Topological **M**ethods in **N**onlinear **A**nalysis
Volume 54, No. 2A, 2019, 665–683
DOI: 10.12775/TMNA.2019.060

SEMICLASSICAL STATES FOR SINGULARLY PERTURBED SCHRÖDINGER–POISSON SYSTEMS WITH A GENERAL BERESTYCKI–LIONS OR CRITICAL NONLINEARITY

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Abstract. This paper is concerned with two classes of singularly perturbed Schrödinger–Poisson systems of the form

$$\begin{cases}-\varepsilon^2\Delta u+u+\phi u=f(u), & x\in\mathbb{R}^3,\\ -\Delta\phi=u^2, & x\in\mathbb{R}^3,\end{cases}$$

and

$$\begin{cases}-\varepsilon^2\Delta u+V(x)u+\phi u=g(x,u)+K(x)u^5, & x\in\mathbb{R}^3,\\ -\Delta\phi=u^2, & x\in\mathbb{R}^3,\end{cases}$$

where $\varepsilon>0$ is a small parameter. We prove that: (1) the first system admits a concentrating bounded state for small ε, where $f\in\mathcal{C}(\mathbb{R},\mathbb{R})$ satisfies Berestycki–Lions assumptions which are almost necessary; (2) there exists a constant $\varepsilon_0>0$ determined by V,K and g such that for any $\varepsilon\in(0,\varepsilon_0]$ the second system has a nontrivial solution, where $V,K\in\mathcal{C}(\mathbb{R}^3,\mathbb{R})$, $V(x)\geq 0$, $K(x)>0$, $g\in\mathcal{C}(\mathbb{R}^3\times\mathbb{R},\mathbb{R})$ is an indefinite function. Our results improve and complement the previous ones in the literature.

2010 *Mathematics Subject Classification.* 35J60, 35J20, 35Q40.

Key words and phrases. Semiclassical states; Schrödinger–Poisson system; Berestycki–Lions type assumptions; critical Sobolev exponent; variational method.

This work was partially supported by the National Natural Science Foundation of China (11571370, 11701487,11626202) and Hunan Provincial Natural Science Foundation of China (2016JJ6137).

1. Introduction

In recent years, the following nonlinear Schrödinger–Poisson system

$$\begin{cases} -\varepsilon^2\Delta u + a(x)u + \phi u = h(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

has been an object of interest for many authors. Here $\varepsilon > 0$ is a parameter, $a\colon \mathbb{R}^3 \to \mathbb{R}$ and $\phi\colon \mathbb{R}^3 \to \mathbb{R}$ represent the effective potential and the electric potential respectively, and the nonlinear term $h\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ simulates the interaction between many particles or external nonlinear perturbations. Such a system, also known as Schrödinger–Maxwell equations, arises in many mathematical physics contexts. It was introduced in [7], [24] as a model used in the Thomas–Fermi-von Weizsäcker theory in quantum mechanics; it also appeared in semiconductor theory [5], [6] to describe solitary waves for nonlinear stationary equations of Schrödinger type interacting with an electrostatic field. For the case that $\varepsilon = 1$, there exists a lot of results on the existence of nontrivial solutions for problems like (1.1), see e.g. [11]–[13], [33], [34], [39]–[41] etc. In quantum physics, the parameter ε is generically quite small. For small $\varepsilon > 0$, the solutions are called semiclassical states, which describe a kind of transition from Quantum Mechanics to Newtonian Mechanics, see [28], [30].

There has been a lot of research on the existence of solutions for the following system:

$$\begin{cases} -\varepsilon^2\Delta u + u + \phi u = f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{$(\mathcal{BP})_\varepsilon$}$$

where $\varepsilon > 0$ is a small parameter. For example, in the case of $\varepsilon = 1$ and $f(u) = |u|^{p-1}u$ with $1 < p < 5$, the existence of radially symmetric positive solutions of $(\mathcal{BP})_1$ was obtained by D'Aprile and Mugnai [16] for $3 < p < 5$ and Ruiz [31] for $2 \le p \le 3$; Azzollini and Pomponio [3] proved the existence of ground state solutions for $2 < p < 5$; in [15] and [31], it is showed that $(\mathcal{BP})_1$ does not have any nontrivial solution for $p \le 1$ or $p \ge 5$ and for $1 < p \le 2$, respectively. If $\varepsilon > 0$ is sufficiently small, D'Aprile and Wei [17] constructed a family of positive radially symmetric bound states for $f(u) = |u|^{p-1}u$ with $1 < p < 11/7$ which concentrate around a sphere in $\mathbb{R}^3$ as $\varepsilon \to 0$. Later, these existence results were extended by many authors to more general nonlinearities which satisfy some global growth assumptions such as the Ambrosetti–Rabinowitz type condition or the Nehari-type monotonicity condition and so on, see e.g. [18], [29], [34]–[37], [39]–[41]. When f satisfies Berestycki–Lions assumptions:

