Dependent Choices and Weak Compactness

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Abstract   We work in set theory without the Axiom of Choice ZF. We prove that the Principle of Dependent Choices (DC) implies that the closed unit ball of a uniformly convex Banach space is weakly compact and, in particular, that the closed unit ball of a Hilbert space is weakly compact. These statements are not provable in ZF and the latter statement does not imply DC. Furthermore, DC does not imply that the closed unit ball of a reflexive space is weakly compact.

1 Introduction   We work in set theory without Axiom of Choice ZF and we denote by ω the set of natural numbers. In this paper, normed spaces (as defined, for example, in [2], Definition 1.2, p. 63) are real normed spaces and they are endowed with the norm metric. A metric space is said to be complete when every Cauchy filter of this space converges (see Remarks 2.9 and 2.10). A Banach space is a normed space which is complete. The continuous dual of a normed space (E, ||.||) is the vector space E* of real linear functionals on E which are bounded on the closed unit ball of E, and E* is endowed with the dual norm ||.||*: for every f ∈ E*, ||f||* := sup{ f(x) : ||x|| ≤ 1}. The second dual of E is the normed space E**. For every x ∈ E, we denote by ˆx the evaluation at point x, that is, the mapping E* → R such that for every f ∈ E*, ˆx(f) = f(x). The natural map jE : E → E**, given by jE(x) = ˆx, is linear and continuous since ||jE(x)||** ≤ ||x||. Using the Hahn-Banach axiom, jE can be proved isometric, that is, ∀x ∈ E ||jE(x)||** = ||x|| (see [2], Corollary 6.7, p. 79), but this is not provable in ZF, since there are models of ZF with infinite dimensional normed spaces E such that E** = {0} (see Remark 2.8). The usual definition of “reflexivity” for a normed space E (see [2], p. 89, Definition 11.2) relies on the fact that jE is isometric, so we will formulate this definition in ZF and we will call it simple reflexivity or reflexivity. The weak topology of E is the coarsest topology on E for which every f ∈ E* is continuous: it is generated by the sets {x ∈ E : f(x) < λ}, λ ∈ R, and f ∈ E*, and it is denoted by σ(E, E*) (see [2], Definition 1.1, p. 124). The weak* topology of E* (see [2], Definition 1.1, pp. 124–5) is the coarsest topology on E* such that for every x ∈ E, ˆx is continuous: it is generated by the sets {f ∈ E* : f(x) < λ},

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\( \lambda \in \mathbb{R}, \) and \( x \in E, \) and it is denoted by \( \sigma(E^*, E) \). A topological space \( X \) is \emph{compact} if every nonempty set of closed subsets of \( X \) with the finite intersection property has a nonempty intersection. In set theory with the Axiom of Choice \( \text{ZFC} \), the reflexivity of \( E \) is known to be equivalent to the compactness of its closed unit ball for its weak topology (see [2], Theorem 4.2, p. 132), but this equivalence is not provable in \( \text{ZF} \) (see Remarks [1,2,2,7] and [2,8]), so we shall consider another notion of reflexivity which we call \emph{compact reflexivity}.

Let us state these two notions of reflexivity for a normed space \( E \):

1. **(Simple) Reflexivity:** The natural mapping \( j_E \) from \( E \) to its second dual \( E^{**} \) is onto and isometric.

2. **Compact Reflexivity:** The closed unit ball of \( E \) is compact for the weak topology.

Note that the classical proof of the following statement of Reflexive Compactness relies on Alaoglu’s Theorem (see [2], Theorem 3.1, pp. 130–1) which is equivalent (within \( \text{ZF} \)) to many other classical statements, for example, the Boolean Prime Ideal Theorem (see Howard and Rubin [4], pp. 21–7); this last statement is not provable in \( \text{ZF} \) (see Jech [5]), hence Alaoglu’s Theorem is not provable in \( \text{ZF} \) either.

