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ON TIME-PERIODIC SOLUTIONS OF SOME NONLINEAR PARABOLIC EQUATIONS WITH NONMONOTONE MULTIVALUED TERMS

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Dedicated to the memory of Professor Ioan I. Vrabie

Abstract. In this paper we study the existence of time periodic solutions to a class of nonlinear parabolic equations with multivalued nonlinear terms subject to the homogeneous Dirichlet boundary condition. We give two types of existence results: one for large periodic solutions with any large data, and the other for small periodic solutions with small data. Both concern the case where the nonlinear terms contain either a upper semicontinuous multivalued term or a lower semicontinuous multivalued term. Some applications of our results are also given.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and let $Q_T := \Omega \times [0,T]$ and $\Gamma_T := \partial\Omega \times [0,T]$ with $T > 0$. Consider the following

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time-periodic problem:

$$
\text{(P)}\quad \begin{cases} \dfrac{\partial}{\partial t} u(x,t) - \Delta u(x,t) \in -\partial\phi(u(x,t)) + G(x,t,u(x,t)), & (x,t) \in Q_T, \\ u(x,t) = 0, & (x,t) \in \Gamma_T, \\ u(x,0) = u(x,T), & x \in \Omega, \end{cases}
$$

where $\partial\phi$ is the subdifferential of the proper lower semicontinuous convex function $\phi\colon \mathbb{R} \to [0,\infty]$ and $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ is a nonmonotone multivalued mapping.

Such inclusions appear naturally in the study of the existence of time periodic solutions to parabolic equations with discontinuous nonlinearities, usually when one extends the discontinuous nonlinearity to a multivalued map by filling the jumps at the discontinuity points of the nonlinearity, and one passes to the problem of finding solutions to the corresponding differential inclusion. We refer to [25] where a time periodic problem with the homogeneous Dirichlet condition is studied for the equation

$$
Lu(x,t) = g(x,t,u(x,t)) \tag{1.1}
$$

driven by an uniformly parabolic operator

$$
Lu(x,t) := \frac{\partial}{\partial t} u(x,t) - \sum_{i,j=1}^{n} (a_{ij}(x,t)u_{x_i})_{x_j} + \sum_{j=1}^{n} b_j(x,t)u_{x_j} + c(x,t)u(x,t)
$$

with a discontinuous nonlinearity $g(x,t,u)$. To prove the existence of strong solutions for (1.1), one considers the problem of proving the existence of solutions for the differential inclusion

$$
Lu(x,t) \in [g_-(x,t,u(x,t)), g_+(x,t,u(x,t))] \tag{1.2}
$$

with $g_-(x,t,u) = \liminf_{v\to u} g(t,x,v)$ and $g_+(x,t,u) = \limsup_{v\to u} g(t,x,v)$.

Conditions on the points of discontinuity of the nonlinearity under which any solution of the inclusion (1.2) satisfies the equation (1.1) almost everywhere, were obtained in [8], [9]. In [25] the author gave the principle of lower and upper solutions for the existence of strong solutions without additional constraints on the jumping-up discontinuities of the nonlinearity.

On the other hand, the existence of periodic solutions in Banach spaces has been studied in many papers. As applications of general theorems, some of them dealt with the existence of periodic solutions of the differential inclusions (1.2) obtained by filling the jumps.

In [6], nonlinear parabolic equations are studied by using the method of upper and lower solutions, truncation and penalization techniques as well as results from the theory of operators of monotone type and a fixed point theorem for set-valued maps defined in ordered metric spaces due to [13]. Deuel and Hess [10] used the upper and lower solutions method to establish the existence of periodic solutions for a class of nonlinear parabolic problem.

In [18] a nonlinear periodic problem with a discontinuous nonmonotone nonlinearity is considered and using a lifting result for operators of type (S_+), a general surjectivity theorem for operators of monotone type and an auxiliary problem defined by truncation and penalization, the authors proved the existence of a solution lying between an upper solution and a lower solution.

The periodic problem in the context of abstract evolution equations was also addressed by Vrabie [28] and Hirano [15], but they both assumed that the nonlinear perturbation term are sublinear and single-valued. The abstract treatment for the periodic problems for parabolic equations is also given in [22], which can cover a wide class of nonlinear heat equations with multivalued perturbations as well as Navier–Stokes equations.

We mention that problems with maximal monotone terms (unilateral constraints), both for finite and infinite dimensional cases, can be found in the book of Vrabie [27].

The goal of this paper is to adapt and improve the techniques and arguments developed in [24] in order to apply them to (P) and obtain existence results for time periodic solutions, which improves some applications given in [22]. Our approach uses tools from multivalued analysis, together with the theory of nonlinear operators of monotone type and methods from the theory of nonlinear evolution equations.

We prove two types of existence results: one for large periodic solutions with any large data, and the other for small periodic solutions with small data. In both cases, we are concerned with the case where G is upper semicontinuous as well as the case where G is lower semicontinuous.

This paper is organized as follows. In Section 2, we recall some notations and basic definitions used in the following sections. In Section 3, we formulate our main results. Section 4 contains a few auxiliary results and Section 5 is devoted to the proof of main results. In Section 5, we exemplify the applicability of our results.

2. Notations and preliminaries

In this section we recall some notations and basic definitions from the nonlinear operator theory and the multivalued analysis which we shall use in the sequel. For further details we refer to [2], [3], [5], [16] and [24].

Let X and Y be topological spaces and 2^Y be the family of all subsets of Y. A multivalued mapping $F\colon X \to 2^Y$ is said to be *upper semi-continuous* (u.s.c., for short) on X if for every $x_0 \in X$ and for each open subset $V \subset Y$ such that $F(x_0) \subset V$, there exists a neighbourhood U of x_0 such that $F(x) \subset V$ for all $x \in U$. A multivalued map $F\colon X \to 2^Y$ is said to be *lower semi-continuous* (l.s.c., for short) on X if for each $x_0 \in X$ and for every $y_0 \in F(x_0)$

and any neighbourhood V of y_0, there exists a neighbourhood U of x_0 such that $F(x) \cap V \neq \emptyset$ for all $x \in U$.

Equivalently, a multivalued map $F\colon X \to 2^Y$ is upper semicontinuous (respectively, lower semicontinuous) on X if and only if, the set $F^-(C) := \{x \in X : F(x) \cap C \neq \emptyset\}$ (respectively, $F^+(C) := \{x \in X : F(x) \subset C\}$) is closed in X for each closed subset C of Y.

It is well known that if $F\colon X \to 2^Y$ is upper semicontinuous on X with closed values, then its *graph* $\mathrm{Gr}(F) := \{(x,y) \in X \times Y : y \in F(x)\}$ is closed in $X \times Y$. Conversely, if $F\colon X \to 2^Y$ has a closed graph and if for each $x \in X$, there exists a neighbourhood U of x such that $F(U) := \bigcup_{x \in U} F(x)$ is precompact, then F is u.s.c. on X.