(F1) $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$ and $\lim_{|t|\to 0} f(t)/t = 0$ and $\lim_{|t|\to\infty} f(t)/t^5 = 0$;

(F2) there exists $s_0 > 0$ such that $F(s_0) > s_0^2/2$, where $F(t) := \int_0^t f(s)\,ds$, Azzollini, Avenia and Pomponio [2] considered a class of Schrödinger–Poisson systems of the form:

$$\begin{cases} -\Delta u + u + \omega\phi u = f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = \omega u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\omega > 0$ is a small parameter. Note that (F1) and (F2) were introduced by Berestycki and Lions in the celebrated paper [8] to obtain a radially symmetric positive solution for the following Schödinger equation:

$$-\Delta u + u = f(u), \quad u \in H^1(\mathbb{R}^3).$$

Using Pohožaev identity they showed that these conditions are almost necessary to get an existence result. We would like to point out that it is much more difficult to generalize the existence result to Schrödinger–Poisson systems due to the appearance of the term ϕu. To overcome this difficulty, Azzollini, Avenia and Pomponio combined a monotonicity trick in [20] and a truncation argument in [21] to prove that there exists $\omega_0 > 0$ such that (1.2) admits a nontrivial positive radial solution for any $\omega \in (0, \omega_0)$. For more existence results on Schrödinger equations under Berestycki–Lions assumptions, we also mention [10], [14], [22].

However, to the best of our knowledge, there seems to be no result on semiclassical states for $(\mathcal{BP})_\varepsilon$ with a Berestycki–Lions nonlinearity so far. Our first purpose of this paper is to fill this gap, and establish the existence and concentration behavior of semiclassical states for $(\mathcal{BP})_\varepsilon$. Different from [2], our main tool is the critical point theory developed by Jeong and Seok in [23, Theorem 1]. Our first result is as follows.

THEOREM 1.1. *Assume f satisfies* (F1) *and* (F2). *Then there exists $\overline{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \overline{\varepsilon}]$, problem $(\mathcal{BP})_\varepsilon$ admits a nontrivial solution $(\overline{v}_\varepsilon, \phi_{\overline{v}_\varepsilon})$ in $H^1_r(\mathbb{R}^3) \times \mathcal{D}^{1,2}_r(\mathbb{R}^3)$. In addition, for any sequence $\{\varepsilon_j\}$ converging to* 0, *the sequence $\{\overline{v}_{\varepsilon_j}(\varepsilon_j x)\}$ converges to a radial least energy solution of the following equation*

$$-\Delta u + u = f(u). \tag{1.3}$$

Another purpose is to study the following Schrödinger–Poisson system with critical growth:

$$(\mathcal{CP})_\varepsilon \qquad \begin{cases} -\varepsilon^2\Delta u + V(x)u + \phi u = g(x,u) + K(x)u^5, & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $V, K\colon \mathbb{R}^3 \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ satisfy the following basic conditions:

(V1) $V \in \mathcal{C}(\mathbb{R}^3, [0,\infty))$, and there is a constant $b > 0$ such that the set $\mathcal{V}_b := \{x \in \mathbb{R}^3 : V(x) < b\}$ has finite Lebesgue measure;

(K1) $K \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$ and $0 < \inf_{x\in\mathbb{R}^3} K(x) \le K(x) \le K_\infty := \sup_{x\in\mathbb{R}^3} K(x)$ for all $x \in \mathbb{R}^3$;

(G1) $g \in \mathcal{C}(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and $g(x,t) = o(|t|)$ as $|t| \to 0$ uniformly in $x \in \mathbb{R}^3$ and $g(x,t) = o(|t|^5)$ as $|t| \to \infty$ uniformly in $x \in \mathbb{R}^3$.

It seems that the first result on the existence of semiclassical states for $(\mathcal{CP})_\varepsilon$ satisfying (V1), (K1) and (G1) is due to Yang, Sheng and Ding [38]. If V and g further satisfy the following assumptions:

(V0) $V(0) = \min_{x\in\mathbb{R}^3} V(x) = 0$;

(AR) there exists a constant $\mu > 4$ such that

$$g(x,t)t - \mu G(x,t) \ge 0 \quad \text{for all } (x,t) \in \mathbb{R}^3 \times \mathbb{R},$$

where $G(x,t) = \int_0^t g(x,s)\,ds$;

(AQ) there exist constants $a_0 > 0$ and $q_0 > 4$ such that

$$G(x,t) \ge a_0 t^{q_0} \quad \text{for all } (x,t) \in \mathbb{R}^3 \times \mathbb{R},$$

they proved that for any $\sigma > 0$ there is $\varepsilon_\sigma > 0$ such that $(\mathcal{CP})_\varepsilon$ has at least one nontrivial solution for $\varepsilon \in (0, \varepsilon_\sigma]$. Subsequently, Huang and Tang [19] obtained a similar result under the weaker assumptions:

