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Set Theory with Indeterminacy of Identity

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Abstract We presume a background theory which allows for indeterminacy of states of affairs involving objects, extending even to indeterminacy of identity between objects. A sentence reporting such an indeterminate state of affairs lacks truth-value. We extend this to a theory of sets, similar to ZFU, in which membership in, and identity between, sets may also be indeterminate.

1 Introduction In the philosophical literature there are a number of identity puzzles which persist without agreed-upon solutions. These include, for example, the ship puzzle (if a ship has its parts all replaced, and if the discarded parts are reassembled into an exact replica of the original, is the original ship identical with the ship with new parts, or with the newly assembled ship?) and the disruptive change puzzle (if a person has *n*% of his/her brain replaced, is the original person identical with the recovered person?). Recently there has been increased interest in the idea that these puzzles have no answer, not because of flaws in the language or conceptual scheme within which they are formulated, but because the world itself is partially indeterminate, where this indeterminacy extends even to identity. In response to this it has been claimed (Salmon [6]) that the idea that identity itself might be indeterminate conflicts with basic principles of set theory. This is a natural worry, since having determinate conditions for membership is sometimes taken as the most basic requirement for set existence, and this suggests that sets are inherently incompatible with indeterminacy.[1] This criticism has been partly met in Woodruff and Parsons [9] where it is shown that a simple theory of objects and sets of objects can accommodate indeterminacy of identity. But the question remains open regarding whether such consistency extends to a hierarchy of sets which themselves contain sets as members. In the present paper we answer this in the affirmative, showing that it is consistent (relative to ZFU) to combine a theory that admits indeterminacy with the existence of a hierarchy of sets, such as that posited by ZFU.[2] (We choose ZFU as a basis for comparison because there is widespread familiarity with it, and confidence in its consistency, and because of the

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variety of its known applications.) There are some significant technical difficulties to be faced, but they can be overcome, and there are coherent ways to allow for indeterminacy of membership in, and identity between, sets which form a hierarchy such as that of ZFU. The present paper formulates some theories that accomplish this.[3]

1.1 Symbolism The indeterminate theories are formulated in a nonbivalent version of the first-order predicate calculus with identity. The symbolism includes variables $x, y, z, \ldots$ that range over all entities, as well as arbitrary names $a, b, c, \ldots$ for such entities. It includes arbitrary predicates $P, Q, R, \ldots$ of any number of places, including the two-place predicates '$\in$' for set membership and '$=$' for identity. It includes the Łukasiewicz connectives: '$\neg$', '&', '$\vee$', '$\Longrightarrow$', '$\Longleftrightarrow$', along with '$\triangleright$' (for truth), '$\triangleleft$' (for nonfalsity), and '$\triangledown$' (for indeterminacy).[4] They are interpreted as follows. '$\neg A$' is true (false) if and only if 'A' is false (true), and lacks truth-value if 'A' does. A conjunction is true if both conjuncts are true, false if either is false, and otherwise it lacks truth-value. A disjunction is the dual of conjunction; it is false if both disjuncts are false, true if either is true, and otherwise it lacks truth-value. The Łukasiewicz conditional '$\Longrightarrow$' is "sustaining if-then"; the truth-value status of such a conditional is determined by how far the consequent drops below the antecedent in truth-value status, counting T as highest and F lowest: if there is no drop at all the level is fully sustained, and the conditional is true; if it drops all the way from T to F the conditional is false; otherwise it lacks truth-value. This truth-condition validates most of the simple laws one naturally expects from a conditional, such as modus ponens, modus tollens, hypothetical syllogism, and contraposition. With '$A \Longleftrightarrow B$' defined as '$(A \Longrightarrow B)$ & $(B \Longrightarrow A)$', one obtains a biconditional that is true when A and B have the same truth-value status, false when they differ in truth-value, and otherwise indeterminate. In addition to the above we have a connective '$\triangleright$' for truth: '$\triangleright S$' is true if S is true, and false if S is either false or truth-valueless. Falsehood is then expressed as '$\triangleright\neg S$'. It is also convenient to have a connective '$\triangleleft$' for nonfalsity, defined as '$\neg\triangleright\neg S$'. Finally, '$\triangledown S$' will mean that 'S' is indeterminate (lacks truth-value); it can be defined as '$\neg\triangleright S$ & $\neg\triangleright\neg S$'.

The quantifiers $\exists$ and $\forall$ are understood as generalizations of $\vee$ and &: a formula $\exists x\Phi x$ is true if Φx is true for at least one assignment to x, false if Φx is false on every assignment to x, and otherwise $\exists x\Phi x$ is lacking in truth-value. $\forall x\Phi x$ is defined as $\neg\exists x\neg\Phi x$; it is true if Φx is true for every assignment to x, false if Φx is false on at least one assignment to x, and otherwise $\forall x\Phi x$ is lacking in truth-value.[5]

1.2 Inferences Valid inferences are ones that preserve truth. Most simple inferences are obvious from the explanations of the meanings of the connectives. Truth ('$\triangleright$') and nonfalsehood ('$\triangleleft$') conveniently commute with the quantifiers and distribute across '&' and '$\vee$'. Neither Indirect Proof nor Conditional Proof hold in their classical forms. Since a proof is a sequence of lines asserted as true if the initial assumptions are true, a subproof resulting in a contradiction establishes only that the hypothesis of the subproof is not true, not that it is false (not that the negation of the hypothesis is true). Thus if H leads to a contradiction, we may infer $\neg\triangleright H$ by indirect proof, but not $\neg H$. Likewise, if A leads to B, conditional proof lets us infer $\triangleright A \Longrightarrow B$, not the simpler $A \Longrightarrow B$. The traditional subproof rules hold, however, whenever the hypothesis of the subproof is bivalent in virtue of its form. (A formula that begins with any of '$\triangleright$', '$\triangleleft$', '$\bigtriangledown$' is bivalent in virtue of its form, and complexes of bivalent formulas are bivalent.) In our exposition we will often give informal proofs, for brevity, but they will respect the strictures due to possible lack of truth-value.

Another restriction to bear in mind is that if an inference pattern is valid, its contrapositive need not be valid. For example, this pattern is valid:

$$S \vdash \triangleright S,$$

but this is not:

$$\neg \triangleright S \vdash \neg S.$$

1.3 Identity The identity sign is postulated to obey these laws (where s and t are arbitrary terms):

Reflexivity:	$s = s$
Symmetry:	$s = t \Longrightarrow t = s$
Leibniz's Law (LL):	$s = t, \Phi s \vdash \Phi t$

We do *not* have the following "contrapositive" version of Leibniz's Law:

$$\Phi s, \neg\Phi t \vdash \neg s = t.$$

This does not follow from LL, and it is essential that it not hold if the coherence of indeterminate identity is to be maintained. For otherwise a statement of indeterminate identity could be disproved by this simplification of the argument of Evans [2]:

1.	$\neg \triangleright a = b \;\&\; \neg \triangleright \neg a = b$	Hypothesis for refutation
2.	$\neg \triangleright a = b$	1, Simplification
3.	$b = b$	Reflexivity
4.	$\triangleright b = b$	3, Logic of $\triangleright$
5.	$\neg a = b$	2,4, Contrapositive LL
6.	$\triangleright \neg a = b$	5, Logic of $\triangleright$, Contradicting 1

The plausibility of a theory that abandons contrapositive LL is much debated in the literature cited earlier; in the present paper we take for granted that contrapositive LL must be abandoned and that theories without it are worth studying.

Contrapositive LL does hold under certain special conditions. When it holds for a formula Φ we say that Φ satisfies the "Condition of Definite Difference" (DDiff for short), which is expressed by the following formula:

[DDiff] $$\forall x \forall y[\triangleright \Phi x \;\&\; \triangleright \neg \Phi y \Longrightarrow \triangleright \neg x = y].$$

There are formulas for which this condition can be shown not to hold (such as the one in the proof above). For the present application we will need to postulate whether DDiff holds for certain formulas containing set-theoretic relations; this will be discussed.

2 Options We focus on a conception of set according to which a set is nothing more than its members. This is meant in the sense that identities between sets are completely settled by settling the truth-value status (true, false, neither) of $x \in S$ for each entity x and each set S. More explicitly,

(i) if two sets have exactly the same determinate members and the same determinate nonmembers, they are identical;
(ii) if one determinately has a member that the other determinately lacks, they are distinct;
(iii) otherwise, it is indeterminate whether or not they are the same set.

Sets that are indeterminately identical (sets such that it is indeterminate whether they are identical) will be such that one has a determinate member or nonmember when it is indeterminate whether that thing is a member of the other set. On these assumptions, a set identity

$$X = Y$$

has exactly the same truth-value status (true, false, or neither) as

$$\forall z(z \in X \Longleftrightarrow z \in Y).$$

We call this equivalence "set essence":[6]

[Set Essence] $\quad X = Y \Longleftrightarrow \forall z(z \in X \Longleftrightarrow z \in Y).$

It is possible to formulate theories about sorts of entities that do not obey this equivalence, but such entities are not sets, as we understand the word. [Set Essence] is equivalent to a conjunction of the conditionals:

[Extensionality] $\quad \forall z(z \in X \Longleftrightarrow z \in Y) \Longrightarrow X = Y;$
[Set Indiscernability][7] $\quad X = Y \Longrightarrow \forall z(z \in X \Longleftrightarrow z \in Y).$

The first of these yields condition (i) above, the first and second together yield (ii), and the second yields (iii). We adopt [Set Essence] for all theories of sets. Decisions about other basic questions, such as the following, yield alternative set theories.