**RC** (Reflexive Compactness): The closed unit ball of a reflexive normed space is compact for the weak topology.

**A (Alaoglu):** The closed unit ball of the continuous dual of a normed space is compact for the weak* topology.

We now consider some geometric properties of normed spaces. A normed space \((E, \| \cdot \|)\) is a \emph{prehilbert space} when there exists an inner product \( \langle \cdot, \cdot \rangle : E \times E \to \mathbb{R} \) such that for every \( x \in E, \| x \| = \sqrt{\langle x, x \rangle} \). A \emph{Hilbert space} is a complete prehilbert space. A normed space \( E \) is \emph{uniformly convex} (see [1], p. 189) if the modulus of uniform convexity of \( E, \delta_E : \mathbb{R}_+^* \to \mathbb{R}_+ \) defined below, satisfies \( \varepsilon > 0 \Rightarrow \delta_E(\varepsilon) > 0. \)

\[
\delta_E(\varepsilon) := \inf \left\{ 1 - \frac{x + y}{2} : \| x \| \leq 1, \| y \| \leq 1 \text{ and } \| x - y \| \geq \varepsilon \right\}
\]

Every prehilbert space is uniformly convex (see [1], pp. 189–90). We now consider Reflexive Compactness particularized to uniformly convex Banach spaces and particularized further to Hilbert spaces:

**RCuc** (Reflexive compactness for uniformly convex Banach spaces): The closed unit ball of a uniformly convex Banach space is weakly compact.

**RCh** (Reflexive Compactness for Hilbert spaces): The closed unit ball of a Hilbert space is weakly compact.

**Remark 1.1** Using projections on closed convex subsets in a Hilbert space (see Lemma 3 in [3]), one can prove in \( \text{ZF} \) that every Hilbert space is simply reflexive. Hence \( \text{RC} \) implies \( \text{RCh} \); in particular, \( \text{RCh} \) does not imply \( \text{DC} \).

In Fossy and Morillon [3], it is proved that \( \text{RCh} \) implies the following set-theoretic axiom \( \text{AC}^{\text{fin}}_\omega^\text{fin} \) which is not provable in \( \text{ZF} \) (see [5]); in particular, \emph{the statements RC and RCh are not provable in ZF either}. 

(Countable Axiom of Choice for finite sets): If $(A_n)_{n \in \omega}$ is a sequence of non-empty finite sets, then $\prod_{n \in \omega} A_n \neq \emptyset$.

**Remark 1.2** Thus, though in ZF every Hilbert space is simply reflexive, there are models of ZF in which some Hilbert spaces are not compact reflexive. Hence simple reflexivity does not imply compact reflexivity.

Now the following question is natural.

**Question 1.3** Is there a principle of “Countable Choice” which implies the axiom RCh?

In this paper, we prove that the following Principle of Dependent Choices implies RCuc (thus it implies RCh too):

**DC** (Principle of Dependent Choices): If $E$ is a nonempty set and $R$ is a binary relation on $E$ satisfying

\[ \forall x \in E \ \exists y \in E \ xRy, \]

then there exists a sequence $(x_n)_{n \in \omega}$ such that for every $n \in \omega$, $x_nRx_{n+1}$.

We shall also observe that DC does not imply RC (see Remark [2.7]). Note that BPI does not imply DC and that DC does not imply BPI (see [4] or [5]).

### 2 The Principle of Dependent Choices implies RCuc

**Notation 2.1** Consider a (real) normed space $(E, \| \cdot \|)$. For each nonnegative real number $r$, $\Gamma(0, r)$ denotes the closed ball of center 0 and radius $r$, that is, $\{ z \in E : \| z \| \leq r \}$; the closed unit ball $\Gamma(0, 1)$ is denoted by $\Gamma_E$. Given two real numbers $r$ and $r'$ such that $0 \leq r < r'$, the crown $\{ z \in E : r \leq \| z \| \leq r' \}$ is denoted by $D(0; r, r')$.

Given a normed space $E$, we denote by $T_E$ the set of finite unions of closed convex subsets of $\Gamma_E$. Notice that $(T_E, \cap, \cup)$ is a lattice of subsets of $\Gamma_E$ and that each closed set of $\Gamma_E$ for the weak topology is an intersection of elements of $T_E$. A filter of $T_E$ is any nonempty set $F$ of nonempty elements of $T_E$ such that the intersection of any two elements of $F$ is in $F$ and such that any element of $T_E$ which is a superset of an element of $F$ is in $F$ too.