(G2) $g(x,t)t/4 - G(x,t) + K(x)t^6/12 \ge 0$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$;

(AQ1) there exist constants $a_1, R_1 > 0$ and $\kappa_1 > 3/2$ such that
- (i) $G(x,t) \ge a_1 t^4$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$;
- (ii) $|tg(x,t)| \le b/3$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ and $|t| \le R_1$;
- (iii) $|g(x,t)|^{\kappa_1} \le |t|^{\kappa_1}[g(x,t)t - G(x,t)/4 + K(x)t^6/12]$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ and $|t| > R_1$.

It is worth emphasizing that (V0), (AR) and (AQ) or (V0) and (AQ1) used in [38] or [19] are very crucial to prove that the mountain pass level value falls in a range in which the Palais–Smale sequence converges weakly to some nontrivial solution. In the second part of this paper, we shall improve the result in [38], [19] by removing the condition (V0) and relaxing (AR), (AQ) and (AQ1) to the following condition:

(G3) $g(x,t)t - 2G(x,t) \ge 0$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$,

and give a more accurate upper bound $\varepsilon_0 > 0$ of ε determined by V, K and g to ensure the existence of semiclassical states to $(\mathcal{CP})_\varepsilon$ for $\varepsilon \in (0, \varepsilon_0]$. In this direction, we have the following theorem.

Theorem 1.2. *Assume V, K and g satisfy* (V1), (K1) *and* (G1)–(G3). *Then there exists a constant $\varepsilon_0 > 0$ determined by V, K and g* (*see* (3.13) *below*) *such that for any $\varepsilon \in (0, \varepsilon_0]$, problem $(\mathcal{CP})_\varepsilon$ admits a nontrivial solution $(\widetilde{u}_\varepsilon, \phi_{\widetilde{u}_\varepsilon})$*

which satisfies

$$(1.4)\qquad 0<\frac{1}{2}\int_{\mathbb{R}^3}\big(\varepsilon^2|\nabla\widetilde{u}_\varepsilon|^2+V(x)\,\widetilde{u}_\varepsilon^2\big)\,dx+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{\widetilde{u}_\varepsilon}(x)\,\widetilde{u}_\varepsilon^2\,dx \\ -\int_{\mathbb{R}^3}G(x,\widetilde{u}_\varepsilon)\,dx-\frac{1}{6}\int_{\mathbb{R}^3}K(x)\,\widetilde{u}_\varepsilon^6\,dx\le\frac{1}{126\sqrt{6K_\infty}(\gamma_0\gamma_6)^3}\,\varepsilon,$$

where $\gamma_0,\gamma_6>0$ *are embedding constants given by* (3.1) *and* (3.2) *below.*

Note that finding of semiclassical states for $(\mathcal{CP})_\varepsilon$, (G2) plays an important role in showing the boundedness of Palais–Smale sequences. It is an interesting question whether (G2) can be further weakened. We shall consider this problem, and introduce the following assumption weaker than (G2):

(G2′) there exists a constant $\theta>0$ such that

$$\frac{g(x,t)t}{4}-G(x,t)+\frac{K(x)t^6}{12}\ge-\theta t^2\quad\text{for all }(x,t)\in\mathbb{R}^3\times\mathbb{R}.$$

When V satisfies the coercive assumption:

(V2) $V\in\mathcal{C}(\mathbb{R}^3,[0,\infty))$ and there exists a constant $r>0$ such that

$$\lim_{|y|\to\infty}\operatorname{meas}\big\{x\in\mathbb{R}^3:|x-y|\le r,\ V(x)\le M\big\}=0,\quad\text{for all }M>0,$$

K and g satisfies (K1), (G1) and (G2′), we establish the existence of semiclassical states for $(\mathcal{CP})_\varepsilon$.

Our last result is as follows.

Theorem 1.3. *Assume V, K and g satisfy* (V2), (K1), (G1) *and* (G2′). *Then there exists a constant $\varepsilon_0>0$ determined by V, K and g (see* (3.13) *below) such that for any $\varepsilon\in(0,\varepsilon_0]$, problem $(\mathcal{CP})_\varepsilon$ admits a nontrivial solution $(\widetilde{u}_\varepsilon,\phi_{\widetilde{u}_\varepsilon})$ which satisfies the second inequality of* (1.4).

To prove Theorems 1.2 and 1.3, we use some new analytic techniques to show that the Palais–Smale sequence under some suitable level value converges weakly to a nontrivial solution of $(\mathcal{CP})_\varepsilon$ (see Lemmas 3.6 and 4.3). In particular, to obtain the above desired range of the mountain pass level, we construct a special mountain pass curve (see Lemma 3.2), which is different from the one of [19], [38].