2.1 DDiff for set membership We have seen that we cannot in general infer from $\triangleright\Phi a$ and $\triangleright\neg\Phi b$ that a and b are distinct objects, since Φx may say something at a conceptual level that may not reflect a genuine difference in the world. But what if Φ is set membership? If we are given $\triangleright a \in X$ and $\triangleright\neg b \in X$ should we be able to infer from this that a and b are genuinely different? That depends on whether or not you view sets as things in the world, as opposed, for example, to concepts. If sets are in the world, the principle is plausible; if not, this may be just another counterexample to the contrapositive version of Leibniz's Law discussed above. Clearly, we have two options to explore. Setlike things that are immune to the principle just described we call "conceptual sets." Consideration of conceptual sets seems called for because they satisfy [Set Essence], and thus they are enough like sets that we should understand what they are like in detail. We describe conceptual sets in Section 5. For all other versions of set theory discussed in this paper we adopt this principle of definite difference for set members:

[DDiff for $\in$ Set] $\quad \triangleright x \in Z \;\&\; \triangleright\neg y \in Z \Longrightarrow \neg x = y.$

2.2 Restricting comprehension by DDiff for set membership The comprehension principle of naive set theory must be restricted in some way so as to avoid inconsistency due to Russell's paradox. We avoid such paradoxes by adopting a version of Zermelo-Fraenkel set theory which abandons Comprehension in favor of some structure-building operations together with Separation (which yields a subset of a given set) or Replacement (which "projects" a set from a given set). But we also need to address other limitations on set formation that are due to the framework of indeterminacy within which we work. In particular, having adopted [DDiff for $\in$ Set] as a principle obeyed by sets, we cannot allow Separation and Replacement to yield sets which violate it. So Separation and Replacement will need to be restricted to cases in which they yield sets with defining conditions that satisfy Ddiff.

2.3 Indeterminate identity for sets If one wishes to combine set theory with a theory encompassing indeterminacy, then it appears that one must allow sets to be indeterminately identical to one another even if there is no indeterminacy of identity of individuals at all. This is because one can devise distinct defining conditions for sets such that it is indeterminate whether the defining conditions specify the same members or not; [Set Essence] then entails that it is indeterminate whether the sets defined from those conditions are the same. Here is a sketch of a proof that addresses this phenomenon.

Proof Sketch: Suppose there is no indeterminacy of identity among objects at all, but there is *some* indeterminacy; for example, there is a formula Φ true only of individuals and such that it is indeterminate whether Φa is true:

1. $\neg\Phi a$.

Since there is no indeterminacy of identity of individuals, both 'Φx' and '$\Phi x \vee x = a$' satisfy DDiff. Applying Separation (restricted as above) to the set of individuals, there are sets A and B such that

2. $\forall x(x \in A \Longleftrightarrow \Phi x)$;
3. $\forall x(x \in B \Longleftrightarrow \Phi x \vee x = a)$.

It is easy to establish that a is a determinate member of B but not of A; so it is not determinately true that $A = B$. Likewise, there is nothing that is determinately a member of one set which is determinately not a member of the other, so it is not determinately false that $A = B$. We conclude that it is indeterminate whether $A = B$. □

We may avoid this consequence by restricting Separation even more severely, so as not to allow the initial steps (2) and (3) in the above proof. But the theory that results ("Fusion Sets") is not particularly useful; we describe it briefly in Section 6. For the bulk of this paper we admit the possibility of indeterminacy of set identity.

2.4 Unforced indeterminacy If we accept DDiff for set membership, then anything that is indeterminately identical to a member of a set must itself be either a determinate or indeterminate member of the set. This is "forced" determinate-or-indeterminate membership. A remaining question is whether we should allow sets to have indeterminate members *other than* things that are indeterminately identical to their determinate members[8] This is a fairly basic question and not one that we are prepared to settle here. Either answer seems natural to us, and thus we consider them both. If we allow sets to have indeterminate members that are not indeterminately identical to any of their determinate members, we will have broader comprehension principles than on the other option. In the next two sections we explore the option of allowing any entities whatsoever (of appropriate rank) as indeterminate members of a set. In Section 5 we explore the tighter option in which no indeterminate members are permitted except for those that are forced by DDiff.

2.5 Summary [Set Essence] is presupposed by all set theories discussed here. (We sketch one nonset option, which we call "Status Patterns", in Section 6.) [DDiff $\in$ Set] is adopted for all other theories except Conceptual Sets. Indeterminate identity is possible between sets in all *set* theories except Fusion Sets. (Conceptual Sets and Fusion Sets are summarized in Section 6.) Unforced indeterminate members occur in the theory of Indeterminate ZFU described in the next two sections and are prohibited in the theory of Hereditarily Tight Sets (Section 5).

3 Indeterminate Zermelo-Fraenkel set theory with ur-elements (IZFU) Here we develop a version of indeterminate set theory patterned after Zermelo-Fraenkel set theory with ur-elements;[9] we call it IZFU. We take for granted these previously discussed principles:

[Set Essence]	$X = Y \Longleftrightarrow \forall z(z \in X \Longleftrightarrow z \in Y)$;
[DDiff for $\in$ Set]	$\rhd x \in Z \; \& \; \rhd \neg y \in Z \Longrightarrow \rhd \neg x = y$.

Can one get a consistent (relative to ZFU) version of indeterminate set theory by writing down natural formulations of axioms for ZFU making no changes other than restricting Separation and Replacement to formulas that satisfy the DDiff condition for set membership? The answer is: almost. Foundation needs a slight reformulation (given below) and there is a further constraint (involving rank) to be imposed on Separation and Replacement. Some additional housekeeping axioms are also needed.

3.1 The rank constraint In ZF the complement of a set is too big to be a set. In indeterminate ZF we have an additional worry about size. Suppose that the determinate members of a set are few enough to form a set; what about the indeterminate members? We cannot ignore this question altogether, for if we do, the ordinary principles of ZFU coupled with indeterminacy will lead to inconsistency. This is because they will permit one to generate a set that is small vis-a-vis its determinate members (it has none at all) but that has too many indeterminate members (*everything* is an indeterminate member). Such a set allows one to separate out a kind of "indeterminate Russell-set", which leads to contradiction.

Proof: Suppose there is a sentence S which is indeterminate. Let x be any entity at all. We prove (1) that it is nonfalse that $x \in \cup\wp(\varnothing)$, where $\varnothing$ is the empty set. Then we show (2) that this leads to inconsistency.

1. Let x be any entity and let S be a sentence without truth-value. By separation, there is a set X defined by the formula:

$$z \in X \Longleftrightarrow z \in \{x\} \; \& \; S.$$

Clearly, X has no determinate members; thus it is nonfalse that $\forall z(z \in X \Longrightarrow z \in \varnothing)$. So it is nonfalse that $X \in \wp(\varnothing)$. Since it is also nonfalse that $x \in X$, this is nonfalse:

$$\text{For some } Y : x \in Y \ \& \ Y \in \wp(\varnothing).$$

Thus it is nonfalse that $x \in \cup\wp(\varnothing)$.

2. We separate out a set R from $\cup\wp(\varnothing)$, defined by

$$z \in R \text{ iff } z \in \cup\wp(\varnothing) \ \& \ \rhd\neg z \in z.$$

Suppose it is nonfalse that $R \in R$. Then it is nonfalse that $R \in \cup\wp(\varnothing) \ \& \ \rhd\neg R \in R$. But then $R \in R$ is false, contrary to hypothesis. So it is false that $R \in R$. But then $\rhd\neg R \in R$ is true, and since anything is not falsely a member of $\cup\wp(\varnothing)$, we infer that it is not false that $R \in \cup\wp(\varnothing) \ \& \ \rhd\neg R \in R$. So it is not false that $R \in R$, contradicting what we just proved. □

This proof uses Separation only when the DDiff condition for set membership is satisfied.[10] It appeals to the set-theoretic principles of Empty Set, Pair Set (to generate unit sets), Union, Power Set, and Separation, so one of these must be restricted in order to block the inconsistency. A theory without one of Empty Set, Pair Set, Union, or Power Set would be possible but quite unorthodox;[11] instead, we will restrict Separation and Replacement.