By a *selection* of a multivalued map $F\colon X \to 2^Y$ we mean any function $f\colon X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.

Let (M,Σ) be a measurable space. We are particularly interested in the case where Σ is the σ-algebra $\mathcal{L}(Q_T)$ of Lebesgue measurable subsets of $Q_T := \Omega \times [0,T]$ and $M = Q_T$, where $T > 0$ and $\Omega \subseteq \mathbb{R}^N$ is a given open set, as well as the case where $M = X$ is a Banach space and $\Sigma := \mathcal{L}(Q_T) \otimes \mathcal{B}(X)$, where $\mathcal{L}(Q_T) \otimes \mathcal{B}(X)$ is the product $\sigma-$algebra on $Q_T \times X$ generated by sets of the form $A \times B$ with $A \in \mathcal{L}(Q_T)$ and $B \in \mathcal{B}(X)$ with $\mathcal{B}(X)$ being the Borel σ-algebra of X.

Let $\big(X, \|\cdot\|_X\big)$ be a separable Banach space. A closed valued multifunction $\Psi\colon M \to 2^X$ is said to be Σ-measurable (or simply, measurable) if, for every open set $U \subset X$, we have

$$\Psi^-(U) := \{\omega \in M : \Psi(\omega) \cap U \neq \emptyset\} \in \Sigma.$$

It is known that $\Psi\colon M \to 2^X$ is measurable if and only if for every $x \in X$, the map

$$\omega \mapsto d(x, \Psi(\omega)) := \inf\{\|z - x\|_X : z \in \Psi(\omega)\}$$

is a measurable $\overline{\mathbb{R}^+} = \mathbb{R}^+ \cup \{\infty\}$-valued function (see [16, Corollary 19, p. 142]). A multifunction $\Psi\colon M \to 2^X$ with nonempty valued is said to be graph measurable if

$$\mathrm{Gr}(\Psi) := \{(\omega, z) \in M \times X : z \in \Psi(\omega)\} \in \Sigma \otimes \mathcal{B}(X).$$

For multifunctions with closed values, the measurability implies the graph measurability, while the converse is true if Σ is complete.

Let Y be a Banach space with norm $\|\cdot\|_Y$ and $F\colon I := [0,T] \to 2^Y$ be a multivalued map. Then for $1 \leq p \leq \infty$ by S_F^p, we denote the set of all selections of F which belong to the Lebesgue–Bochner space $L^p(I,Y)$ that is

$$S_F^p = \{v \in L^p(I,Y) : v(t) \in F(t) \text{ a.e. } t \in I\}.$$

It is easy to check that for a graph measurable multifunction $F\colon I \to 2^Y$, the set S_F^p is nonempty if and only if $t \mapsto \inf\{\|x\|_Y : x \in F(t)\}$ is majorized by a $L^p(\Omega)$-function (see [16, Lemma 3.2, p. 175]).

In the remaining of this section we collect some definitions and properties concerning maximal monotone mappings.

Assume that H is a real Hilbert space with inner product $(\,\cdot\,,\,\cdot\,)_H = (\,\cdot\,,\,\cdot\,)$ and norm $\|\cdot\|_H$. Assume $A\colon H \to 2^H$ is a maximal monotone operator. The *minimal section* (or *minimal selection*) of A is the function $A^0\colon H \to H$ satisfying the following conditions:

$$A^0(x) \in A(x) \quad \text{and} \quad \|A^0(x)\|_H = \inf\{\|\xi\|_H : \xi \in A(x)\} \quad \text{for all } x \in D(A).$$

Recall that, the graph of any maximal monotone operator is *demiclosed*, that is, closed in $H \times H_w$, where H_w denote the space H furnished with the weak topology.

Let $\varphi\colon H \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function. We say that φ is *proper* if its *effective domain*

$$D(\varphi) := \{x \in H : \varphi(x) < +\infty\}$$

is nonempty. The multivalued map $\partial\varphi\colon H \to 2^H$ defined by

$$\partial\varphi(x) = \{g \in H : \varphi(y) - \varphi(x) \geq (g, y - x)_H \text{ for all } y \in H\} \tag{2.1}$$

is called the *subdifferential of* φ (in the sense of convex analysis).

It is known that if the *subdifferential* $\partial\varphi$ of a proper lower semicontinuous convex function φ is a maximal monotone operator and

$$D(\partial\varphi) := \{x \in H : \partial\varphi(x) \neq \emptyset\} \subset D(\varphi).$$

We shall use $\partial^0\varphi$ instead of $(\partial\varphi)^0$ to denote the minimal section of the maximal monotone operator $\partial\varphi$.

3. Main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset of $\mathbb{R}^N$, $T > 0$, $Q_T := \Omega \times [0, T]$ and $\Gamma_T := \partial\Omega \times [0, T]$. For a Banach space X, we denote by $C_\pi([0, T]; X)$ the set of all continuous maps $u\colon [0, T] \to X$ such that $u(0) = u(T)$.

To formulate our main results we introduce some conditions on the multivalued map $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$:

(GC) (Growth condition) There exist nonnegative numbers $k \in [0, 1)$, $q \in (0, 2^*)$, $C_q \geq 0$ and a function $a(\,\cdot\,,\,\cdot\,) \in L^1(Q_T)$ such that

$$|||G(x, t, u)|||^2 \leq |a(x, t)| + k|\partial^0\phi(u)|^2 + C_q|u|^{2(q-1)}, \tag{3.1}$$

for all $(x,t) \in Q_T$ and for all $u \in D(\partial\phi)$, where $|||G(x,t,u)||| := \sup\{|\xi| : \xi \in G(x,t,u)\}$ and

$$2^* := \begin{cases} \infty & \text{if } N = 1 \text{ or } 2, \\ \dfrac{2N}{N-2} & \text{if } N \geq 3. \end{cases} \tag{3.2}$$

(H_G^1) (u.s.c. condition) $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multivalued map with closed convex values such that:
 (a) for almost all $(x,t) \in Q_T$, $G(x,t,\,\cdot\,)\colon \mathbb{R} \to 2^{\mathbb{R}}$ is upper semicontinuous,
 (b) For each $u \in \mathbb{R}$, $G(\,\cdot\,,\,\cdot\,,u)\colon Q_T \to 2^{\mathbb{R}}$ is $\mathcal{L}(Q_T)$- measurable.

