(x, y)$ elements where g is a recursive function and $x = |Y|$ and y the number of points of depth at most m in $\mathfrak{M}$.

The proof of this theorem uses the selection procedure. It shows that if ρ is refutable in the n-characterizing model then we can construct a model whose size we can estimate a priori and in which ρ is refuted as well. This model also has the so-called view-realizing property. Conversely, if such a model exists, ρ is refutable in the n-characterizing model. The proof of the latter statement is the most involved, but it seems that it can be simplified using the technique of homogenization proposed in [7].

As it turns out, the standard modal systems, $\mathsf{K4}$, $\mathsf{S4}$, GL, Grz, $\mathsf{S5}$, with or without an axiom of finite width, all satisfy the conditions of this theorem, and the problem of admissibility is therefore decidable in them. By the Translation Theorem, admissibility of inference rules is decidable for the logics Int and LC and many more. The remainder of Chapter 3 is devoted to some questions related to the decidability of admissibility. For example, if a logic L has the disjunction property and the admissibility problem is decidable, then the universal theory of $\mathfrak{F}_L(\omega)$ is decidable. However, as is also shown, mostly the elementary theory of this algebra is undecidable for the standard systems (K, $\mathsf{K4}$, GL). This gets even worse when we consider the logic of *schemes*. Schemes are introduced to study the admissibility problem for rules in first-order theories. A *scheme* is a formula S formed from variables z_i, $i \in \omega$, for first-order formulas using the Boolean connectives and the quantifiers $(\forall x_i)$, $(\exists x_i)$ where the x_i $(i \in \omega)$ are first-order variables. A scheme $S(z_0, \ldots, z_{n-1})$ is *valid* in a first-order theory T if $S(\alpha_0, \ldots, \alpha_{n-1})$ is derivable in T for all first-order formulas α_i, $i < n$. A valid scheme (for all first-order theories) is, for example,

$$(\forall x_0)(z_0 \wedge z_1) \longleftrightarrow (\forall x_0)z_0 \wedge (\forall x_1)z_1 \ .$$

A *rule* is a pair $\langle \Sigma, S\rangle$ where Σ is a set of schemes and S a single scheme. It is *admissible* in a first-order theory T if for all substitutions s of formulas for scheme variables, if every member from Σ^s is valid in T, then so is S^s. An example of a rule is the rule $\langle\{z_0\}, (\forall x_i)z_0\rangle$; it is admissible in all first-order theories. It turns out that if a first-order theory T has infinite models, then the set of T-valid schemes is not decidable; if T is in addition decidable, its set of valid schemes is not even recursively enumerable. This means that in such cases the admissibility of a first-order rule is undecidable. It is decidable if and only if T has only finitely many finite models. The chapter closes with examples of logics (due to Alexander Chagrov) which are decidable but for which the problem of admissibility of rules is undecidable. Furthermore, it is shown how to prove admissibility using the so-called reduced form of a rule.

The fourth chapter deals with the problem of finding axiomatic bases for the set of admissible inference rules. It defines an infinite series of frames, whose precise definition we will not give here, and shows that if these frames are L-frames and L has the finite model property, the property of branching below m for some m and the effective m-drop point property, then L has no basis of admissible rules in finitely many variables.[4] This covers K4, S4, GL, Grz, and their extensions of finite depth, and consequently also the logic Int (and its extensions of finite depth). This answers negatively the question of Harvey Friedman whether or not Int has a finite basis of admissible rules. Crucially, the logics for which this technique works must be of infinite width, so there is hope that logics of finite width behave differently. Indeed, it is shown in Section 4.3 that S4.3 is much different in this respect. The main result is here that if $L \supseteq \mathsf{S4.3}$ then $\vdash^m_L$ is axiomatizable over $\Vdash_L$ by the single rule $\Diamond p, \Diamond\neg p/q$. The quasi variety generated by a finite number of finite, subdirectly irreducible algebras has a finite basis for its set of quasi identities, provided these are algebras for K.T. This means that the consequence determined by these algebras is finitely axiomatizable. This is false if the algebras are not K.T algebras. If we return to the question of admissibility of inference rules, the picture changes again. There is a finite Grz-frame of depth 3, whose logic has no basis for admissible rules in finitely many variables. This is the best possible result, since Remazky has shown that all tabular logics of depth 2 have a finite basis for admissible rules.