Remark 1.4. In Theorem 1.2, condition (G3) is much weaker than (AR) and (AQ1); moreover, $V(0)$ may not be equal to zero and $G(x,u)$ is allowed to be sign-changing. In this sense, Theorem 1.2 improves and complements the corresponding results in [19], [38]. In addition, Theorem 1.3 seems to be the first result on Schrödinger–Poisson systems with critical exponent since g in $(\mathcal{CP})_\varepsilon$ only need to satisfies (G1) and (G2′).

The rest of the paper is organized as follows. In Section 2, we establish the existence and concentration of semiclassical states for $(\mathcal{BP})_\varepsilon$, and give the proof of Theorem 1.1. In Sections 3 and 4, we consider the existence of semiclassical states for $(\mathcal{CP})_\varepsilon$, and complete the proofs of Theorems 1.2 and 1.3, respectively.

Throughout this paper, $H^1(\mathbb{R}^3)$ is the usual Sobolev space equipped with the inner product and norm

$$(u,v)=\int_{\mathbb{R}^3}(\nabla u\cdot\nabla v+uv)\,dx,\quad \|u\|=(u,u)^{1/2},\quad \text{for all } u,v\in H^1(\mathbb{R}^3);$$

for any $1\le s<\infty$, we denote by $\|\cdot\|_s$ the usual norm of the Lebesgue space $L^s(\mathbb{R}^3)$; $B_r(x)=\{y\in\mathbb{R}^3:|y-x|<r\}$, and positive constants possibly different in different places, by $C_1,C_2,\ldots$

2. Proof of Theorem 1.1

In this section, we study the existence and concentration of semiclassical states for $(\mathcal{BP})_\varepsilon$ and give the proof of Theorem 1.1. As observed in [5], [6], by the Lax–Milgram theorem for every $u\in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u\in\mathcal{D}^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta\phi_u=u^2\quad\text{with}\quad \phi_u(x)=\int_{\mathbb{R}^3}u^2(y)/|x-y|\,dy.$$

Moreover, by using the Hardy–Littlewood–Sobolev inequality (see [25] or [26, p. 98]), one has

$$\int_{\mathbb{R}^3}\phi_u(x)u^2=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}\,dx\,dy\le\frac{8\sqrt[3]{2}}{3\sqrt[3]{\pi}}\|u\|_{12/5}^4:=\mathcal{C}_0\|u\|_{12/5}^4, \tag{2.1}$$

for $u\in L^{12/5}(\mathbb{R}^3)$.

Substituting ϕ_u in $(\mathcal{BP})_\varepsilon$ and performing the scaling $u(x)=v(\varepsilon x)$, we can rewrite $(\mathcal{BP})_\varepsilon$ as the following equivalent equation:

$$-\Delta u+u+\varepsilon^2\phi_u(x)u=f(u). \tag{$(\mathcal{B})_\varepsilon$}$$

Obviously, (v,ϕ_v) is a solution of $(\mathcal{BP})_\varepsilon$ if and only if u is a solution of $(\mathcal{B})_\varepsilon$. As we all know, under (F1), $(\mathcal{B})_\varepsilon$ is variational and its solutions are the critical points of the $\mathcal{C}^1$-functional $\mathcal{I}_\varepsilon\colon H^1(\mathbb{R}^3)\to\mathbb{R}$ defined by

$$\mathcal{I}_\varepsilon(u)=\frac{1}{2}\int_{\mathbb{R}^3}\big(|\nabla u|^2+u^2\big)\,dx+\frac{\varepsilon^2}{4}\int_{\mathbb{R}^3}\phi_u(x)u^2\,dx-\int_{\mathbb{R}^3}F(u)\,dx. \tag{2.2}$$

We look for a critical point of $\mathcal{I}_\varepsilon$ as a solution of $(\mathcal{B})_\varepsilon$ by applying the following critical point theory developed by Jeong and Seok [23].

Proposition 2.1 ([23, Theorem 1]). *Let H be a separable Hilbert space with norm $\|\cdot\|$ and let $I_0\colon H\to\mathbb{R}$ be a $\mathcal{C}^1$ functional of the form*

$$I_0(u)=\frac{1}{2}\|u\|^2-P(u),\quad \textit{where } P\colon H\to\mathbb{R} \textit{ and } P'\colon H\to H^* \textit{ are compact.}$$

Consider a functional I_ε perturbed from the limit functional I_0: $I_\varepsilon(u):=I_0(u)+J_\varepsilon(u)$, where $\varepsilon>0$ denotes a small parameter and $J_\varepsilon\colon H\to\mathbb{R}$ is a $\mathcal{C}^1$ functional such that