(H_G^2) (l.s.c. condition) $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multivalued map with closed values such that:
 (a) for almost all $(x,t) \in Q_T$, $G(x,t,\,\cdot\,)\colon \mathbb{R} \to 2^{\mathbb{R}}$ is lower semicontinuous,
 (b) $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ is $\mathcal{L}(Q_T) \otimes \mathcal{B}(\mathbb{R})$-measurable.

Then our main results are stated as follows.

Theorem 3.1 (Large periodic solution). *Let condition* (GC) *be satisfied with $q = 2$ and $C_q = C_2 < \lambda_1^2(1-k)$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ with domain $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. Assume further that condition* (H_G^1) *or* (H_G^2) *is satisfied. Then there exists a solution $u \in C_\pi([0,T]; H_0^1(\Omega))$ of* (P). *More precisely, it holds that*

$$\exists\, b(x,t) \in \partial\phi(u(x,t)), \quad \exists\, g(x,t) \in G(x,t,u(x,t)) \text{ such that} \tag{3.3}$$

$$\frac{\partial u}{\partial t} - \Delta u + b - g = 0 \quad \text{for a.e. } (x,t) \in Q_T, \tag{3.4}$$

$$\frac{\partial u}{\partial t}, \Delta u, b, g \in L^2\big(0,T; L^2(\Omega)\big). \tag{3.5}$$

Theorem 3.2 (Small periodic solution). *Let condition* (GC) *be satisfied with $2 < q < 2^*$. Assume further that* (H_G^1) *or* (H_G^2) *is satisfied. Then there exists a* (*sufficiently small*) *number $r_0 > 0$ such that, if $\|a\|_1 \leq r_0$, then problem* (P) *admits a solution $u \in C_\pi\big([0,T]; H_0^1(\Omega)\big)$ satisfying* (3.3)–(3.5). *Here*

$$\|a\|_1 = \sup_{0 \leq t < \infty} \int_t^{t+1} \big|\widetilde{a}(\,\cdot\,,s)\big|_{L^1(\Omega)}\, ds,$$

where $\big|\widetilde{a}(\,\cdot\,,s)\big|_{L^1(\Omega)}$ is the zero extension of $|a(\,\cdot\,,s)|_{L^1(\Omega)}$ to $[0,\infty)$.

4. Auxiliary results

In this section we assume that Ω is a bounded open subset of $\mathbb{R}^N$ with finite Lebesgue measure denoted by $|\Omega|$, $N \geq 1$, $\phi\colon \mathbb{R} \to [0,\infty]$ is a proper convex and lower semicontinuous function, $T > 0$, $Q_T := \Omega \times [0,T]$, and $\mathcal{H} := L^2(0,T;L^2(\Omega))$.

We here prepare two kinds of notions for the multivalued maps.

DEFINITION 4.1. Let $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ be a multivalued mapping. The multivalued map $\widetilde{G}\colon \mathcal{H}\to 2^{\mathcal{H}}$ defined by

$$\widetilde{G}(u) = \{g \in \mathcal{H} : g(x,t) \in G(x,t,u(x,t)) \text{ for a.e. } (x,t) \in Q_T\} \tag{4.1}$$

is called the realization of G in $\mathcal{H}$.

DEFINITION 4.2. We say that the realization $\widetilde{G}$ of G in $\mathcal{H}$ is a.e.-demiclosed if for any sequence of functions $(u_n)_{n\in\mathbb{N}}$ from Q_T into $\mathbb{R}$ which converges almost everywhere in Q_T to a function $u\colon Q_T \to R$ and for any sequence of functions $(g_n)_{n\in\mathbb{N}}$ from Q_T into $\mathbb{R}$ such that $g_n(x,t) \in G(x,t,u_n(x,t))$ for each $n \in \mathbb{N}$ and almost all $(x,t) \in Q_T$, which converges weakly in $\mathcal{H}$ to a function $g \in \mathcal{H}$, then one has $g \in \widetilde{G}(u)$, that is,

$$g(x,t) \in G(x,t,u(x,t)) \quad \text{for almost all } (x,t) \in Q_T.$$

Proposition 1 of [24] with x and Ω replaced by (x,t) and Q_T, respectively assures that condition (H^1_G) gives a sufficient condition for the a.e.-demiclosedness of $\widetilde{G}$.

PROPOSITION 4.3. *Let* (H^1_G) *be satisfied. Then* $\widetilde{G}\colon \mathcal{H} \to 2^{\mathcal{H}}$ *defined by* (4.1) *is a.e.-demiclosed.*

5. Proofs of main results

Let $\phi\colon \mathbb{R} \to [0,+\infty]$ be a proper lower semicontinuous convex function with $\phi(0) = 0$ and put

$$\varphi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \left|\nabla u(x)\right|^2 dx + \int_\Omega \phi(u(x))\,dx & \text{if } u \in D(\varphi), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus D(\varphi), \end{cases} \tag{5.1}$$

where

$$D(\varphi) = \left\{u \in H^1_0(\Omega) : \widetilde{\phi}(u) := \int_\Omega \phi(u(x))\,dx < +\infty\right\}.$$

Then φ becomes a proper lower semicontinuous convex functional from $L^2(\Omega)$ into $[0,+\infty]$ and its subdifferential is given by

$$\partial\varphi(u) = -\Delta u + \partial\phi(u) \tag{5.2}$$

with domain

$$\begin{aligned} D(\partial\varphi) = \{&u \in D(\varphi) : u \in H^2(\Omega) \cap H^1_0(\Omega), \\ &\text{there exists } b \in L^2(\Omega) \text{ such that } b(x) \in \partial\phi(u(x)) \text{ for a.e. } x \in \Omega\}. \end{aligned} \tag{5.3}$$

Moreover, for any $b \in \partial\phi(u)$ such that

$$z = -\Delta u + b, \tag{5.4}$$

it holds that

$$\|z\|_{L^2}^2 \geq \|\Delta u\|_{L^2}^2 + \|b\|_{L^2}^2 \tag{5.5}$$

whence follows

$$(-\Delta u, b)_{L^2} \geq 0 \quad \text{for all } b \in \partial\phi(u) \text{ and all } u \in D(\partial\varphi) \tag{5.6}$$

(see Lemma 1 in [23]). Then we prepare the following standard result.

Proposition 5.1. *For any $h \in \mathcal{H} := L^2(0,T;L^2(\Omega))$, the problem*

$$(\mathrm{P})_h \qquad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \Delta u(x,t) \in -\partial\phi(u(x,t)) + h(x,t), & (x,t) \in Q_T, \\ u(x,t) = 0, & (x,t) \in \Gamma_T, \\ u(x,0) = u(x,T), & x \in \Omega, \end{cases}$$

admits a unique solution $u_h \in C_\pi([0,T];H_0^1(\Omega))$ satisfying

$$\frac{\partial u_h}{\partial t},\ \Delta u_h \in L^2(0,T;L^2(\Omega)). \tag{5.7}$$

Proof. Since $\varphi(\,\cdot\,)$ is coercive in $L^2(\Omega)$, i.e. $\varphi(u) \geq \lambda_1\|u\|_{L^2}^2/2$, the existence of periodic solution $u_h \in C_\pi([0,T];L^2(\Omega))$ is assured by Corollary 3.4 in [5] and the uniqueness follows from the strict monotonicity of $\partial\varphi$.