For each set $F$ of subsets of $\Gamma_E$, let

\[ R(F) := \inf \left\{ r \in \mathbb{R} : 0 \leq r \leq 1 \text{ and } \forall F \in F, \Gamma(0, r) \cap F \neq \emptyset \right\}. \]

When $F$ has the finite intersection property, $F \cup \{ \Gamma(0, r) : R(F) < r \leq 1 \}$ generates a filter $F_c$ of $T_E$ called the circled filter associated to $F$.

The following lemma is an easy consequence of the definitions.

**Lemma 2.2** Let $E$ be a normed space and $F$ be a filter of $T_E$.

1. For every real numbers $r$ and $r'$ such that $0 \leq r < R(F) < r'$, there exists $F \in F_c$ such that $F \subseteq D(0; r, r')$.
2. For any filter $F'$ of $T_E$ extending $F_c$: $R(F') = R(F)$ and $(F')_c = F'$.
Lemma 2.3  Given a uniformly convex normed space $E$ with modulus of uniform convexity $\delta_E$, consider real numbers $\eta > 0$ and $r, r'$ such that $0 < r < r'$ and $r \geq \frac{1}{2} \frac{\eta}{\delta_E(\frac{\eta}{r})}$. Then the diameter of any convex subset $C$ of the crown $D(0; r, r')$ is less than or equal to $\eta$.

Proof: Assume by contradiction that some convex subset $C$ of $D(0; r, r')$ contains two points $x$ and $y$ such that $\|x - y\| > \eta$. Then, from the definition of $\delta_E$, it follows that $\|\frac{x + y}{2}\| < (1 - \delta_E(\frac{\eta}{r}))r'$; but $\|\frac{x + y}{2}\| \geq r$, since $C$ is convex.

Theorem 2.4  Given a uniformly convex Banach space $E$, let $\mathcal{F}$ be a filter of $T_E$. The Principle of Dependent Choices DC implies the nonemptiness of the set $\cap \mathcal{F}$.

Proof: We prove the existence of a Cauchy filter $\mathcal{G}$ of $T_E$ (i.e., a filter containing sets of arbitrary small diameter) extending the circled filter $\mathcal{F}_c$ associated to $\mathcal{F}$ (thus, since the elements of $\mathcal{G}$ are closed and $E$ is complete, $\cap \mathcal{G} \neq \emptyset$ and a fortiori, $\cap \mathcal{F} \neq \emptyset$).

Denote by $\delta_E$ the modulus of uniform convexity of $E$. Let $R := R(\mathcal{F})$. If $R = 0$, then $\mathcal{F}_c$ is already Cauchy. Now assuming that $R > 0$, for each $n \in \omega$, let $r_n$ and $r'_n$ be positive real numbers such that $r_n < R < r'_n$ and $r_n \geq (1 - \frac{1}{n+1})r'_n$ (e.g., consider $\alpha_n > 0$ such that $\frac{2\alpha_n}{\alpha_n + R} \leq \delta_E(\frac{1}{n+1}R)$ and let $r_n := R - \alpha_n$ and $r'_n := R + \alpha_n$). Let $S$ denote the set of finite mappings $s \subseteq \omega \times T_E$ such that

(i) for every $n \in \text{domain}(s)$, $s(n) \subseteq D(0; r_n, r'_n)$,

(ii) $\mathcal{F}_c \cup \text{range}(s)$ has the finite intersection property.

Every element of $\mathcal{S}$ admits a proper extension in $\mathcal{S}$: given $s \in \mathcal{S}$ and $n \notin \text{domain}(s)$, it follows from Lemma 2.2 that some element $F$ of the filter $\mathcal{F}_c$ of $T_E$ generated by $\mathcal{F}_c \cup \text{range}(s)$ is a subset of $D(0; r_n, r'_n)$; then, given closed convex subsets $C_1, \ldots, C_m$ of $E$ such that $F = C_1 \cup \cdots \cup C_m$, observe that for some $i \in \{1, \ldots, m\}$, $C_i$ meets every element of $\mathcal{F}_c$, so that $s \cup \{(n, C_i)\}$ is a proper extension of $s$ in $\mathcal{S}$. Now, invoking DC, get an increasing sequence (with respect to proper extension) $(s_n)_{n \in \omega}$ of $T_E$, and observe that, given $s := \cup \{s_n : n \in \omega\}$, Lemma 2.2 implies that $\mathcal{F}_c \cup \text{range}(s)$ generates a Cauchy filter of $T_E$.