In any reasonable version of ZFU there will be ordinals that can somehow index the ranks of sets. Suppose that the rank of a set S is defined as the smallest ordinal greater than the ranks of all the members *and* indeterminate members of S. The difficulty focused on in the proof above comes from generating a set that has arbitrarily high rank solely because of its indeterminate members. It appears that we must impose some conditions that will prevent the generation of sets whose indeterminate members have no maximum rank. So it is natural to require that both the determinate members and the indeterminate members of a set must be limited in rank by some ordinal. Such a principle can be invoked in general, and we can limit Separation and Replacement to those instances in which the resulting set satisfies the principle. However, this limitation all by itself is not sufficient to avoid inconsistency. For it still lets us generate a sequence of sets indexed by all the ordinals, each of which satisfies the limitation and such that each set in the sequence is indeterminately identical to each of the others. This conflicts with the Pairing principle that lets us form unit sets.

Let A be a set of some rank β, and let A_1 result from A by adding some entity of rank β as an indeterminate member; A_1 will then have rank $\beta + 1$. In general, let $A_{\alpha+1}$ result from A_α by adding something of the same rank as A_α as an indeterminate member. For limit ordinal λ, let A_λ have as determinate members the determinate members of A and as indeterminate members the indeterminate members of each of the A_β's such that $\beta < \lambda$. The result is a sequence of sets indexed by the ordinals of unbounded rank, each member of which (by [Set Essence]) is indeterminately identical to each of the others. But this conflicts with our forming the unit set of A. According to the Pairing axiom, the indeterminate members of $\{A\}$ are the things indeterminately identical to A. So each A_α in the sequence must be an indeterminate member of $\{A\}$. As a result, $\{A\}$ itself cannot be formed as a set.

It appears that in order to avoid this difficulty we must assume some limit L on how far the ranks of the indeterminate members of a set may go above the ranks of the determinate members.[12] The option we will explore is $L = 0$: the option that no indeterminate member of a set may have higher rank than that of every determinate member. This constraint is already preserved by the structure-building axioms of Pair Sets, Unions, and Power Set, and it yields a tidy theory. On this option it turns out that if it is indeterminate whether $x \in y$, then $\mathrm{rank}(x) < \mathrm{rank}(y)$, and if it is indeterminate whether $x = y$ then $\mathrm{rank}(x) = \mathrm{rank}(y)$. This is the option developed below; we do not explore other options in this paper.[13]

3.2 The axioms Our vocabulary includes at least the primitive two-place predicates '$\in$' and '$=$'. We assume a primitive name 'I' for the set of individuals and '$\varnothing$' for the set with no determinate or indeterminate members. Hereafter we understand capital letters to be variables restricted to sets (to nonindividuals), so '$\forall X[\ldots X \ldots]$' is short for '$\forall x[\neg x \in I \rightarrow \ldots x \ldots]$'. In order to formulate the rank restriction it is convenient to have a primitive two-place predicate 'ranks', where 'α ranks x' means that the ordinal α is the rank of entity x.[14] The two axioms above, [Set Essence] and [Ddiff for $\in$Set], along with the following, are our axioms for IZFU.[15] (These axioms are not all mutually independent; we have chosen explicitness over austerity.)

There is nothing in the logical principles themselves to prove that there is any indeterminacy at all. If there is no indeterminacy, then the proper set theoretic axioms reduce to ordinary bivalent ZFU, and the project of establishing the relative consistency of indeterminate ZFU becomes trivial. So we include the following "soft" axiom, which, unlike the others, is understood to be contingently true. If there is any indeterminacy at all, then, by the sort of reasoning discussed in Section 2, there will be some indeterminacy of identity. So the logically weakest assertion of indeterminacy we can make is that identity is sometimes indeterminate:

[NonTriviality] $\qquad \exists x \exists y \triangledown x = y$.

We present the remaining axioms beginning with things of the lowest rank.

[Bivalence of Individuals] $\qquad x \in I \vee \neg x \in I$

[Individuals Lack Members][16] $\qquad x \in I \Longrightarrow \neg \exists y y \in x$

[Empty Sets] $\qquad \neg \exists y \triangleright \Phi y \Longrightarrow \exists S \forall y [y \in S \Longleftrightarrow y \in I \mathbin{\&} \Phi y]$

[Empty Sets] generates sets with no determinate members and with arbitrary sets of individuals as indeterminate members. Selecting 'Φy' to be '$\neg y = y$' entails the usual empty set axiom of ZF:

$$\exists S \forall y \neg y \in S \, .$$

This yields an "emptiest set", $\varnothing$, a set with no determinate *or* indeterminate members:

[Emptiest Set] $\qquad \neg \varnothing \in I \mathbin{\&} \forall y \neg y \in \varnothing$.

If there are any individuals at all, there will be sets indeterminately identical to $\varnothing$; they, unlike it, have indeterminate (individual) members.

The following structural axioms are straightforward:

[Pairs] $\exists S \forall u[u \in S \Longleftrightarrow u = x \vee u = y]$,

[Union][17] $\exists S \forall z[z \in S \Longleftrightarrow \exists u(z \in u \,\&\, u \in X)]$,

[Power Set] $\exists S \forall z[z \in S \Longleftrightarrow z \subseteq X]$, where '$x \subseteq y$' $=_{\mathrm{df}}$ '$\neg x \in I \,\&\, \neg y \in I \,\&\, \forall u(u \in x \Longrightarrow u \in y)$' ,

[Infinity] $\exists S[\varnothing \in S \,\&\, \forall Y(\rhd Y \in S \Longrightarrow \exists Z[Z \in S \,\&\, \forall U(U \in Z \Longleftrightarrow U = Y \vee U \in Y)])]$.

Our Replacement scheme will require rank restrictions. To formulate the rank restrictions we first define the ordinals. As a preliminary, we call a set 'tight' if its indeterminate members are all "forced" by being indeterminately identical to some determinate member:

$$\mathrm{Tight}(x) =_{\mathrm{df}} \neg x \in I \,\&\, \forall y[\lhd y \in x \Longrightarrow \exists z(\rhd z \in x \,\&\, \lhd\, y = z)] \,.$$

Transitive sets are ones whose determinate members are determinate subsets:

$$\mathrm{Transitive}(x) =_{\mathrm{df}} \neg x \in I \,\&\, \forall y(\rhd y \in x \Longrightarrow \rhd y \subseteq x) \,.$$

Ordinals are defined as tight transitive sets whose determinate members are all tight and transitive:

$$\mathrm{Ord}(x) =_{\mathrm{df}} \mathrm{Tight}(x) \,\&\, \mathrm{Trans}(x) \,\&\, \forall y[\rhd y \in x \Longrightarrow \mathrm{Tight}(y) \,\&\, \mathrm{Trans}(y)] \,.$$

So-defined, being an ordinal is a bivalent condition.

Theorem 3.1 $\mathit{Ord}(x) \vee \neg \mathit{Ord}(x)$.

To express ordinal comparison, we define

$$x < y =_{\mathrm{df}} \mathrm{Ord}(x) \,\&\, \mathrm{Ord}(y) \,\&\, x \in y \,,$$

$$x \leq y =_{\mathrm{df}} x < y \vee x = y \,.$$

In order for the ordinals to be well-ordered we take this as an axiom:

[Least Ordinal] $\exists x[\mathrm{Ord}(x) \,\&\, \rhd x \in S] \Longrightarrow \exists x[\mathrm{Ord}(x) \,\&\, \rhd x \in S \,\&\, \forall y[\mathrm{Ord}(y) \,\&\, \rhd y \in S \Longrightarrow x \leq y]]$.

Hereafter we use small Greek letters to range over the ordinals. To complete the structuring of the ordinals we adopt[18]

[Ordinal Non-Self-$\in$] $\neg \alpha \in \alpha$.

The following axioms constrain rankings. The first two resemble parts of a familiar definition of ranking by recursion, but they are stated here as axioms because 'ranks' is primitive notation:

[Individuals Not Ranked] $x \in I \Longrightarrow \neg y$ ranks x ;

[Ranking for Sets] z ranks $X \Longleftrightarrow z$ is the least ordinal such that $\forall y[\exists x(\rhd x \in X \,\&\, y \text{ ranks } x) \Longrightarrow y < z]$.

As discussed earlier, we posit that no indeterminate member of a set exceeds all the determinate members in rank:

[Rank Limitation of Indeterminate Members]

$$\lhd x \in X \;\&\; \alpha \text{ ranks } x \Longrightarrow \exists y \exists \beta[\rhd y \in X \;\&\; \beta \text{ ranks } y \;\&\; \alpha \leq \beta] \,.$$

Our final axiom scheme is Replacement. Suppose we have a

(i) bivalent relational formula $x\Psi y$ which
(ii) is functional on a set S, and suppose that $x\Psi y$ "projects" from S a condition satisfying
(iii) the rank constraint and
(iv) DDiff for set membership.

Then its range for domain S is a set:

[Replacement]

If

(i) $u\Psi y \vee \neg u\Psi y$;
(ii) $\lhd u \in S \;\&\; \lhd v \in S \;\&\; u\Psi y \;\&\; v\Psi z \;\&\; u = v \Longrightarrow y = z$;
(iii) $\lhd u \in S \;\&\; u\Psi Y \Longrightarrow \exists v \exists z[\rhd v \in S \;\&\; v\Psi z \;\&\; \exists\alpha\exists\beta[\alpha \text{ ranks } Y \;\&\; \beta \text{ ranks } z \;\&\; \alpha \leq \beta]]$;
(iv) $\rhd\exists x(x \in S \;\&\; x\Psi u) \;\&\; \rhd\neg\exists y(y \in S \;\&\; y\Psi v) \Longrightarrow \neg u = v$.