Furthermore the regularity (5.7) assures the absolute continuity of the mapping: $t \mapsto \varphi(u(t))$ on $[0,T]$ (see Lemma 3.3 of [5]), which together with the fact $u_h \in C_\pi([0,T];L^2(\Omega))$ assures $u_h \in C_\pi([0,T];H_0^1(\Omega))$. $\square$

Hence, we can well define a multivalued mapping

$$\begin{gathered} \mathcal{G}\colon \mathcal{H} := L^2(0,T;L^2(\Omega)) \to 2^{\mathcal{H}}, \\ \mathcal{G}(h) := \{v \in \mathcal{H} : v(x,t) \in G(x,t,u_h(x,t)) \text{ a.e. } (x,t) \in Q_T\}, \end{gathered} \tag{5.8}$$

where u_h is the solution of $(\mathrm{P})_h$. Here we set

$$|||\mathcal{G}(h)|||_{\mathcal{H}} := \sup\{\|v\|_{\mathcal{H}} : v \in \mathcal{G}(h)\}, \tag{5.9}$$

and, for $R > 0$, put $\mathbf{K}_R := \{h \in \mathcal{H} : \|h\|_{\mathcal{H}} \leq R\}$. Then the upper semicontinuity and the lower semicontinuity of $\mathcal{G}$ in $\mathcal{H}_w$, is assured by the following results, where $\mathcal{H}_w$ is the space $\mathcal{H} := L^2(0,T;L^2(\Omega))$ endowed with the weak topology.

Lemma 5.2. *Let $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ satisfy (H_G^1) and suppose that there exists $R > 0$ such that $\mathcal{G}$ maps $\mathbf{K}_R$ into itself. Then $\mathcal{G}\colon \mathbf{K}_R \to 2^{\mathbf{K}_R}$ is upper semicontinuous with respect to the weak topology of $\mathcal{H}$.*

Lemma 5.3. *Let $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ satisfy (H_G^2) and suppose that there exists $R > 0$ such that $\mathcal{G}$ maps $\mathbf{K}_R$ into itself. Then $\mathcal{G}\colon \mathbf{K}_R \to 2^{\mathbf{K}_R}$ is lower semicontinuous with respect to the weak topology of $\mathcal{H}$.*

Proof of Lemma 5.2. Since $\mathbf{K}_R$ is a compact subset of $\mathcal{H}_w$, in order to show the upper semicontinuity of $\mathcal{G}$, it suffices to show that its graph is closed in $\mathcal{H}_w \times \mathcal{H}_w$, as is stated in Section 2. This part is essentially done in Lemma 4.3 of [22] or Lemma 3.1 of [21]. However, for the sake of completeness, we repeat the same reasoning reflecting the present specific setting.

Let $(h_n, g_n)_{n\in\mathbb{N}}$ be an arbitrary sequence such that $h_n \rightharpoonup h$, $g_n \rightharpoonup g$ weakly in $\mathcal{H}$, $h_n \in \mathbf{K}_R$, $g_n \in \mathcal{G}(h_n)$, i.e. $g_n(x,t) \in G(x,t,u_{h_n}(x,t))$ and $u_{h_n}(x,t)$ is the unique solution of

$$(5.10)\qquad \begin{cases} \dfrac{\partial}{\partial t} u_{h_n}(x,t) - \Delta u_{h_n}(x,t) \in -\partial\phi(u_{h_n}(x,t)) + h_n(x,t), & (x,t) \in Q_T, \\ u_{h_n}(x,t) = 0, & (x,t) \in \Gamma_T, \\ u_{h_n}(x,0) = u_{h_n}(x,T), & x \in \Omega. \end{cases}$$

Multiplying (5.10) by $z_n = -\Delta u_{h_n} + b_n$ with $b_n(x,t) \in \partial\phi(u_{h_n}(x,t))$, we have

$$(5.11)\qquad \begin{aligned} \frac{d}{dt}\varphi(u_{h_n}(x,t)) + \|z_n(t)\|_{L^2}^2 &\le \|h_n(t)\|_{L^2}^2 \|z_n(t)\|_{L^2}^2 \\ &\le \frac{1}{2}\|z_n(t)\|_{L^2}^2 + \frac{1}{2}\|h_n(t)\|_{L^2}^2. \end{aligned}$$

Since $h_n \in \mathbf{K}_R$, integrating (5.11) over $[0,T]$, we get by (5.5)

$$(5.12)\qquad \int_0^T \|\Delta u_{h_n}(t)\|_{L^2}^2\,dt + \int_0^T \|b_n(t)\|_{L^2}^2\,dt \le \int_0^T \|z_n(t)\|_{L^2}^2\,dt \le R^2.$$

Since

$$\|z\|_{L^2}\|u\|_{L^2} \ge |(z,u)_{L^2}| \ge \|\nabla u\|_{L^2}^2 \ge \sqrt{\lambda_1}\|\nabla u\|_{L^2}\|u\|_{L^2},$$

we get

$$(5.13)\qquad 2\lambda_1\varphi(u) \le \|z\|_{L^2}^2 \quad \text{for all } u \in D(\partial\varphi) \text{ and for all } z \in \partial\varphi(u).$$

Hence we find that there exists $t_0 \in [0,T]$ such that

$$(5.14)\qquad \varphi(u_{h_n}(t_0)) \le \frac{1}{T}\int_0^T \varphi(u_{h_n}(t))\,dt \le \frac{1}{2\lambda_1 T} R^2.$$

Then integrating (5.11) over $[t_0, t]$ with $t \in [t_0, t_0+T]$, we obtain

$$(5.15)\qquad \sup_{0\le t\le T}\varphi(u_{h_n}(t)) + \int_0^T \|\Delta u_{h_n}(t)\|_{L^2}^2\,dt + \int_0^T \|b_n(t)\|_{L^2}^2\,dt \le C_0,$$

where C_0 is a constant depending on T, λ_1 and R but not on n. Hence we can derive the boundedness of $\|du_{h_n}(t)/dt\|_{L^2(0,T;L^2(\Omega))}$ from the equation and the pre-compactness of $\{u_{h_n}(t)\}_{n\in\mathbb{N}}$ in $L^2(\Omega)$. Then, by Ascoli's theorem, there exists a subsequence of $(u_{h_n})_{n\in\mathbb{N}}$, denoted again by $(u_{h_n})_{n\in\mathbb{N}}$, such that

$$\begin{aligned}
u_{h_n} &\to u && \text{strongly in } C([0,T];L^2(\Omega)) \text{ and a.e. in } Q_T,\\
\frac{\partial u_{h_n}}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t} && \text{weakly in } L^2(0,T;L^2(\Omega)),\\
-\Delta u_{h_n} &\rightharpoonup -\Delta u && \text{weakly in } L^2(0,T;L^2(\Omega)),\\
b_n &\rightharpoonup b \in \partial\phi(u) && \text{weakly in } L^2(0,T;L^2(\Omega)).
\end{aligned}$$