(J1) *$J_\varepsilon(u)$ and $J'_\varepsilon(u)$ converge to* 0 *locally uniformly for u, i.e. for any $M>0$,*

$$\lim_{\varepsilon\to 0}\sup_{\|u\|\le M}|J_\varepsilon(u)| = \lim_{\varepsilon\to 0}\sup_{\|u\|\le M}|J'_\varepsilon(u)| = 0;$$

(J2) *$J_\varepsilon\colon H\to\mathbb{R}$ and $J'_\varepsilon\colon H\to H^*$ are compact.*

Assume that I_0 satisfies the following conditions:

(I1) *$I_0(0)=0$, there exist $c,r>0$ such that if $\|u\|=r$, then $I_0(u)\ge c$ and there exists $v_0\in H$ such that $\|v_0\|>r$ and $I_0(v_0)<0$;*

(I2) *there exists a critical point $u_0\in H$ of I_0 such that*

$$I_0(u_0)=c_0:=\min_{\gamma\in\Gamma}\max_{s\in[0,1]} I_0(\gamma(s)),$$

where $\Gamma=\{\gamma\in\mathcal{C}([0,1],H):\gamma(0)=0,\ \gamma(1)=v_0\}$;

(I3) *$c_0=\inf\{I_0(u)\mid u\in H\setminus\{0\}, I'_0(u)=0\}$;*

(I4) *the set $\mathcal{K}:=\{u\in H\setminus\{0\}\mid I'_0(u)=0,\ I_0(u)=c_0\}$ is compact in H;*

(I5) *there exists a curve $\gamma_0\in\Gamma$ passing through u_0 at $s=s_0$ and satisfying $I_0(u_0)>I_0(\gamma_0(s))$ for all $s\ne s_0$.*

Then there exists $\overline{\varepsilon}>0$ such that for any $\varepsilon\in(0,\overline{\varepsilon})$, the functional I_ε admits a nontrivial critical point $u_\varepsilon\in H$. In addition, for any sequence $\{\varepsilon_j\}$ converging to 0*, the sequence of critical points $\{u_{\varepsilon_j}\}$ found above converges to some $w\in\mathcal{K}$ up to a subsequence.*

To use Proposition 2.1, in this section we work in the radial functions space $H^1_r(\mathbb{R}^3)$, and consider the limit equation (1.3) of $(\mathcal{B})_\varepsilon$. It is well known that solutions of (1.3) are critical points of the $\mathcal{C}^1$-functional $\mathcal{I}_0\colon H^1_r(\mathbb{R}^3)\to\mathbb{R}$ defined by

$$\mathcal{I}_0(u)=\frac{1}{2}\int_{\mathbb{R}^3}\big(|\nabla u|^2+u^2\big)\,dx-\int_{\mathbb{R}^3}F(u)\,dx. \tag{2.3}$$

Let

$$P(u)=\int_{\mathbb{R}^3}F(u)\,dx \quad\text{and}\quad \mathcal{J}_\varepsilon(u)=\frac{\varepsilon^2}{4}\int_{\mathbb{R}^3}\phi_u(x)u^2\,dx,$$

for all $u\in H^1_r(\mathbb{R}^3)$. Then

$$\mathcal{I}_0(u)=\frac{1}{2}\|u\|^2-P(u),\quad \mathcal{I}_\varepsilon(u)=\mathcal{I}_0(u)+\mathcal{J}_\varepsilon(u),\quad \text{for all } u\in H^1_r(\mathbb{R}^3).$$

It is easy to see that P and P' are compact on $H^1_r(\mathbb{R}^3)$, and $\mathcal{J}_\varepsilon$ satisfies (J1) and (J2) since the Sobolev embedding $H^1_r(\mathbb{R}^3)\hookrightarrow L^q(\mathbb{R}^3)$ for any $q\in(2,6)$ is compact.

PROOF OF THEOREM 1.1. We apply Proposition 2.1 with $H=H^1_r(\mathbb{R}^3)$, $I_0=\mathcal{I}_0$, $J_\varepsilon=\mathcal{J}_\varepsilon$ and $I_\varepsilon=\mathcal{I}_\varepsilon$. To prove Theorem 1.1, it suffices to show that $\mathcal{I}_0$ satisfies (I1)–(I5). In fact, Berestycki and Lions [8] proved that (1.3) has a radially symmetric least energy solution $u_0\in H^1_r(\mathbb{R}^3)$ under (F1) and (F2). Moreover, Jeanjean and Tanaka [22] verified that $\mathcal{I}_0$ has a mountain pass geometry and the least energy solution u_0 of (1.3) is a mountain pass solution.

Therefore, (I1)–(I3) are clearly satisfied. Since $\mathcal{I}_0$ satisfies the Palais–Smale condition in $H_r^1(\mathbb{R}^3)$, then (I4) holds.