Then

$$\exists X \forall y[y \in X \Longleftrightarrow \exists z[z \in S \;\&\; z\Psi y]] \,.$$

3.3 Theorems The following are some useful theorems.

Theorem 3.2 $\exists\alpha[\alpha \textit{ ranks } X]$.

Theorem 3.3 $x \textit{ ranks } y \vee \neg x \textit{ ranks } y$.

Theorem 3.4 $\alpha \textit{ ranks } x \;\&\; \beta \textit{ ranks } x \Longrightarrow \alpha = \beta$.

Theorem 3.5 $\lhd Y \in X \Longrightarrow \exists\alpha\exists\beta[\alpha \textit{ ranks } Y \;\&\; \beta \textit{ ranks } X \;\&\; \alpha < \beta]$.

Theorem 3.6 $\lhd X = Y \Longrightarrow \exists\alpha[\alpha \textit{ ranks } X \;\&\; \beta \textit{ ranks } Y]$.

Separation follows from [Replacement]. Given a set S and formula Φ, a separated set exists if Φ satisfies DDiff with respect to the members of S and if the appropriate rank constraint is satisfied.

Theorem 3.7 ([Separation])

If

(i) $\rhd(x \in Z \;\&\; \Phi x) \;\&\; \rhd\neg(y \in Z \;\&\; \Phi y) \Longrightarrow \neg x = y$,
(ii) $\lhd(Y \in Z \;\&\; \Phi Y) \Longrightarrow \exists x[\rhd(x \in Z \;\&\; \Phi x) \;\&\; \exists\alpha\exists\beta[\alpha \textit{ ranks } x \;\&\; \beta \textit{ ranks } Y \;\&\; \alpha \geq \beta]]$,

then $\quad \exists S \forall x[x \in s \Longleftrightarrow x \in Z \,\&\, \Phi x]$.

Foundation does not hold in its transliteral formulation;[19] instead it holds with membership replaced by greater membership.

Theorem 3.8 $\quad \exists y \lhd y \in X \Longrightarrow \exists y[\lhd y \in X \,\&\, \neg\exists z(\lhd z \in y \,\&\, \lhd z \in X)]$.

(Most, though not all, of the $\lhd$'s are redundant. The theorem is logically equivalent to the following.

Theorem 3.9 $\quad \exists y y \in X \Longrightarrow \exists y[\lhd y \in X \,\&\, \neg\exists z(z \in y \,\&\, z \in X)])$.

Determinate Tight Separation: Separation is restricted by the DDiff and rank constraints which can be irksome in practice. If we want a separated set and we care only about getting a set with a certain determinate membership, then the constraints may be ignored. For any set S and formula Φ we have the following.

Theorem 3.10 ([Determinate Tight Separation]) $\quad \exists X[\mathit{Tight}(X) \,\&\, \forall z[\rhd z \in X \Longleftrightarrow \rhd(z \in S \,\&\, \Phi z)]]$.

Proof: Let the 'Φx' in [Separation] be '$\exists y[\rhd(y \in S \,\&\, \Phi y) \,\&\, x = y]$'). $\square$

Tightenings: As a corollary, every set has a "tightening".

Theorem 3.11 $\quad \exists X[\mathit{Tight}(X) \,\&\, \forall z[\rhd z \in X \Longleftrightarrow \rhd z \in S]]$.

3.4 Bivalent applications The above is the full general theory, designed to accommodate both objects and sets that may be indeterminately identical. There are complications, such as the constraints on Separation and Replacement, but for familiar purposes, familiar techniques remain valid. For example, for considering the foundations of classical mathematics, one usually works with a pure version of ZF. Classical ZF is equivalent to a subtheory of IZFU. That is, within IZFU one can define the *hereditarily pure tight sets*, which correspond to the pure sets of ZF. Define a hereditarily pure tight set of rank α as follows.

$$\mathrm{HPT}_\alpha(x) \Longleftrightarrow \mathrm{Tight}(x) \,\&\, \forall y[\rhd y \in x \Longrightarrow \exists \beta[\beta < \alpha \,\&\, \mathrm{HPT}_\beta(y)]] \,.$$

The hereditarily pure tight sets are then given by

$$\mathrm{HPT}(x) =_{\mathrm{df}} \exists \alpha : \mathrm{HPT}_\alpha(x) \,.$$

The HPT sets resemble the usual ones from ZF; using bracket notation (not officially introduced) for finite sets, the first few are

One of rank 0: $\varnothing$
One of rank 1: $\{\varnothing\}$
Two of rank 2: $\{\varnothing, \{\varnothing\}\}, \{\varnothing\}$
Four of rank 3: $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \{\varnothing, \{\varnothing, \{\varnothing\}\}\}, \{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \{\{\varnothing, \{\varnothing\}\}\}$
$\vdots$

One can easily establish by induction that both identity and membership are bivalent relations between HPT sets. Further, suppose that our primitive predicates are limited to '$\in$', '$=$', and 'ranks'. Then if all quantifiers are relativized to HPT sets, the axioms given above all hold, and yield HPT sets. For example, if '$x \subseteq y$' is redefined as '$\forall z(\mathrm{HTP}(z) \Longrightarrow (z \in x \Longrightarrow z \in y))$' then the power set axiom holds and yields an HPT set. Most importantly, the restrictions on Separation and Replacement concerning DDiff and ranks vanish. So one can simply take over all classical results of ZF into this theory.

4 A classical interpretation of IZFU In this section we define a class of structures within classical ZFU to represent the objects and sets of IZFU. Then we describe how to convert formulas of IZFU into formulas of ZFU, converting descriptions of the objects and sets of IZFU into descriptions of their representatives in ZFU. The axioms of IZFU convert into theorems of ZFU, thus establishing the consistency of IZFU relative to ZFU.

4.1 Representing indeterminate sets We suppose we have a set OB which we think of as consisting of *object representatives*, on which is given a symmetric and irreflexive relation ¿. The idea is that '$a = b$' represents determinate identity of the objects represented by 'a' and 'b', 'a ¿ b' represents indeterminacy of identity of the objects represented by 'a' and 'b', and '$\neg a = b$ & $\neg a$ ¿ b' represents determinate nonidentity of the things represented by 'a' and 'b'.[20] The goal is to extend this domain OB of object representatives by adding set-representatives on top of it, things that represent sets whose properties and relations may themselves be indeterminate. In addition to identity, we will need to represent determinate membership, determinate nonmembership, and indeterminacy of membership.

We work entirely within a classical bivalent logic with the principles of ZFU; the ur-elements of ZFU are the object representatives mentioned above (we assume that OB is a set). We use uppercase letters to range over sets and lowercase letters to range over both individuals and sets.

Below we will define a *set representative* as a pair consisting of a set representing the determinate members of the represented set and a set representing the determinate *or* indeterminate members of the represented set. On analogy with talk about organizations, we call a determinate *or* indeterminate member of a set a "greater member" of the set. So if $\langle A, B\rangle$ is such a set representative, the classical members of A represent the determinate members of the indeterminate set represented by $\langle A, B\rangle$, and the classical members of B represent the greater members of the set represented by $\langle A, B\rangle$. An entity not in B (and thus also not in A) represents a determinate nonmember of the set represented by $\langle A, B\rangle$ (or represents nothing at all). Although these are the guiding intuitions, nothing is said here literally about the indeterminate sets themselves which are represented *by* our set-representatives; the theory is formulated purely in terms of the set-representatives themselves, which are ordinary pairs of classical sets in ZFU. We characterize the domain of "representatives of indeterminate sets" on top of the domain of object representatives by a recursive definition of *being a set-representative of a given leve*l. In the definition to be given, '$x \in^{+} A$' will mean that the entity represented by x is determinately a member of the set represented by A, and '$x \in^{\#} A$' will stand for greater membership of the things represented. It will also be handy to have similar notions for identity: '$x =^{+} y$' will mean that x and y represent determinately identical sets or individuals, and '$x =^{\#} y$' will mean that x and y represent things that are determinately or indeterminately identical.

We call the set-representatives “set-reps” for short. Set-reps, together with the entities in OB that represent objects, are “rep-entities”. Set-reps are defined by “levels” which we call “Rlevels”. The Rlevels are the classical ordinals. Members of OB have no Rlevel. Set-reps of Rlevel 0 are set-reps of object representatives. Set-reps of Rlevel 1 are those that contain some set-reps of Rlevel 0 and also perhaps some object-representatives. And so on, for all transfinite levels.

In the following, ‘α’ and ‘β’ range over the classical ordinals. We read ‘Set-$\text{rep}_\alpha(S)$’ as ‘S is a set-rep of Rlevel α’. Recall that when x and y are object-representatives, we are already given the relation ‘x ¿ y’. This relation does not relate any object-representative to any set-rep. We define Set-$\text{rep}_\alpha(x)$ and $x =^{\#}_{\alpha} y$ by simultaneous recursion.