Passing to the limit in (5.10), we find that u gives the unique solution of solution of $(\mathrm{P})_h$. Furthermore, since $g_n(x,t) \in G(x,t,u_{h_n}(x,t))$ almost everywhere in Q_T and $g_n \in \mathbf{K}_R$, there exists a subsequence of $(g_n)_{n\in\mathbb{N}}$, denoted again by $(g_n)_{n\in\mathbb{N}}$, which converges weakly to some g in $\mathcal{H}$.

On the other hand, $u_{h_n} \to u$ almost everywhere in Q_T. Therefore, by Proposition 4.3 we find that $g \in \mathcal{G}(h)$.

Since the argument above does not depend on the choice of the subsequences, we conclude that the original sequence $(h_n, g_n)_{n\in\mathbb{N}}$ converges to (h,g) which belongs to the graph of $\mathcal{G}$, that implies the closedness of the graph of $\mathcal{G}$ in $\mathcal{H}_w \times \mathcal{H}_w$. $\square$

Proof of Lemma 5.3. We repeat almost the same argument as in the proof of Lemma 10 in [24]. In order to check the lower semicontinuity of $\mathcal{G}$, we have only to show that for any weakly closed subset C in $\mathcal{H}$,

$$\mathcal{G}^+(C) := \{h \in \mathbf{K}_R : \mathcal{G}(h) \subset C\}$$

forms a weakly closed subset of $\mathcal{H}$. Let C be a weakly closed in $\mathcal{H}$ and let $(h_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{G}^+(C)$ weakly convergent in $\mathcal{H}$ to some $h_0 \in \mathcal{H}$. To prove that $\mathcal{G}^+(C)$ is weakly closed, it suffices to show that $g_0 \in C$ for any $g_0 \in \mathcal{G}(h_0)$. Let $g_0 \in \mathcal{G}(h_0)$, i.e. $g_0(x,t) \in G(x,t,u_{h_0}(x,t))$ for almost every $(x,t) \in Q_T$. Then we are going to show that $g_0 \in C$. For each $n \in \mathbb{N}$ we define $\varphi_n \colon Q_T \to \mathbb{R}$ by

$$\varphi_n(x,t) = d(g_0(x,t), G(x,t,u_{h_n}(x,t))) + \frac{\rho(x,t)}{n},$$

where $d(g_0(x,t), G(x,t,u_{h_n}(x,t))) = \inf\{|g_0(x,t) - y| : y \in G(x,t,u_{h_n}(x,t))\}$ and $\rho(x,t) > 0$ almost everywhere in Q_T, and $\|\rho\|_{L^2(Q_T)} = 1$. Then by virtue of (b) of (H^2_G), we can show that for each $n \in \mathbb{N}$, $\varphi_n(\,\cdot\,,\,\cdot\,)$ is a measurable function, and the multifunction $(x,t) \mapsto \Gamma_n(x,t)$ where

$$\Gamma_n(x,t) := \{y \in \mathbb{R} : |g_0(x,t) - y| \le \varphi_n(x,t)\} \cap G(x,t,u_{h_n}(x,t)) \tag{5.16}$$

is also a measurable mapping with nonempty values (see the proof of Lemma 2 in [24]). Then, by the Kuratowski and Ryll-Nardzewski theorem (see Theorem 5.1 in [14]), there exists a measurable selection $g_n(x,t)$ of $\Gamma_n(x,t)$, i.e. $g_n \colon Q_T \to \mathbb{R}$ is measurable and

$$g_n(x,t) \in \Gamma_n(x,t) \quad \text{for a.e. } (x,t) \in Q_T. \tag{5.17}$$

Since $\Gamma_n(x,t) \subset G(x,t,u_{h_n}(x,t))$ and $\mathcal{G}$ maps $\mathbf{K}_R$ into itself, $(g_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$. Then there exists a subsequence of $(g_n)_{n\in\mathbb{N}}$ again denoted by $(g_n)_{n\in\mathbb{N}}$ such that

$$(g_n)_{n\in\mathbb{N}} \text{ converges weakly in } \mathcal{H} \text{ to some } g^0. \tag{5.18}$$

On the other hand, since $(h_n)_{n\in\mathbb{N}}$ is also bounded in $\mathcal{H}$, repeating the same argument as for (5.15), we obtain the same estimate (5.15) and find that

$$u_{h_n} \to u_{h_0} \text{ strongly in } C([0,T], L^2(Q_T)) \text{ and for a.e. } (x,t) \in Q_T, \text{ as } n \to \infty.$$

In particular,

$$u_{h_n}(x,t) \to u_{h_0}(x,t) \quad \text{as } n \to \infty \text{ for a.e. } (x,t) \in Q_T.$$

Hence, by virtue of the fact that the lower semicontinuity of $u \mapsto G(x,t,u)$ implies the upper semicontinuity of $u \mapsto d(v, G(x,t,u))$ for every $v \in \mathcal{H}$ (see Proposition 2.26, Chapter 1 of [16]), we see that

$$\varphi_n(x,t) \to 0 \quad \text{as } n \to \infty \text{ for a.e. } (x,t) \in Q_T.$$

Then, in view of (5.16) and (5.17), we get

$$g_n(x,t) \to g_0(x,t) \text{ as } n \to \infty \text{ a.e. } (x,t) \in Q_T. \tag{5.19}$$

Then, by virtue of Egorov's theorem together with (5.18) and (5.19) we get that $g^0 = g_0$ (see the proof of Lemma 10 of [24]). Consequently, (5.18) implies that $(g_n)_{n\in\mathbb{N}}$ converges weakly in $\mathcal{H}$ to g_0. Since $g_n \in \mathcal{G}(h_n) \subset C$ and C is assumed to be weakly closed in $\mathcal{H}$, we conclude that $g_0 \in C$. $\square$

5.1. Proof of Theorem 3.1. Large periodic solutions. In this subsection, we give a proof of Theorem 3.1. To this end we prepare the following lemma.