Chapter 5 deals with questions of structural completeness. It introduces a technique originally due to Citkin, which is an analogue of Jankov's technique of splittings (see [3] and [4]). Before we can introduce this technique, it is worthwhile to recall a few facts. If L is a logic that has the finite model property then the free algebra $\mathfrak{F}_L(\omega)$ is a subalgebra of the product of the finite subdirectly irreducible L-algebras. Under this condition, a logic L is structurally complete if and only if every finite subdirectly irreducible L-algebra is embeddable into the algebra $\mathfrak{F}_L(\omega)$ (or some $\mathfrak{F}_L(n)$, n a finite number). Now let $\mathfrak{A}$ be a finite, subdirectly irreducible K4-algebra. Then there exists a largest element $\omega \neq 1$ such that $\omega \wedge \tau\omega = \omega$. Call this element the *opremum*. Now, take for each element a of $\mathfrak{A}$ a propositional variable p_a and let $r(\mathfrak{A})$ be the following rule.

$$r(\mathfrak{A}) := \frac{\{p_{a*b} \longleftrightarrow p_a * p_b : a, b \in A\} \cup \{p_{\circ a} \longleftrightarrow \circ p_a : a \in A\} \cup \{p_1\}}{p_\omega}$$

where $*$ runs through all the basic binary connectives and $\circ$ through all the basic unary connectives. This is the *quasi-characteristic* inference rule of $\mathfrak{A}$. Now the following holds.

Theorem 5 *Let $\mathfrak{A}$ be a finite, subdirectly irreducible* K4*-algebra. Then for any* K4*-algebra $\mathfrak{B}$, $r(\mathfrak{A})$ is invalid in $\mathfrak{B}$ if and only if $\mathfrak{A}$ is isomorphically embeddable into $\mathfrak{B}$.*

It is not hard to show that no K4-algebra with at least two elements is embeddable into $\mathfrak{F}_{\mathsf{K4}}(\omega)$. Armed with this result one can show that there are infinitely many admissible rules which are independent from each other. One has to show only that there are infinitely many simple, finite K4-algebras. On the other hand, the set of admissible quasi-characteristic rules of S4 and Grz have a finite basis. In the latter case the generalized Mints' rule alone forms a basis.

$$\frac{[(p \to q) \to (q \vee r)] \vee u}{[((p \to q) \to p) \vee ((p \to q) \to r)] \vee u}$$

For S4 we need in addition to the modal translation of this rule two more, one of which is the quasi-characteristic rule of the two element cluster, which is equivalent to the rule $\Diamond p, \Diamond\neg p/q$, which we have already met above.

Indeed, the results on extensions of S4.3 can be understood quite easily now. All we need are the following two facts which are not hard to establish:

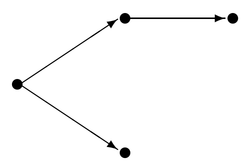
Lemma 6 *Let L be a modal logic containing* S4.3 *and $\mathfrak{A}$ a finite, subdirectly irreducible L-algebra. Then $\mathfrak{A} \times \mathbf{2}$ is a subalgebra of $\mathfrak{F}_L(\omega)$ where $\mathbf{2}$ is the two-element* S4*-algebra.*

Lemma 7 *The rule $\Diamond p, \Diamond\neg p/q$ is valid in $\mathfrak{A}$ if and only if the wrapping algebra of the two element cluster is not embeddable into $\mathfrak{A}$.*

Now, any extension L of S4.3 is finitely axiomatizable and has the finite model property, by results of Fine and Bull ([1] and [5]). L has the property of branching below 1 and the effective m-drop point property for some m. It follows that the admissibility of inference rules is decidable for L. Second, if we add the rule $\Diamond p, \Diamond\neg p/q$ then the resulting consequence relation axiomatizes the quasi variety containing all finite L-algebras of the form $\mathfrak{A} \times \mathbf{2}$. Since L is determined by such algebras, we see that this quasi variety contains $\mathfrak{F}_L(\omega)$. Moreover, since the smallest quasi variety containing $\mathfrak{F}_L(\omega)$ must contain these algebras, the two are equal. Hence we have axiomatized $\vdash_L^m$.

The chapter continues with the investigation of intrinsically complete modal logics. Call a logic L *hereditarily structurally complete* if all its extensions are structurally complete. L is *structurally precomplete* if it is not structurally complete but all its proper extensions are. It is shown that there are exactly 20 structurally precomplete logics containing K4 and they are all tabular. (The Kripke-frames for these logics are explicitly given.) A logic is hereditarily structurally complete if and only if neither of these frames is a frame for the logic. Consequently, there is a least hereditarily structurally complete logic, and this logic is the join of twenty splitting logics. From this, results on S4 and Int are immediately derived (since the frames are explicitly known). All these logics must be of width 2.

Chapter 6 rounds off the book. It covers a number of related issues. The first section deals with rules that have parameters and extends the results obtained so far to such rules. The second and third section characterizes those logics containing S4 or Int in which all rules admissible for S4 (Int) are also admissible. It turns out that these logics closely resemble the hereditarily structurally complete logics (indeed, they form a subset of these logics). The condition is that they are of width 2 and tightness 1 (in the terminology of [7]). The latter means that the following frame is not a subframe of an L-Kripke frame:



The remaining two sections are devoted to the study of noncompact logics where a logic L is called *compact* if the following is the case. Let φ be given. If for each finite subset of L, Δ, there is a Kripke-frame $\mathfrak{F}$ such that $\mathfrak{F} \models \Delta$ but $\mathfrak{F} \not\models \varphi$ then there is an L-Kripke frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$.