4.2 Set-reps by Rlevel Set-$\text{rep}_0(S)$ if and only if S is any pair $\langle A, B\rangle$ such that

(i) $A \subseteq B \subseteq \text{OB}$;

(ii) $\forall x[x \in \text{OB} \;\&\; \exists y[y \in A \;\&\; y$ ¿ $x] \supset x \in B]$.

If $\alpha > 0$, then Set-$\text{rep}_\alpha(S)$ if and only if S is any pair $\langle A, B\rangle$ such that

(i) α is the least ordinal such that A and B are sets whose members are rep- entities of Rlevel $< \alpha$;

(ii) $\forall x[(\exists \beta < \alpha)[(\text{Set-rep}_\beta(x) \vee x \in \text{OB}) \;\&\; \exists y(y \in A \;\&\; y =^{\#}_{\beta} x)] \supset x \in B]$;

(iii) B has no member with an Rlevel greater than that of every member of A.

4.2.1 Determinate-or-indeterminate identity for Set-reps

$X =^{\#}_{\alpha} Y =_{\text{df}}$ Set-$\text{rep}_\alpha(X)$ & Set-$\text{rep}_\alpha(Y)$ & the first member of X is a subset of the second member of Y, and vice versa.

Based on the above definition we can give these characterizations independently of Rlevel.

4.2.2 Set-reps in general

$$\text{Set-rep}(S) =_{\text{df}} \exists \alpha \; \text{Set-rep}(S, \alpha) \; .$$

4.2.3 R-membership in general

$x \in^{+} y =_{\text{df}}$ Set-rep(y) & $(x \in \text{OB} \vee \text{Set-rep}(x))$ & $x \in$ the first member of y .

$x \in^{\#} y =_{\text{df}}$ Set-rep(y) & $(x \in \text{OB} \vee \text{Set-rep}(x))$ & $x \in$ the second member of y .

4.2.4 R-Identity in general

$x =^{+} y =_{\text{df}}$ $[x \in \text{OB} \;\&\; y \in \text{OB} \;\&\; x = y] \vee [\text{Set-rep}(x) \;\&\; \text{Set-rep}(y) \;\&\; x = y]$.

$x =^{\#} y =_{\text{df}}$ $[x \in \text{OB} \;\&\; y \in \text{OB} \;\&\; (x = y \vee x$ ¿ $y)] \vee [\text{Set-rep}(x) \;\&\; \text{Set-rep}(y) \;\&\; \exists \alpha : x =^{\#}_{\alpha} y]$.

Suppose A is a set of rep-entities. We define $A\circ$ to be the set obtained by supplementing A with those rep-entities that are indeterminately identical to members of A:

$$A\circ =_{\mathrm{df}} \{x|\exists y(y \in A \,\&\, x =^{\#} y)\}.$$

Using this notion we can give a compact statement characterizing set-reps.

Theorem 4.1 *Suppose $\langle A, B\rangle$ is a pair of sets of Rep-entities. Then $\langle A, B\rangle$ is a set-rep if and only if $A\circ \subset B$ and no member of B has a higher Rlevel than every member of A.*

4.3 R-ordinals To save syntactic complexity we use bold letters to represent variables ranging over rep-entities:

'$\forall\mathbf{A}\ldots$' abbreviates '$\forall A(A$ is a set-rep $\supset \cdots)$'
'$\forall\mathbf{x}\ldots$' abbreviates '$\forall x(x \in \mathrm{OB} \vee x$ is a set-rep $\supset \cdots)$'

Definition 4.2 (R-Ordinals) An ordinary ordinal is a transitive set of transitive sets. We will represent these by Rordinals, which are one kind of tight set-rep. We call a set-rep *tight* if its greater members are exactly its determinate members plus things indeterminately identical with them:

$$\mathrm{Tight}(\mathbf{x}) =_{\mathrm{df}} \mathrm{Set\text{-}rep}(\mathbf{x}) \,\&\, \forall\mathbf{z}[\mathbf{z} \in^{\#} \mathbf{x} \supset \exists\mathbf{y}(\mathbf{y} \in^{+} \mathbf{x} \,\&\, \mathbf{z} =^{\#} \mathbf{y})]\,.$$

Rsubset relations are defined as:

$$\mathbf{x} \subseteq^{+} \mathbf{y} =_{\mathrm{df}} \mathrm{Set\text{-}rep}(\mathbf{x}) \,\&\, \mathrm{Set\text{-}rep}(\mathbf{y}) \,\&\, \forall\mathbf{z}(\mathbf{z} \in^{+} \mathbf{x} \supset \mathbf{z} \in^{+} \mathbf{y}) \,\&\, \forall\mathbf{z}(\mathbf{z} \in^{\#} \mathbf{x} \supset \mathbf{z} \in^{\#} \mathbf{y});$$
$$\mathbf{x} \subseteq^{\#} \mathbf{y} =_{\mathrm{df}} \mathrm{Set\text{-}rep}(\mathbf{x}) \,\&\, \mathrm{Set\text{-}rep}(\mathbf{y}) \,\&\, \forall\mathbf{z}(\mathbf{z} \in^{+} \mathbf{x} \supset \mathbf{z} \in^{\#} \mathbf{y}).$$

A set-rep is transitive if its determinate members are determinate subsets:

$$\mathrm{Trans}(\mathbf{x}) =_{\mathrm{df}} \mathrm{Set\text{-}rep}(\mathbf{x}) \,\&\, \forall\mathbf{z}(\mathbf{z} \in^{+} \mathbf{x} \supset \mathbf{z} \subseteq^{+} \mathbf{x})\,.$$

An Rordinal is a tight transitive set-rep whose determinate members are tight transitive set-reps:

$$\mathrm{Rordinal}(\mathbf{x}) =_{\mathrm{df}} \mathrm{Tight}(\mathbf{x}) \,\&\, \mathrm{Trans}(\mathbf{x}) \,\&\, \forall\mathbf{y}[\mathbf{y} \in^{+} \mathbf{x} \supset \mathrm{Tight}(\mathbf{y}) \,\&\, \mathrm{Trans}(\mathbf{y})]\,.$$

The Rordinals are isomorphic to the ordinary ordinals (including 0): for any ordinal α there can be correlated a unique Rordinal of Rlevel α. Notice that the first member of any tight set-rep uniquely determines its second member. If A is a set of set-reps, let $\oplus A$ be the unique pair $\langle A, A\circ\rangle$. Then $\oplus A$ will be a set-rep, indeed, a tight set-rep. Let us say that a classical ordinal α is *correlated with* Rordinal X if and only if the following conditions are met:

Basis:	0 corr $\langle\varnothing, \varnothing\rangle$
Successor:	$\alpha + 1$ corr $\oplus\{\mathbf{Y}|\exists\beta(\beta \leq \alpha \,\&\, \beta$ corr $\mathbf{Y})\}$
Limit:	If λ is a limit ordinal then λ corr $\oplus\{\mathbf{Y}|\exists\beta(\beta < \lambda \,\&\, \beta$ corr $\mathbf{Y})\}$.

Then each set correlated with an ordinal is an Rordinal, and there are no other Rordinals. The first few Rordinals are:

0	$\oplus\varnothing$	$\langle\varnothing,\varnothing\rangle$
1	$\oplus\{\oplus\varnothing\}$	$\langle\{\langle\varnothing,\varnothing\rangle\},\{\langle\varnothing,A\rangle \mid A\subseteq \mathrm{OB}\}\rangle$
2	$\oplus\{\oplus\varnothing,\oplus\{\oplus\varnothing\}\}$	...

Hereafter we will speak as much as possible in terms of set-reps. In keeping with this, we define the Rrank of a set-rep as the unique Rordinal correlated with α where α is the Rlevel of the set- rep. We will speak hereafter of Rranks instead of Rlevels.

4.4 Analogues of axioms of IZFU We give two transformations: $\tau_{\triangleright}$ and $\tau_{\triangleleft}$. The former produces a classical formula stating the truth-conditions of the formula of IZFU to which it is applied, and the latter specifies the nonfalsehood conditions. The *analogues* of axioms are taken to be their truth conditions, but we need to define both transformations because they interact.