Next, we prove that (I5) also holds. Let $(u_0)_t(x) = u_0(x/t)$ for $t > 0$. Then

$$\mathcal{I}_0((u_0)_t) = \frac{t}{2}\|\nabla u_0\|_2^2 + \frac{t^3}{2}\|u_0\|_2^2 - t^3 \int_{\mathbb{R}^3} F(u_0)\,dx. \tag{2.4}$$

It is well known that the solution u_0 of (1.3) satisfies the following Pohožaev identity (see [8, (2.1)]):

$$\mathcal{P}_0(u_0) := \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{3}{2}\|u_0\|_2^2 - 3\int_{\mathbb{R}^3} F(u_0)\,dx = 0. \tag{2.5}$$

Using (F1), (2.4) and (2.5), it is easy to check that there exists $T > 0$ such that $\mathcal{I}_0((u_0)_t) < 0$ for all $t \geq T$. Setting

$$\gamma_0(t) = \begin{cases} (u_0)_{(tT)} & \text{for } t > 0, \\ 0 & \text{for } t = 0. \end{cases} \tag{2.6}$$

Then $\gamma_0 \in \Gamma$, where Γ is defined by Proposition 2.1 (I2). By a simple calculation, we can deduce that $\mathcal{I}_0(\gamma_0(t))$ has unique maximum value at $t = 1/T$, that is $\max_{t\in[0,1]} \mathcal{I}_0(\gamma_0(t)) = \mathcal{I}_0(u_0) > I_0(\gamma_0(t))$ for all $t \neq 1/T$. Hence, (I1)–(I5) are satisfied.

In view of Proposition 2.1, there exists $\overline{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \overline{\varepsilon})$, $\mathcal{I}_\varepsilon$ admits a nontrivial critical point $\overline{u}_\varepsilon \in H_r^1(\mathbb{R}^3)$. Let $\overline{v}_\varepsilon(x) = \overline{u}_\varepsilon(x/\varepsilon)$. Then $(\overline{v}_\varepsilon, \phi_{\overline{v}_\varepsilon}) \in H_r^1(\mathbb{R}^3) \times \mathcal{D}_r^{1,2}(\mathbb{R}^3)$ is a nontrivial solution of $(\mathcal{BP})_\varepsilon$ for any $\varepsilon \in (0, \overline{\varepsilon})$. In addition, for any sequence $\{\varepsilon_j\}$ converging to 0, the sequence of critical points $\{\overline{v}_{\varepsilon_j}(\varepsilon_j x)\}$ converges to a radial least energy solution of (1.3) up to a subsequence.

3. Proof of Theorem 1.2

In this section, we consider the existence of semiclassical states for $(\mathcal{CP})_\varepsilon$ satisfying (V1), and give the proofs of Theorem 1.2. For this purpose, we define the function space

$$E := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2\,dx < \infty \right\}$$

with the scalar product and norm

$$(u, v)_{E,\lambda} = \int_{\mathbb{R}^3} [\nabla u \cdot \nabla v + \lambda V(x)uv]\,dx,$$

$$\|u\|_{E,\lambda} = \left\{ \int_{\mathbb{R}^3} \left[|\nabla u|^2 + \lambda V(x)u^2\right] dx \right\}^{1/2}.$$

In view of [32, Lemma 1], under (V1), there exists a constant $\gamma_0 > 0$ independent of λ such that

$$\|u\| \le \gamma_0 \|u\|_{E,\lambda}, \quad u \in E,\ \lambda \ge 1. \tag{3.1}$$

It is easy to see that $(E, \|\cdot\|_{E,\lambda})$ is a Hilbert space for $\lambda \ge 1$. Furthermore, by the Sobolev embedding theorem and (3.1), for $s \in [2,6]$ there exists a constant $\gamma_s > 0$ independent of λ such that

$$\|u\|_s \le \gamma_s \|u\| \le \gamma_0 \gamma_s \|u\|_{E,\lambda}, \quad u \in E,\ \lambda \ge 1. \tag{3.2}$$

Substituting ϕ_u in $(\mathcal{CP})_\varepsilon$, we reduce $(\mathcal{CP})_\varepsilon$ to the following single equation

$$(\widetilde{\mathcal{CP}})_\varepsilon \qquad -\varepsilon^2 \Delta u + V(x)u + \phi_u(x)u = g(x,u) + K(x)u^5.$$

Solutions of $(\widetilde{\mathcal{CP}})_\varepsilon$ are the critical points of the $\mathcal{C}^1$-functional $\widetilde{I}_\varepsilon \colon E \to \mathbb{R}$ defined by

$$\begin{aligned} \widetilde{I}_\varepsilon(u) = {}& \frac{1}{2}\int_{\mathbb{R}^3} \big(\varepsilon^2 |\nabla u|^2 + V(x)u^2\big)\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u(x)u^2\,dx \\ & - \int_{\mathbb{R}^3} G(x,u)\,dx - \frac{1}{6}\int_{\mathbb{R}^3} K(x)u^6\,dx. \end{aligned} \tag{3.3}$$