Lemma 5.4. *Let* (GC) *be satisfied with* $q = 2$ *and* $C_q = C_2 < \lambda_1^2(1-k)$, *where* $\lambda_1 > 0$ *is the first eigenvalue of* $-\Delta$. *Then there exists* $R > 0$ *such that* $\mathcal{G}$ *maps* $\mathbf{K}_R$ *into itself.*

Proof. Let $h \in \mathbf{K}_R$ and u_h be the unique solution of $(\mathrm{P})_h$ whose existence is assured by Proposition 5.1. Therefore u_h satisfies the equation

$$\frac{\partial}{\partial t} u_h(x,t) - \Delta u_h(x,t) + b_h(x,t) = h(x,t), \quad (x,t) \in Q_T, \tag{5.20}$$

where $b_h(x,t) \in \partial\phi(u_h(x,t))$ for $(x,t) \in Q_T$. Multiply (5.20) by u_h, then we get

$$\frac{1}{2}\frac{d}{dt}\|u_h(t))\|_{L^2}^2 + \|\nabla u_h(t))\|_{L^2}^2 + \widetilde{\phi}(u_h(x,t)) \le \|h(t)\|_{L^2}\|u_h(t)\|_{L^2}.$$

Integration of this over $[0,T]$ gives

$$\int_0^T \|\nabla u_h(x,t)\|_{L^2}^2\,dt + \int_0^T \widetilde{\phi}(u_h(x,t))\,dt \le \int_0^T \|h(t)\|_{L^2}\|u_h(t)\|_{L^2}\,dt.$$

Then, by Poincaré's inequality, we have

$$\begin{aligned}(5.21)\qquad \lambda_1 \int_0^T \|u_h(t)\|_{L^2}^2\,dt &\le \int_0^T \|\nabla u_h(x,t)\|_{L^2}^2\,dt \\ &\le \frac{\lambda_1}{2}\int_0^T \|u_h(x,t)\|_{L^2}^2\,dt + \frac{1}{2\lambda_1}\int_0^T \|h(t)\|_{L^2}^2\,dt.\end{aligned}$$

Hence, we obtain

$$\int_0^T \|u_h(x,t)\|_{L^2}^2\,dt \le \frac{R^2}{\lambda_1^2}. \tag{5.22}$$

We next multiply (5.20) by $b_h(x,t)$ and we get

$$\frac{d}{dt}\,\widetilde{\phi}(u_h(x,t)) + \|b_h(t)\|_{L^2}^2 \le \frac{1}{2}\|b_h(t)\|_{L^2}^2 + \frac{1}{2}\|h(t)\|_{L^2}^2,$$

where we used the fact that $(-\Delta u_h(t), b_h(x,t))_{L^2} \ge 0$ (see (5.6)). Then we obtain

$$\int_0^T \|b_h(t)\|_{L^2}^2\,dt \le R^2. \tag{5.23}$$

We set now

$$R^2 = \max\bigg\{\frac{\|a\|_{L^1(Q_T)}}{1-k-C_2/\lambda_1^2}, 1\bigg\}. \tag{5.24}$$

Then we obtain by (GC), (5.9), (5.22), (5.23) and (5.24)

$$|||\mathcal{G}(h)|||_{\mathcal{H}}^2 \le \|a\|_{L^1(Q_T)} + k\|b_h\|_{\mathcal{H}}^2 + C_2\|u_h\|_{\mathcal{H}}^2 \le \|a\|_{L^1(Q_T)} + \bigg(k + \frac{C_2}{\lambda_1^2}\bigg)R^2 \le R^2$$

which implies that $\mathcal{G}$ maps $\mathbf{K}_R$ into itself. □

Now we are ready to give a proof of Theorem 3.1.

Proof of Theorem 3.1. Suppose that G satisfies (H_G^1) or (H_G^2). Then Lemma 5.4 together with Lemma 5.2 or Lemma 5.3 assures that $\mathcal{G}$ maps $\mathbf{K}_R$ into itself and $\mathcal{G}$ becomes upper semicontinuous or lower semicontinuous multivalued maps. Then applying the Kakutani-Tikhonov fixed point theorem for $\mathcal{G}$ (for the upper semicontinuous case) or the Schauder-Tikhonov's fixed point theorem for a continuous selection $g\colon \mathbf{K}_R \to \mathbf{K}_R$ of $\mathcal{G}$ whose existence is assured by Fryszkowski's theorem [11] (for the lower semicontinuous case), we can obtain a fixed point $\overline{h} \in \mathbf{K}_R$ of $\mathcal{G}$ (i.e. $\overline{h} \in \mathcal{G}(\overline{h})$. Then $u_{\overline{h}}$ gives the desired solution. □

5.2. Proof of Theorem 3.2. Small periodic solutions. In this subsection, we give a proof of Theorem 3.2. In parallel with Lemma 5.4 we now have the following:

Lemma 5.5. *Let* (GC) *be satisfied with* $2 < q < 2^*$. *Then there exists a* (*sufficiently small*) *number* $R > 0$ *such that* $\mathcal{G}$ *maps* $\mathbf{K}_R$ *into itself.*

To prove this lemma we use the following result (for a proof, see Lemma 6 of [24]).

Lemma 5.6. *If $q \in (2, 2^*)$, then for any $\eta > 0$, there exists $K_q(\eta) > 0$ such that*

$$\|u\|_{L^{2(q-1)}}^{2(q-1)} \le \eta \, \|\Delta u\|_{L^2}^2 + K_q(\eta) \, \|\nabla u\|_{L^2}^{\gamma}, \tag{5.25}$$

for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$, where

$$\gamma = \begin{cases} 2(q-1) & \text{for } N = 1, 2 \text{ or for } N \ge 3 \text{ and } q \le \dfrac{2(N-1)}{N-2}, \\ 2 + \dfrac{4(q-2)}{2N-(N-2)q} & \text{for } N \ge 3 \text{ and } q > \dfrac{2(N-1)}{N-2}. \end{cases}$$

Proof of Lemma 5.5. Let $h \in \mathbf{K}_R$ and u_h be the solution of $(\mathrm{P})_h$. Then by (5.21) we can obtain

$$\int_0^T \|\nabla u_h(t)\|_{L^2}^2 \, dt \le \frac{1}{\lambda_1} R^2. \tag{5.26}$$

Next, multiplying (5.20) by $-\Delta u_h(t)$ and $z_h(t) = -\Delta u_h(t) + b_h(t)$, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_h(t)\|_{L^2}^2 + \|\Delta u_h(t)\|_{L^2}^2 \le \|h(t)\|_{L^2}\|\Delta u_h(t)\|_{L^2}, \tag{5.27}$$

$$\frac{d}{dt}\,\varphi(u_h(t)) + \|z_h(t)\|_{L^2}^2 \le \|h(t)\|_{L^2}\|z_h(t)\|_{L^2}, \tag{5.28}$$

which implies

$$\int_0^T \|\Delta u_h(t)\|_{L^2}^2 \, dt + \int_0^T \|b_h(t)\|_{L^2}^2 \, dt \le \int_0^T \|z_h(t)\|_{L^2}^2 \, dt \le R^2. \tag{5.29}$$