Atomic Formulas

If s and r are terms, then

$\tau_{\triangleright}[r\in s]=r'\in^{+}s'$
$\tau_{\triangleleft}[r\in s]=r'\in^{\#}s'$

$\tau_{\triangleright}[r=s]=r'=^{+}s'$
$\tau_{\triangleleft}[r=s]=r'=^{\#}s'$

$\tau_{\triangleright}[s \text{ ranks } t]=s' \text{ RRanks } t'$
$\tau_{\triangleleft}[s \text{ ranks } t]=s' \text{ RRanks } t'$

$\tau_{\triangleright}[P(s)]=P^{+}(s')$ for arbitrary predicate P other than ‘ $\in$’, ‘$=$’, ‘ranks’[21]
$\tau_{\triangleleft}[P(s)]=P^{\#}(s')$ for arbitrary predicate P other than ‘ $\in$ ’, ‘$=$’, ‘ranks’

where

$s'=\mathbf{s}$ if s is a variable
$\varnothing'=\langle\varnothing,\varnothing\rangle$
$\mathrm{OB}'=\langle\mathrm{OB},\mathrm{OB}\rangle$

Complex Formulas

$\tau_{\triangleright}[\forall x\Phi x]=\forall\mathbf{x}\tau_{\triangleright}[\Phi\mathbf{x}]$ (similarly with large-letter variables)
$\tau_{\triangleleft}[\forall x\Phi x]=\forall\mathbf{x}\tau_{\triangleleft}[\Phi\mathbf{x}]$ (similarly with large-letter variables)
$\tau_{\triangleright}[\exists x\Phi x]=\exists\mathbf{x}\tau_{\triangleright}[\Phi\mathbf{x}]$ (similarly with large-letter variables)
$\tau_{\triangleleft}[\exists x\Phi x]=\exists\mathbf{x}\tau_{\triangleleft}[\Phi\mathbf{x}]$ (similarly with large-letter variables)

$\tau_{\triangleright}[\Phi \mathbin{\&} \Psi]=\tau_{\triangleright}[\Phi] \mathbin{\&} \tau_{\triangleright}[\Psi]$ (similarly for $\vee$)
$\tau_{\triangleleft}[\Phi \mathbin{\&} \Psi]=\tau_{\triangleleft}[\Phi] \mathbin{\&} \tau_{\triangleleft}[\Psi]$ (similarly for $\vee$)

$\tau_{\triangleright}[\Phi\Longrightarrow\Psi]=(\tau_{\triangleright}[\Phi]\supset\tau_{\triangleright}[\Psi]) \mathbin{\&} (\tau_{\triangleleft}[\Phi]\supset\tau_{\triangleleft}[\Psi])$
$\tau_{\triangleleft}[\Phi\Longrightarrow\Psi]=(\tau_{\triangleright}[\Phi]\supset\tau_{\triangleleft}[\Psi])$

$\tau_{\triangleright}[\Phi\Longleftrightarrow\Psi]=(\tau_{\triangleright}[\Phi]\equiv\tau_{\triangleright}[\Psi]) \mathbin{\&} (\tau_{\triangleleft}[\Phi]\equiv\tau_{\triangleleft}[\Psi])$

$$\tau_{\triangleleft}[\Phi \Longleftrightarrow \Psi] = (\tau_{\triangleright}[\Phi] \supset \tau_{\triangleleft}[\Psi]) \ \&\ (\tau_{\triangleright}[\Psi] \supset \tau_{\triangleleft}[\Phi])$$

$$\tau_{\triangleright}[\neg\Phi] = \neg\tau_{\triangleleft}[\Phi]$$
$$\tau_{\triangleleft}[\neg\Phi] = \neg\tau_{\triangleright}[\Phi]$$

$$\tau_{\triangleright}[\triangleright\Phi] = \tau_{\triangleleft}[\triangleright\Phi] = \tau_{\triangleright}[\Phi]$$
$$\tau_{\triangleright}[\triangleleft\Phi] = \tau_{\triangleleft}[\triangleleft\Phi] = \tau_{\triangleleft}[\Phi]$$
$$\tau_{\triangleright}[\nabla\Phi] = \tau_{\triangleleft}[\nabla\Phi] = \tau_{\triangleleft}[\Phi] \ \&\ \neg\tau_{\triangleright}[\Phi]$$

The logical principles for IZFU described earlier are such that for any formula Φ, $\tau_{\triangleright}[\Phi]$ classically entails $\tau_{\triangleleft}[\Phi]$.

4.5 Checking the analogues of the axioms and definitions To show that the principles of IZFU transform into theorems of ZFU one merely needs to check them, case by case. This is tedious but straightforward. For illustration, we review here only some of the basic principles for sets.

4.5.1 Set Essence As noted in Section 2, the principle

[Set Essence] $\quad X = Y \Longleftrightarrow \forall z(z \in X \Longleftrightarrow z \in Y).$

is equivalent to the conjunction of

[Extensionality] $\quad \forall z(z \in X \Longleftrightarrow z \in Y) \Longrightarrow X = Y;$
[Set Indiscernability] $\quad X = Y \Longrightarrow \forall z(z \in X \Longleftrightarrow z \in Y).$

4.5.2 Extensionality The analogue of [Extensionality] is equivalent to the conjunction of

[EXT+] $\quad \forall \mathbf{z}[\mathbf{z} \in^{+} \mathbf{X} \equiv \mathbf{z} \in^{+} \mathbf{Y}] \ \&\ \forall \mathbf{z}[\mathbf{z} \in^{?} \mathbf{X} \equiv \mathbf{z} \in^{?} \mathbf{Y}] \supset \mathbf{X} =^{+} \mathbf{Y}.$
[EXT#] $\quad \forall \mathbf{z}[\mathbf{z} \in^{+} \mathbf{X} \supset \mathbf{z} \in^{\#} \mathbf{Y}] \ \&\ \forall \mathbf{z}[\mathbf{z} \in^{+} \mathbf{Y} \supset \mathbf{z} \in^{\#} \mathbf{X}] \supset \mathbf{X} =^{\#} \mathbf{Y}.$

Each of these is easily established from the recursive definition of set-reps and of the definitions of determinate and greater membership and of determinate and determinate-or-indeterminate identity.

4.5.3 Set Indiscernability The analogue of [Set Indiscernability] is equivalent to the conjunction of

[SI+] $\quad \mathbf{X} =^{+} \mathbf{Y} \supset \forall \mathbf{z}[[\mathbf{z} \in^{+} \mathbf{X} \equiv \mathbf{z} \in^{+} \mathbf{Y}] \ \&\ [[\mathbf{z} \in^{\#} \mathbf{X} \equiv \mathbf{z} \in^{\#} \mathbf{Y}]] \,.$
[SI#] $\quad \mathbf{X} =^{\#} \mathbf{Y} \supset \forall \mathbf{z}[[\mathbf{z} \in^{+} \mathbf{X} \supset \mathbf{z} \in^{\#} \mathbf{Y}] \ \&\ [[\mathbf{z} \in^{+} \mathbf{X} \supset \mathbf{z} \in^{\#} \mathbf{Y}]] \,.$

The first of these is a logical truth; the second is immediate from the definitions of determinate and greater membership and determinate-or-indeterminate identity.

4.5.4 DDiff for $\in$ set The principle [DDiff for $\in$ Set]

$$\triangleright x \in Z \ \&\ \triangleright \neg y \in Z \Longrightarrow \triangleright \neg x = y$$

transforms into

$$\mathbf{x} \in^{+} \mathbf{Z} \ \&\ \neg \mathbf{y} \in^{\#} \mathbf{Z} \supset \neg \mathbf{x} =^{\#} \mathbf{y}.$$

This is provable from the clauses labeled ‘(ii)’ in the definition of set-reps.

The remaining principles follow a similar pattern.

5 Hereditarily tight sets We turn now to the option, noted in Section 2, of allowing as indeterminate members of a set only those entities which are forced to be such by our espousal of the principle of Definite Difference. A *tight* set we defined to be one whose indeterminate members are precisely the entities indeterminately identical to some determinate member but definitely identical to none. The present theory then is the theory of sets which are "hereditarily tight", in the sense that they, their members, members of members, and so on, are all tight.

5.1 Motivation As we know from Section 2, "naive" indeterminate ZFU is inconsistent with the existence of indeterminacy. To avoid this result, we adopted in Section 3 the Rank Constraint: that the indeterminate members of a set can be of rank no higher than the determinate members. While we have shown that the constraint successfully avoids the trivialization of IZFU, this justification is in a certain sense purely negative. Furthermore, it may appear to be ad hoc in the following sense: while it imposes a restriction on the indeterminate members of a set by reference to its determinate members, no more fundamental account of the relation between determinate and indeterminate members is given which would explain this connection. For hereditarily tight sets a motivation is available which meets the two objections just given. It is a familiar part of the motivation for the hierarchical approach to set theory taken by ZF that sets are formed by "collecting" previously given objects.[22] If we suppose that what are primarily "collected" are the definite members of a set, and that objects are indeterminately collected only by being indeterminately identical to something primarily collected, then collection will always produce a tight set, and its iteration will produce the hierarchy of hereditarily tight sets.

In the following we give axioms for hereditarily tight sets and prove from these axioms that a tight set always has the same rank as any tight set indeterminately identical to it. Since the indeterminate members of a tight set simply are the entities indeterminately identical to some member, it follows at once that the rank constraint is satisfied.