Let $\lambda = \varepsilon^{-2}$, then $(\widetilde{\mathcal{CP}})_\varepsilon$ becomes the following equation

$$(\mathcal{C})_\lambda \qquad -\Delta u + \lambda V(x)u + \lambda \phi_u(x)u = \lambda g(x,u) + \lambda K(x)u^5.$$

Solutions of $(\mathcal{C})_\lambda$ are critical points of the $\mathcal{C}^1$-functional $\Phi_\lambda \colon E \to \mathbb{R}$ defined by

$$\begin{aligned} \Phi_\lambda(u) = {}& \frac{1}{2}\int_{\mathbb{R}^3} \big[|\nabla u|^2 + \lambda V(x)u^2\big]\,dx + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_u(x)u^2\,dx \\ & - \lambda \int_{\mathbb{R}^3} G(x,u)\,dx - \frac{\lambda}{6}\int_{\mathbb{R}^3} K(x)u^6\,dx. \end{aligned} \tag{3.4}$$

Obviously, any critical point of the functional $\Phi_{\varepsilon^{-1/2}}$ is a solution $(\widetilde{\mathcal{CP}})_\varepsilon$ for $\varepsilon > 0$.

To prove the existence of critical points of $\widetilde{I}_\varepsilon$, we look for critical points of Φ_λ in the following two sections. To this end, we recall a geometrical result due to Brezis and Nirenberg [9] which is an expression of the Ambrosetti–Rabinowitz [1] mountain pass theorem without the (PS) condition.

Lemma 3.1 ([9], Theorem 2.2). *Assume that Φ is a $\mathcal{C}^1$-functional on a Banach space X. Suppose there exists a neighbourhood U of 0 in X and a constant ϱ such that $\Phi(u) \ge \varrho$ for all $u \in \partial U$, and $\Phi(0) < \varrho$ and $\Phi(v) < \varrho$ for some $v \notin U$. Set*

$$c = \inf_{p \in \mathcal{P}} \max_{t \in [0,1]} \Phi(p(t)), \quad \text{where } \mathcal{P} = \{p \in \mathcal{C}([0,1],X) : p(0) = 0, p(1) = v\}.$$

Then there is a sequence $\{u_n\} \subset X$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ in X^*.

Let

$$h_0 := \max\{1, 5375\pi\sqrt{6K_\infty}(\gamma_0\gamma_6)^3\}. \tag{3.5}$$

Inspired by [27], we define the following functions:

$$\psi(x) := \begin{cases} \dfrac{1}{h_0} & \text{if } |x| \le h_0, \\ \dfrac{1}{h_0}\left(2 - \dfrac{|x|}{h_0}\right) & \text{if } h_0 < |x| \le 2h_0, \\ 0 & \text{if } |x| > 2h_0 \end{cases} \tag{3.6}$$

and

$$e_\lambda(x) := \lambda^{1/2}\psi(\lambda^{7/6}x).$$

By a simple calculation, one has

$$\|\nabla\psi\|_2^2 \le \int_{|x|\le 2h_0} \frac{1}{h_0^4}\,dx = \frac{32\pi}{3}\,h_0^{-1}, \tag{3.7}$$

$$\|\psi\|_2^2 \le \int_{|x|\le 2h_0} \frac{1}{h_0^2}\,dx = \frac{32\pi}{3}\,h_0, \tag{3.8}$$

$$\|\psi\|_{12/5}^4 \le \left(\int_{|x|\le 2h_0} \frac{1}{h_0^{12/5}}\,dx\right)^{5/3} = \left(\frac{32\pi}{3}\right)^{5/3} h_0 \tag{3.9}$$

and

$$\|\psi\|_4^4 \ge \frac{4\pi}{3}\,h_0^{-1}. \tag{3.10}$$

From (K1) and (G1), we can deduce that

$$\lim_{|t|\to\infty} \frac{6G(x,t) + K(x)t^6}{t^4} = +\infty \quad \text{uniformly in } x \in \mathbb{R}^3 \tag{3.11}$$

and there exist constants $\theta_1, \theta_2 > 0$ such that

$$G(x,t) + \frac{1}{6}K(x)t^6 \ge \theta_1 t^4 - \theta_2 t^2, \quad \text{for all } (x,t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{3.12}$$

Setting

$$\varepsilon_0^{-2} = \lambda_0 := \max\left\{1, \left[\sup_{|x|\le 2h_0} V(x) + 2\theta_2\right]h_0^2, \left(\frac{2^{26}\pi}{3^5\theta_1^3}\right)^{1/7} h_0^{6/7}\right\}. \tag{3.13}$$