This book gives an exhaustive overview over the problem of admissibility of rules in modal and intuitionistic logics. Most of these results are due to the author himself. Many constructions are delicate and use sophisticated methods of modal logic. Certainly, without the modern inventory of techniques for transitive logics (which have been provided among others by Fine and Zakharyaschev) these results would certainly have been impossible. Nevertheless, the author uses them with ease and imagination.[5]

Despite all this, the book also has its shortcomings. I feel that the author has not tried to make things really simple. His terminology (model, compact logic) is often not standard, and the notation not really suggestive or clumsy. It leads to such constructs as $a^{\leq\leq}$, which denotes the cone generated by a (under the relation denoted by $\leq$). To give another example, $\mathcal{F} \circ 1$ denotes the disjoint union of the frame $\mathcal{F}$ with the one element reflexive frame. But why not write $\mathfrak{F} \sqcup 1$ where $\sqcup$ denotes the disjoint union? And why not use $\bullet$ instead of 1—as is used in pictures? Second, the results on extensions of S4.3 are readily understood if the techniques of Chapter 4 are used, as we have shown above. But this is nowhere mentioned or explained. This means more effort than necessary. There are so many mistakes and typographical errors that an unexperienced reader is easily put off. Even though I have not found any deeply worrying mistakes, I would bet that there is no single flawless page in the book. This concerns both the mathematical formulas as well as the English prose. The author must be held responsible for the errors in the formulas, and for tacitly changing the notation, which occurs not so infrequently. However, the English language is not something that an author is supposed to know well enough to write a book. For that, he should be able to rely on the publishers. Yet, the publisher didn't bother to have this book proofread beforehand. So, even though the author has had other people read the manuscript, the quality of the prose varies considerably. If this (widespread) policy of editing books continues, the literary style of scientific books will deteriorate in the long run. Moreover, writers with a native command of English will have an advantage in publishing books, since they do not need to worry that much about the language. All others run the risk of making stupid mistakes that will appear in print (and upset the critic). Another concern is the layout. Also the layout and typesetting is now entirely a responsibility of the author, which those experienced with it will enjoy. For all others it is a pain in the neck. The present book illustrates what can happen if someone not so experienced with typesetting is left alone with the job.

With all this being said, the book is enjoyable for the experienced reader. It is full of innovative methods and strong results not only about the admissibility of rules. It deepens our understanding of modal logic, and of logic in general. And it may—hopefully—show that modal logic is full of deep and also difficult theorems, and help to advance a topic that is nowadays a rather neglected area of logic, namely, the study of rules and logical consequence. I should mention perhaps that the notion of a rule as considered in the book is—at least in one instance—not general enough. For there are often cases when we want to have several conclusions. Let us therefore write $\rho = \langle \Delta, \Gamma \rangle$ for sets Δ and Γ to denote such a rule and call it a *multiple conclusion rule*. ρ is called *admissible* in L if for every substitution σ: if $\Delta^{\sigma} \subseteq L$ then for some $\gamma \in \Gamma$ $\gamma^{\sigma} \in L$. In the literature there are several examples of multiple conclusion rules, for example, the rule of margins: $\langle \{p \to \Box p\}, \{p, \neg p\} \rangle$. Another example is related to the disjunction property. Recall that an intermediate logic has the disjunction property if whenever $\varphi \vee \psi \in L$ then either $\varphi \in L$ or $\psi \in L$. L has the disjunction property if and only if the multiple conclusion rule $\delta := \langle \{p \vee q\}, \{p, q\} \rangle$ is admissible in L.[6] It would be interesting to develop a theory of such rules. It is clear that multiple conclusion rules are connected with properties of the quasi varieties. For example, the admissibility of multiple conclusion rules is decidable if and only if the universal theory of the algebra $\mathfrak{F}_L(\omega)$ is decidable. Thus, by the results of the book, the problem of admissibility of multiple conclusion rules is decidable for all extensions of $\mathsf{S4.3}$. It is, of course, beyond the scope of this review to present a theory of multiple conclusion rules, but we have at least indicated how such a theory might go and that it is worth its while. Now it is time for others to pick up the book and continue the research.

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NOTES

1. Although the book also treats tense logic in the introductory chapters, there are no results of substance proved about them later. I will therefore ignore tense logic in sequel.

2. In the book, there is an occasional reference to infinitary rules, and some results are proved about them. However, the majority of results concern finitary rules and finitary consequence relations.

3. Actually, the condition of finite model property is lacking from the formulation, which

is clearly false. Otherwise, any logic containing K4 has the fmp, since every generated subframe of $\mathfrak{Ch}_L(n)$ is finite.

4. This implies that there is no finite basis, since this basis has finitely many variables. But it also implies that there is no infinite basis using only finitely many variables.
5. Vladimir Rybakov has emphasized in personal communication that he has developed these techniques independently of Fine and Zakharyaschev.
6. There exist analogous notions for modal logics, but we shall not go into that.

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