5.2 Axioms By [Set Essence] the identity of a set is determined by its determinate and indeterminate members; the identity of a tight set is thus entirely determined by its determinate members. It might therefore seem that we could replace [Set Essence] by a form of extensionality which identifies sets with the same determinate members. This would, however, be to overlook the fact that [Set Essence] also governs the indeterminate identity of sets, and that the reduction of indeterminate membership to determinate membership plus indeterminate identity does not, by itself, involve a reduction of indeterminate identity to anything, even for sets. Thus we will retain [Set Essence], but almost all of our other axioms deal only with determinate membership (the one exception is the reduction principle itself). In view of this, it will be convenient in writing the axioms to use the conditional and biconditional defined as follows:[23]

$$\Phi \rightarrow \Psi =_{\mathrm{df}} \triangleright\Phi \Longrightarrow \triangleright\Psi$$
$$\Phi \longleftrightarrow \Psi =_{\mathrm{df}} (\Phi \rightarrow \Psi)\ \&\ (\Psi \rightarrow \Phi)$$

Our Axioms

[Set Essence]	$X = Y \Longleftrightarrow \forall z(z \in X \Longleftrightarrow z \in Y)$
[Reduction]	$\triangleleft x \in X \longleftrightarrow \exists y(\triangleleft x = y \ \& \ \triangleright y \in X)$
[Bivalence of Individuals]	$x \in I \vee \neg x \in I$
[Individuals Lack Members]	$x \in I \rightarrow \neg \exists y y \in x$
[NonTriviality]	$\exists x \in I \exists y \in I \triangledown x = y$
[Pairs$^{\mathrm{T}}$]	$\exists S \forall u[u \in S \longleftrightarrow u = x \vee u = y]$
[Union$^{\mathrm{T}}$]	$\exists S \forall z[z \in S \longleftrightarrow \exists u(z \in u \ \& \ u \in X)]$
[Power Set$^{\mathrm{T}}$]	$\exists S \forall z[z \in S \longleftrightarrow z \subseteq X]$
[Infinity$^{\mathrm{T}}$]	$\exists S[\varnothing \in S \ \& \ \forall Y(Y \in S \rightarrow \exists Z[Z \in X \ \& \ \forall U(U \in Z \rightarrow U = Y \vee U \in Y)])]$
[Replacement$^{\mathrm{T}}$]	
If	$\forall u \in S(u \Psi y \ \& \ u \Psi z \rightarrow y = z)$
Then	$\exists X \forall y[y \in X \longleftrightarrow \exists z[z \in S \ \& \ z \Psi y]]$
[Foundation$^{\mathrm{T}}$]	$\exists y y \in X \rightarrow \exists y[y \in X \ \& \ \neg \exists z(z \in y \ \& \ z \in X)]$

Aside from Reduction, Set Essence, and the axioms for individuals, the above are (if we read '$\rightarrow$' as material implication) part of a standard set of axioms for ZF. The missing axiom is

[Extensionality] $\quad \forall z(z \in X \longleftrightarrow z \in Y) \rightarrow X = Y$

but this follows from Set Essence. In fact, it can be shown that if one replaces in a standard formulation of ZF each atomic formula Φ by $\triangleright\Phi$, the resulting formula is provable from the above axioms if the original is provable in ZF. In particular, we can show that transfinite recursion in all its many forms is acceptable (for such formulas).[24] In particular, we can define a functor 'Rank' satisfying

$$\mathrm{Rank}(X) = \text{the least ordinal } \alpha \text{ such that for all } Y, Y \in X \rightarrow \mathrm{Rank}(Y) < \alpha.$$

We now prove by recursion on rank the Rank Restriction:

$$\forall X \forall Y(\triangleleft X = Y \rightarrow \mathrm{Rank}(X) = \mathrm{Rank}(Y)) \ .$$

Indeed, suppose that for Z of rank less than X we have

$$\forall Y(\triangleleft Z = Y \rightarrow \mathrm{Rank}(Z) = \mathrm{Rank}(Y)) \ ,$$

and suppose $\triangleright X = Y$. Then if $Z \in Y$, by Set Essence we have $\triangleleft Z \in X$ and hence by Reduction $(\triangleright y \in X \ \& \ \triangleleft y = Z)$ for some y. Now y must be a set and since $\triangleright y \in X$, $\mathrm{Rank}(y) < \mathrm{Rank}(X)$ so by the inductive hypothesis $\mathrm{Rank}(Z) = \mathrm{Rank}(y) < \mathrm{Rank}(X)$. It follows that for all $Z \in Y$, $\mathrm{Rank}(Z) < \mathrm{Rank}(X)$ and hence $\mathrm{Rank}(Y) \leq \mathrm{Rank}(X)$. A similar argument shows that $\mathrm{Rank}(X) \leq \mathrm{Rank}(Y)$ which establishes the conclusion of the inductive step.

Which of the axioms of IZFU continue to hold for tight sets? [Set Essence] remains an axiom, and [DDiff for $\in$ Set] follows immediately from [Reduction]. The axioms for individuals are largely the same as before; the present Nontriviality implies the earlier one. [Empty Sets] is, of course, false, since the sets (save one) it generates are not tight; in its place we have [Emptiest Set] as a theorem. It is not hard to show that [Pairs] and (more surprisingly) [Power Set] are theorems, as is [Infinity]. On the other hand, [Replacement] is not a theorem—not surprising in view of the fact that it is not forced to build tight sets. More surprising is the fact that [Union] fails; even if X, Y and $\{X, Y\}$ are tight sets, there need, in general, be no tight set Z such that $x \in Z \iff \exists U(x \in U \;\&\; U \in \{X, Y\})$; let a, b, c be individuals such that $\triangleleft a = b$ and $\triangleleft b = c$ but $a \neq c$, and let X, Y both be $\{a\}$. Then we have $\triangleleft\{a\} = \{b\}$, so $\triangleleft\{b\} \in \{X, Y\}$, but also $\triangleleft c \in \{b\}$, so if Z satisfies the above condition, then, on the one hand, a is the only definite member of Z, but also c is an indefinite member, yet c is not indefinitely identical to a. Hence Z is not tight.[25]

6 Alternatives: fusion sets, status patterns, and conceptual sets When we considered options worth developing we postponed certain alternatives; they are discussed here. We postponed Fusion Sets because the option is not particularly interesting. We postponed Status Patterns because they do not satisfy our criteria for sets, but they have a simple theory that very closely resembles that of IZFU. We postponed Conceptual Sets merely for convenience; this theory also closely resembles that of IZFU.

6.1 Fusion Sets There are two ways to get universal determinacy of identity for sets. Each way involves making the identity conditions for sets be insensitive in some way to the question of which members they have. At one extreme we determinately distinguish sets even when it is indeterminate whether they have the same members; this yields what we call Status Patterns (Section 6.2), which we have called nonsets for this reason.

At the other extreme, we refuse to permit the existence of any pair of sets when it would be indeterminate whether they have the same members. As a result, when it is indeterminate whether i is j, if we try to make a set containing i we must also include j. So the "unit" set $\{i\}$ does not exist; only $\{i, j\}$ exists, and so for purposes of set theory, i and j are treated as a unit. For this reason we call these "Fusion Sets". This limitation is extreme. Suppose that it is indeterminate whether i is j, and indeterminate whether j is k, but suppose it is determinately true that $i \neq k$. For purposes of making sets we cannot distinguish i from j, and we cannot distinguish j from k; as a result we cannot distinguish i from k even though they are determinately distinct. In interesting cases, the theory is so impoverished that it has few useful applications. We do not develop it here.

6.2 Status patterns In every theory developed so far we assume that if it is indeterminate whether sets A and B have the same members then it is indeterminate whether $A = B$. If this principle is abandoned, one can obtain a theory of what we call "status patterns" that closely resembles set theory in many respects. A status pattern is a classification of all entities into three statuses: definite member, indeterminate member, and nonmember, with the assumption that two such status patterns are definitely distinct if they do not classify every entity in exactly the same way, even if there is nothing that one classifies as a member and the other classifies as a nonmember. Status patterns satisfy the following modification of [Set Essence]:

[Status Pattern Essence] $\quad \forall X \forall Y[X = Y \iff \triangleright\forall z(z \in X \iff z \in Y)]\,.$

This entails the bivalence of identity of status patterns:

$$\forall X \forall Y[X = Y \vee \neg X = Y]\,.$$

Status Patterns are worth comparing with our versions of set theory because although their essence has changed, the rest of the proper axioms governing them are virtually the same as those for sets. For example, if we alter the theory of IZFU by replacing [Set Essence] with [Status Pattern Essence] and make no other changes, the resulting theory is consistent. In order to show this we can repeat the construction of Section 4 except that we define greater identity between set-reps to coincide with determinate identity. That is, in the recursive definition of set-rep we replace

$$X =^{\#}_{\alpha} Y =_{\mathrm{df}} \quad \text{Set-rep}_{\alpha}(X) \ \& \ \text{Set-rep}_{\alpha}(Y) \ \& \ \text{the first member of } X \text{ is a subset of the second member of } Y\text{, and vice versa.}$$

by

$$X =^{\#}_{\alpha} Y =_{\mathrm{df}} \quad \text{Set-rep}_{\alpha}(X) \ \& \ \text{Set-rep}_{\alpha}(Y) \ \& \ X = Y\ .$$

In order for [Nontriviality] to hold, we also need to posit the existence of a pair of members of OB related by '¿'. No other change is needed.