Furthermore, let $t_0 \in [0, T]$ be such that

$$\|\nabla u_h(t_0)\|_{L^2} = \min_{0 \le t \le T} \|\nabla u_h(t)\|_{L^2}.$$

Then (5.26) gives

$$\|\nabla u_h(t_0)\|_{L^2}^2 \le \frac{1}{\lambda_1 T} R^2. \tag{5.30}$$

Then, integrating (5.27) over $[t_0, t]$ with $t \in [t_0, t_0 + T]$, we have

$$\|\nabla u_h(t)\|_{L^2}^2 \le \|\nabla u_h(t_0)\|_{L^2}^2 + R^2 \le \left(\frac{1}{\lambda_1 T} + 1\right) R^2, \tag{5.31}$$

for all $t \in [t_0, t_0 + T]$. Now we apply Lemma 5.6 with $\eta = k/C_q$, and we obtain by (5.26), (5.29) and (5.31)

$$
\begin{aligned}
(5.32)\quad \iint_{Q_T} |||\mathcal{G}(h)|||^2\,dx\,dt
&\le \|a\|_{L^1(Q_T)} + k\int_0^T \|\partial^0\phi(u_h)\|_{L^2}^2\,dt + C_q\int_0^T \|u_h\|_{L^{2(q-1)}}^{2(q-1)}\,dt \\
&\le \|a\|_{L^1(Q_T)} + k\int_0^T \big(\|b_h\|_{L^2}^2 + \|\Delta u_h\|_{L^2}^2\big)\,dt \\
&\quad + C_q K_q\bigg(\frac{k}{C_q}\bigg)\int_0^T \|\nabla u_h\|_{L^2}^{\gamma}\,dt \\
&\le \|a\|_{L^1(Q_T)} + kR^2 + C_q K_q\bigg(\frac{k}{C_q}\bigg)\bigg(\frac{1}{\lambda_1 T}+1\bigg)^{(\gamma-2)/2} R^{\gamma-2}\,\frac{1}{\lambda_1}\,R^2 \\
&\le \|a\|_{L^1(Q_T)} + \bigg(k + C_q K_q\bigg(\frac{k}{C_q}\bigg)\bigg(\frac{1}{\lambda_1 T}+1\bigg)^{(\gamma-2)/2}\frac{1}{\lambda_1}\,R^{\gamma-2}\bigg)R^2.
\end{aligned}
$$

Here, nothing that $\gamma > 2$, we take R and $\|a\|_{L^1(Q_T)}$ sufficiently small so that:

$$
(5.33)\qquad C_q K_q\bigg(\frac{k}{C_q}\bigg)\bigg(\frac{1}{\lambda_1 T}+1\bigg)^{(\gamma-2)/2}\frac{1}{\lambda_1}\,R^{\gamma-2} \le \frac{1}{2}\,(1-k),
$$

$$
(5.34)\qquad \|a\|_{L^1(Q_T)} \le \frac{1}{2}\,(1-k)R^2.
$$

Then, (5.32)–(5.34) assure that $\mathcal{G}$ maps $\mathbf{K}_R$ into itself. □

Proof of Theorem 3.2. We can repeat exactly the same arguments as in the proof of Theorem 3.1. However, in this case, we have to take $\|a\|_{L^1(Q_T)}$ sufficiently small so that

$$
\|a\|_{L^1(Q_T)} \le \frac{1-k}{2}\bigg[\frac{\lambda_1(\lambda_1 T)^{(\gamma-2)/2}(1-k)}{2C_q K_q(k/C_q)(\lambda_1 T+1)^{(\gamma-2)/2}}\bigg]^{2/(\gamma-2)}. \qquad □
$$

6. Examples

In this section we give some examples which can be dealt with in our setting.

6.1. The case where $D(\phi) = \mathbb{R}^1$. Let $\beta(\,\cdot\,) = \partial\phi(\,\cdot\,)$ be a maximal monotone graph in $\mathbb{R}^1 \times \mathbb{R}^1$ which can be multivalued such that $\phi(0) = 0 = \min_{r\in\mathbb{R}^1} \phi(r)$ and

$$
(6.1)\qquad \big|\partial^0\phi(u)\big|^2 \ge |u|^{2(p-1)} - C_1 \quad \text{for all } u \in \mathbb{R}^1,\ 1 < p < \infty,\ C_1 \ge 0.
$$

We also introduce a class of continuous functions $\mathcal{C}_{p,q}$ by the following: $f \in \mathcal{C}_{p,q}$ if and only if $f\colon \mathbb{R}^1 \to \mathbb{R}^1$ is continuous and satisfy

$$
(6.2)\qquad |f(u)|^2 \le C_0 + k_0|u|^{2(p-1)} + C_q^0|u|^{2(q-1)} \quad \text{for all } u \in \mathbb{R}^1,
$$

where $1 < q < \infty$, $C_0 \ge 0$, $C_q^0 \ge 0$ and $k_0 \in [0, 1)$ are constants.

In the following we consider the case where

$$G(x,t,u) = G_0(u) + f_e(x,t), \quad \text{with } f_e \in L^2(Q_T). \tag{6.3}$$

6.1.1. The upper semicontinuous case. Take f_1^+, f_1^-, f_2^+, $f_2^- \in \mathcal{C}_{p,q}$, such that

$$\begin{aligned} &f_1^-(u) < f_2^-(u) \quad \text{for all } u \in (-\infty, 0],\\ &f_1^+(u) < f_2^+(u) \quad \text{for all } u \in [0, +\infty),\\ &f_2^-(0) < f_1^+(0), \end{aligned}$$

and define

$$G_0(u) = \begin{cases} \big[f_1^-(u), f_2^-(u)\big] & \text{if } u < 0,\\ \big[f_1^+(u), f_2^+(u)\big] & \text{if } u > 0,\\ \big[f_1^-(0), f_2^+(0)\big] & \text{if } u = 0. \end{cases} \tag{6.4}$$

Then it is easy to see that $G(x,t,u) = G_0(u) + f_e(x,t)$ is a closed convex and upper semicontinuous multivalued function satisfying (H_G^1). Note that there is no continuous selection of $G_0(\,\cdot\,)$.