Corollary 2.5  $\text{DC} \implies \text{RCuc}$.

Proof: Given a uniformly convex Banach space $E$, let $\mathcal{H}$ be a nonempty set of weakly closed sets of $\Gamma_E$ with the finite intersection property. Since each such closed set is an intersection of elements of $T_E$, $\cap \mathcal{H} = \cap \mathcal{F}$, where $\mathcal{F} = \{F \in T_E : F \supseteq H, \text{for some } H \in \mathcal{H}\}$; but $\mathcal{F}$ is a filter of $T_E$, hence $\cap \mathcal{F} \neq \emptyset$, according to Theorem 2.4.

Remark 2.6  $\text{RCuc}$ does not imply $\text{A}$, since $\text{RCuc}$ follows from $\text{DC}$, which does not imply BPI, and hence, does not imply $\text{A}$ either.

Remark 2.7  Pincus and Solovay [5] have built a model $\mathcal{M}$ of (ZF + DC) in which every finitely additive measure is trivial. This means that, given any set $I$, for every finitely additive mapping $m : \mathcal{P}(I) \to \mathbb{R}$, there exists a family $(\lambda_i)_{i \in I}$ of real numbers such that, for every subset $A$ of $I$, $m(A) = \sum_{i \in A} \lambda_i$. It follows that the continuous dual of $\ell^\infty(I)$ is equal to $\ell^1(I)$. Thus, in this model $\mathcal{M}$, every $\ell^1(I)$ is a reflexive normed space.
Besides, the closed unit ball \( \Gamma \) of \( \ell^1(I) \) is never weakly compact when \( I \) is infinite: in fact, denoting by \( P_f(I) \) the set of finite subsets of \( I \), for each \( H \in P_f(I) \), let

\[
F_H := \left\{ f \in \Gamma : \sum_{k \in I} f(k) = 1 \quad \text{and} \quad \forall k \in H, \ f(k) = 0 \right\}.
\]

Each \( F_H \) is a weakly closed set of \( \Gamma \), and when \( I \) is infinite, the family \( \{ F_H : H \in P_f(I) \} \) has the finite intersection property, but \( \bigcap \{ F_H : H \in P_f(I) \} \) is empty.

Summing up, in the model \( \mathcal{M} \), the space \( \ell^1(\omega) \) is reflexive and separable but its closed unit ball is not weakly compact. Hence simple reflexivity does not imply compact reflexivity even in the case of separable spaces. Moreover, since the model \( \mathcal{M} \) satisfies \( DC \), \( DC \) does not imply \( RC \) even for separable reflexive spaces.

**Remark 2.8**  Compact reflexivity does not imply simple reflexivity since, in every model of \( ZF \) where Hahn-Banach Theorem fails (for instance, the model \( \mathcal{M} \) above), there exists an infinite dimensional normed space \( E \) such that \( E^* = \{0\} \): such an \( E \) fails to be reflexive although \( \Gamma_E \) is weakly compact.

Let us now consider the following consequence of \( DC \):

\( AC_\omega \) (Countable Axiom of Choice): If \( (A_n)_{n \in \omega} \) is a sequence of nonempty sets, then \( \prod_{n \in \omega} A_n \neq \emptyset \).

**Remark 2.9**  Say that a metric space \( (X, d) \) is sequentially complete when every Cauchy sequence converges. So every complete metric space is sequentially complete. In \( (ZF + AC_\omega) \), hence in \( (ZF + DC) \), every sequentially complete metric space is complete.

**Remark 2.10**  A set \( X \) is Dedekind-finite when there exists no one-to-one mapping from \( \omega \) to \( X \). There are models of \( ZF \) (e.g., Cohen’s first model, see \[5\]) with a Dedekind-finite dense subset \( A \) of \( \mathbb{R} \). The metric subspace \( A \) is sequentially complete but it is not complete.
Note: Uniformly convex spaces are simply reflexive (work in progress). Thus RC implies RCuc; in particular, like RCh, RCuc fails to imply DC. (Cf. Abstract and Remark 1.1.)
We know no answer to the following questions:

**Question 2.11** Are the statements $A$ and $RC$ equivalent?

**Question 2.12** Does $AC_{\omega}$ imply $RCh$ or $RCuc$?

**Question 2.13** Are $RCh$ and $RCuc$ equivalent?

REFERENCES