6.3 Conceptual Sets The theory of conceptual sets is the theory of IZFU with DDiff removed as an axiom and also removed as a constraint on Separation and Replacement. This theory can be shown consistent relative to ZFU by a construction similar to that used for IZFU; simply remove the DDiff clauses (the clauses numbered "(ii)") from the recursive definition of set-rep in Section 4.1.

NOTES

1. Cf. Frege [3], Section 54, which specifically concerns what he calls "concepts".

2. Nothing in this paper is aimed at solving the set-theoretic paradoxes.

3. There already exist versions of intuitionistic set theory in which identities may be indeterminate. It is still worth exploring the present approach, for a number of reasons. First, the intuitionistic picture of the world is only one special version that the possibility of indeterminacy of identity may take. Second, intuitionistic logic is a considerably more radical departure from classical logic than the system adopted in this paper; it is, in fact, a sublogic of the logic employed here, (That is, connectives may be defined within the present logic that obey all of the principles of intuitionistic logic, and more besides.) If an indeterminate set theory is developed within intuitionism, then it is difficult to see which unusual properties of the results are due to indeterminacy of identity, and which are due to the special nature of the underlying intiutionistic view.

4. One can take '$\neg$' and '$\Longrightarrow$' as primitive; the rest may then be defined as follows:

'$A \vee B$'	as	'$(A \Longrightarrow B) \Longrightarrow B$'
'$A \ \& \ B$'	as	'$\neg(\neg A \vee \neg B)$'
'$A \Longleftrightarrow B$'	as	'$(A \Longrightarrow B) \ \& \ (B \Longrightarrow A)$'
'$\rhd A$'	as	'$\neg(A \Longrightarrow \neg A)$'
'$\lhd A$'	as	'$\neg \rhd \neg A$'
'$\triangledown A$'	as	'$\neg \rhd A \ \& \ \lhd A$'

5. Throughout the text we use 'Φx' to mean an arbitrary formula with a substitutable free variable 'x', and 'Φt' for the result of substituting all free occurrences of 'x' by the term 't' with the understanding that no variable free in 't' may become bound by a quantifier in Φ.

6. All free variables in the formulas used in this paper are taken to be universally quantified with scope over the whole formula. Recall also that assertion of an axiom is equivalent to asserting it as true. So there is no difference in logical consequences between taking 'AX' as an axiom or taking '$\triangleright AX$' as an axiom.

7. This is not already a truth of our logic. That logic alone (using Leibniz's Law) lets us prove

$$\triangleright X = Y \Longrightarrow \forall z(z \in X \Longleftrightarrow z \in Y) \qquad \textit{(truth of logic).}$$

   But this only addresses the case where X and Y are definitely identical. If we want to also address the case in which it is indeterminate whether they are identical, we need to postulate something more. The principle [Set Indiscernability] adds that if it is indeterminate whether A and B are the same, then it is (true or) indeterminate whether they have the same members. (The 'true or' is ruled out by other considerations.) [Set Indiscernability] essentially adds to our logic the DDiff condition for formulas Φx of the form '$a \in x$'.

8. In the terminology of Van Inwagen [8] the question is whether a set's penumbra may extend beyond its fringe.

9. We include ur-elements because development of this theory is driven by its applications to individuals; see Parsons [4] and Woodruff and Parsons [9]. The theory is consistent with the assumption that the set of ur-elements is empty.

10. When we constrain Separation by DDiff for set membership the constraint applies to the *whole* condition specifying the separated set which is the conjunction:'$x \in Z$ & Φx', with Z the set being separated from, and Φ the separating formula.

11. One might consider weakening Union from '$\exists Y \forall z[z \in Y \Longleftrightarrow \exists u(z \in u \;\&\; u \in X)]$' to '$\exists Y \forall z[z \in Y \Longleftrightarrow \exists u(z \in u \;\&\; \triangleright u \in X)]$'. We have not explored this option.

12. Fine has suggested that this is independently motivated by the fact that differences in ranks are structural differences and structural differences should be definite.

13. We do not know whether it is possible to get elegant theories with looser options. Here is one sort of issue to be faced. Suppose that the indeterminate members of a set may be one rank higher than the determinate members. Let S be a finite set of finite rank α with no indeterminate members. Let R be a set obtained by adding to S an indeterminate member of rank α. Then R will be of rank $\alpha + 1$. Further, R will be indeterminately identical with S. As a result, R will be an indeterminate member of $\{S\}$, so the rank of $\{S\}$ must be at least $\alpha + 2$. So natural assumptions about how sets are rank-related to their unit sets are not preserved. We have not explored the significance of this.

14. Normally one proposes the axioms and then introduces the notion of rank by definition. But such a definition presupposes that Replacement has already been stated. In the present formulation we need to restrict Replacement using an already existing notion of rank. We introduce 'rank' as a primitive notion for this purpose; the conditions that are normally used to define it are posed as axioms instead of as a definition.

15. We have devised a notation and a choice of formulations of the axioms that make it possible to consider axioms that look like normal axioms of ZFU. But nonbivalent logic inevitably has its idiosyncrasies, and the reader should be aware that "slight" reformulations of some of these axioms can invalidate them. For example, the Power Set axiom justifies us in introducing Power Set notation validating

$$X \in \wp(Y) \Longrightarrow \forall z[z \in X \Longrightarrow z \in Y]\,.$$

But the following need not be true (it may be indeterminate):

$$\forall z[X \in \wp(Y)\ \&\ z \in X \Longrightarrow z \in Y].$$

This is an idiosyncrasy of the Łukasiewicz conditional. When reasoning from truths the differences do not usually matter, since when $X \in \wp(Y)$ is true the two forms are equivalent.

16. This axiom together with the previous one entail that I is a set (a nonindividual): $\neg I \in I$.

17. We would like to have a binary union operation on sets that satisfies

$$x \in A \cup B \Longleftrightarrow x \in A \vee x \in B\,.$$

This is usually definable as

$$A \cup B = \cup\{A, B\}\,,$$

but in the present axiomatization these are not equivalent; that is, we do not necessarily have

$$x \in \cup\{A, B\} \Longleftrightarrow x \in A \vee x \in B\,.$$

This is because $\cup\{A, B\}$ may have indeterminate members that are not indeterminate members of A or of B; these will be indeterminate members of sets indeterminately identical to A or to B. To get the usual binary operation $A \cup B$ on sets we can instead use Separation from $\cup\{A, B\}$:

$$\exists S \forall x[x \in S \Longleftrightarrow x \in \cup\{A, B\}\ \&\ (x \in A \vee x \in B)]\,.$$

18. Normally one derives well-ordering of the ordinals and non-self-membership from the Foundation axiom. Because Foundation takes a special form in this indeterminate framework, we have found it more revealing to posit well-ordering and non-self-membership for ordinals and then derive the proper version of Foundation as a theorem.

19. By the "transliteral" formulation of Foundation we mean

$$\forall X[\exists y y \in X \Longrightarrow \exists y(y \in X\ \&\ \neg\exists z(z \in y\ \&\ z \in X))]\,.$$

If there are any individuals at all, this is false. Suppose $i \in I$. By [Empty Sets] there is a set y which has no determinate members and which has i alone as an indeterminte member. By pairing, there is a set $\{y\}$ whose only determinate member is y. Let $X = \{y\} \cup y$. (See earlier note regarding binary unions.) Then X has y as its only determinate member, and X and y share i as an indeterminate member. This provides a nontrue instantiation of the above formula.

The point of a foundation axiom is to posit "blocks" to infinite descending chains. With the option of either determinate or indeterminate membership, a descending chain may fork, and the "natural" formulation above inaccurately says that the block will always be at the determinate side of the fork.

20. In [5] these relations are defined in terms of relations between sets of “ontons”, but the “ontic” details are irrelevant to the present construction, and they will not be discussed here.

21. The motivation for indeterminate set theory comes primarily from its intended applications to theories of individuals. These theories may contain arbitrary predicates in addition to ‘$\in$’, ‘$=$’, ‘ranks’. In our classical interpretation we suppose that such predicates come in pairs, with the axiom: $\forall x(P^{+}x \rightarrow P^{\#}x)$.

22. See, for instance, Shoenfield [7], p. 238.

23. Since the biconditional ‘$\longleftrightarrow$’ so defined is logically weaker than the biconditional ‘$\Longleftrightarrow$’ used in formulating IZFU, several of the set generation axioms stated here are weaker than the corresponding axioms of IZFU. In several cases (e.g., for [Pair Set]) they could be strengthened, but not in all cases; see discussion below.

24. See, e.g., Enderton [1], Chapters 7, 8.

25. In Note 17 we pointed out that [Union] does not deliver (by itself) a set with the appropriate membership conditions for A $\cup$ B. If we replaced [Union] by

$$[\text{Union}^*] \qquad \exists S\forall z[z \in S \Longleftrightarrow \exists u(\triangleright z \in u \;\&\; u \in X)]\,,$$

pairwise union would be directly underwritten; [Union*] is derivable from [Union$^{\mathrm{T}}$].

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