Here we note that $f_1^{\pm}, f_2^{\pm} \in \mathcal{C}_{p,q}$ together with (6.1) and (6.2) implies that, for any small $\eta > 0$, there exists $C_\eta > 0$ such that

$$\begin{aligned} |||G(x,t,u)|||^2 \le C_\eta |f_e(x,t)|^2& \\ + (1+\eta)\big[C_0 + k_0 C_1 + k_0 |\partial^0\phi(u)|^2 + C_q^0 |u|^{2(q-1)}\big]& \end{aligned} \tag{6.5}$$

for almost every $(x,t) \in Q_T$, for all $u \in \mathbb{R}^1$. Then the existence of periodic solutions of our equation with (6.1), (6.2) and (6.3) falls into the following two cases:

(I) *Large periodic solutions.* We can apply Theorem 3.1 for all of the following cases, i.e. problem (P) admits a solution satisfying (3.5) for any $f_e \in L^2(Q_T)$.

(I_1) *The case where* $q < p$. Since the Young inequality assures that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$(1+\eta) C_q^0 |u|^{2(q-1)} \le \varepsilon \big|\partial^0\phi(u)\big|^2 + C_\varepsilon \quad \text{for all } u \in \mathbb{R}^1,$$

$G(x,t,u)$ satisfies (GC) with $k = (1+\eta)k_0 + \varepsilon < 1$, $C_q = 0$ and

$$|a(x,t)| = C_\eta |f_e(x,t)|^2 + (1+\eta)(C_0 + k_0 C_1) + C_\varepsilon.$$

(I_2) *The case where* $q < 2$. Since for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$(1+\eta) C_q^0 |u|^{2(q-1)} \le \varepsilon |u|^2 + C_\varepsilon \quad \text{for all } u \in \mathbb{R}^1,$$

$G(x,t,u)$ satisfies (GC) with $k = \widetilde{k} := (1+\eta)k_0 < 1$, $q = 2$, $C_q = C_2 = \varepsilon < \lambda_1^2(1 - \widetilde{k})$ and

$$|a(x,t)| = C_\eta |f_e(x,t)|^2 + (1+\eta)(C_0 + k_0 C_1) + C_\varepsilon.$$

(I_3) *The case where* $q = 2$, $p \le 2$ *and* $C_2^0 < \lambda_1^2(1 - k_0)$. $G(x, t, u)$ satisfies (GC) with $k = \widetilde{k} = (1 + \eta)k_0 < 1$, $q = 2$, $C_q = C_2 = (1 + \eta)C_2^0 < \lambda_1^2(1 - \widetilde{k})$ and

$$|a(x,t)| = C_\eta |f_e(x,t)|^2 + (1+\eta)(C_0 + k_0 C_1).$$

(II) *Small periodic solutions.* Let $2 < q < 2^*$ and let C_0, $k_0 C_1$ and $\|f_e\|_1^2$ be small enough. Then (P) has a periodic solution satisfying (3.5). In fact, $G(x, t, u)$ satisfies (GC) with $k = (1 + \eta)k_0 < 1$, $q \in (2, 2^*)$, $C_q = (1 + \eta)C_q^0$ and

$$|a(x,t)| = C_\eta |f_e(x,t)|^2 + (1+\eta)(C_0 + k_0 C_1).$$

Then we can apply Theorem 3.2.

6.1.2. The lower semicontinuous case. Let f^+, $f^- \in \mathcal{C}_{p,q}$ and $-\infty \le r_0 < r_1 \le +\infty$ such that $f^-(u) < f^+(u)$ for all $u \in (r_0, r_1)$ and define

$$G_0(u) = \begin{cases} \{f^+(u)\} & \text{if } r_1 \le u \text{ (when } r_1 < +\infty), \\ \{f^-(u)\} & \text{if } u \le r_0 \text{ (when } -\infty < r_0), \\ \big[f^-(u), f^+(u)\big] \cap \mathbb{Q}_n & \text{if } u \in (r_0, r_1), \end{cases} \tag{6.6}$$

where $\mathbb{Q}_n := \{q \in \mathbb{Q} : 10^n q \in \mathbb{Z}\}$ with n sufficiently large so that

$$\big[f^-(u), f^+(u)\big] \cap \mathbb{Q}_n \neq \emptyset \quad \text{for all } u \in (r_0, r_1).$$

Then it is easy to see $G_0(\,\cdot\,)$ is lower semicontinuous and closed valued (but not convex valued) and there is no continuous selection of $G_0(\,\cdot\,)$.

Note that assumption $f^\pm \in \mathcal{C}_{p,q}$ assures (6.5). Then we again obtain two kinds of results just same as: (I) large periodic solutions and (II) small periodic solutions, as before.

6.2. The case where $D(\phi)$ is precompact. Here we give some examples for the case where $D(\phi)$ is precompact, i.e.

$$D(\phi) = \{u \in \mathbb{R}^1 : \phi(u) < +\infty\} \subset [a, b] \quad \text{with } -\infty < a < b < +\infty.$$

Consider the following typical two examples:

Example 6.1.

$$\phi(u) = I_{[a,b]}(u) = \begin{cases} 0 & \text{if } u \in [a, b], \\ +\infty & \text{otherwise.} \end{cases} \tag{6.7}$$

Then we have

$$\partial\phi(u) = \partial I_{[a,b]}(u) = \begin{cases} \{0\} & \text{if } u \in (a, b), \\ (-\infty, 0] & \text{if } u = a, \\ [0, +\infty) & \text{if } u = b, \\ \emptyset & \text{if } u \notin [a, b]. \end{cases}$$

EXAMPLE 6.2.

$$\phi(u) = \phi_h(u) = \begin{cases} h(u) & \text{if } u \in (a,b), \\ +\infty & \text{otherwise}, \end{cases} \tag{6.8}$$

where $h \in C^1((a,b);\mathbb{R}^1)$ is convex and satisfies

$$\lim_{u \to a+} h(u) = \lim_{u \to b-} h(u) = +\infty.$$

Then we have

$$\partial\phi(u) = \partial\phi_h(u) = \begin{cases} \{h'(u)\} & \text{if } u \in (a,b), \\ \emptyset & \text{if } u \notin (a,b). \end{cases}$$

Again define $G(x,t,u)$ by (6.3) together with (6.4) or (6.5). Then since $D(\partial I_{[a,b]}) = [a,b]$ and $D(\partial\phi_h) = (a,b)$, $G(x,t,u)$ satisfies (GC) with $k = C_q = 0$ and

$$|a(x,t)| = C(|f_e(x,t)|^2 + 1) \quad \text{for some } C > 0,$$

provided that $f_i^{\pm}$ $(i = 1,2)$, $f^{\pm} \in C(\mathbb{R}^1;\mathbb{R}^1)$. Hence, Theorem 3.1 assures that (P) admits a solution u satisfying (3.5) for any $f_e \in L^2(Q_T)$.

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