Lemma 3.2. *Assume that* (V1), (K1) *and* (G1) *hold. Then*

(a) $\Phi_\lambda(5\theta_1^{-1/2}e_\lambda) < 0$ *for all* $\lambda \ge \lambda_0$;

(b) *for every* $\lambda \ge \lambda_0$, *there exist constants* $\rho, \delta > 0$ *such that*

$$\Phi_\lambda(u) \ge \delta, \quad \textit{for all } \|u\|_{E,\lambda} = \rho.$$

Proof. (a) By (3.4)–(3.10) and (3.13), we have

$$
\begin{aligned}
\Phi_\lambda(te_\lambda) &= \frac{t^2}{2}\int_{\mathbb{R}^3}\big[|\nabla e_\lambda|^2+\lambda V(x)e_\lambda^2\big]\,dx+\frac{\lambda t^4}{4}\int_{\mathbb{R}^3}\phi_{e_\lambda}e_\lambda^2\,dx \\
&\quad -\lambda\int_{\mathbb{R}^3}\bigg[G(x,te_\lambda)+\frac{1}{6}\,K(x)|te_\lambda|^6\bigg]\,dx \\
&\le \frac{t^2}{2}\int_{\mathbb{R}^3}\big[|\nabla e_\lambda|^2+\lambda V(x)e_\lambda^2\big]\,dx \\
&\quad +\frac{\lambda\mathcal{C}_0 t^4}{4}\|e_\lambda\|_{12/5}^4+\lambda\theta_2 t^2\|e_\lambda\|_2^2-\lambda\theta_1 t^4\|e_\lambda\|_4^4 \\
&= \lambda^{-1/2}\,t^2\bigg[\frac{1}{2}\|\nabla\psi\|_2^2+\frac{\lambda^{-1}}{2}\int_{\mathbb{R}^3}\big[V(\lambda^{-7/6}x)+2\theta_2\big]\psi^2\,dx \\
&\quad +\frac{\mathcal{C}_0}{4}\,\lambda^{-7/3}t^2\|\psi\|_{12/5}^4-\theta_1 t^2\|\psi\|_4^4\bigg] \\
&\le \lambda^{-1/2}t^2\bigg\{\frac{16\pi}{3}\,h_0^{-1}+\frac{16\pi}{3}h_0\lambda^{-1}\bigg[\sup_{|x|\le 2h_0}V(x)+2\theta_2\bigg] \\
&\quad +\frac{\mathcal{C}_0}{4}\bigg(\frac{32\pi}{3}\bigg)^{5/3}h_0\lambda^{-7/3}t^2-\frac{4\theta_1\pi}{3}h_0^{-1}t^2\bigg\} \\
&= \lambda^{-1/2}\,\frac{2\pi}{3h_0}t^2\bigg\{8+8h_0^2\lambda^{-1}\bigg[\sup_{|x|\le 2h_0}V(x)+2\theta_2\bigg] \\
&\quad +\frac{1}{\pi}\bigg(\frac{2}{\pi}\bigg)^{1/3}\bigg(\frac{32\pi}{3}\bigg)^{5/3}h_0^2\lambda^{-7/3}t^2-2\theta_1 t^2\bigg\} \\
&\le \lambda^{-1/2}\,\frac{2\pi}{3h_0}\,t^2(16-\theta_1 t^2),
\end{aligned}
\tag{3.14}
$$

for all $t\ge 0$, $\lambda\ge\lambda_0$, which implies

$$
\Phi_\lambda(te_\lambda)\le -\lambda^{-1/2}\frac{150\pi}{\theta_1 h_0}<0,\quad \text{for all } t\ge 5\theta_1^{-1/2},\ \lambda\ge\lambda_0.
\tag{3.15}
$$

(b) By (K1) and (G1), there exists a constant $\mathcal{C}_1>0$ such that

$$
\int_{\mathbb{R}^3}\bigg[G(x,u)+\frac{1}{6}K(x)u^6\bigg]\,dx\le\frac{1}{4\lambda(\gamma_2\gamma_0)^2}\|u\|_2^2+\mathcal{C}_1\|u\|_6^6,
\tag{3.16}
$$

for all $u\in E$. Then, for every $\lambda\ge\lambda_0$, it follows from (3.1), (3.2), (3.4) and (3.16) that

$$
\begin{aligned}
\Phi_\lambda(u) &\ge \frac{1}{2}\|u\|_{E,\lambda}^2-\frac{1}{4(\gamma_2\gamma_0)^2}\|u\|_2^2-\mathcal{C}_1\lambda\|u\|_6^6 \\
&\ge \frac{1}{4}\|u\|_{E,\lambda}^2\big[1-4\lambda\mathcal{C}_1(\gamma_0\gamma_6)^2\|u\|_{E,\lambda}^4\big],
\end{aligned}
$$

for all $u\in E$, which, together with (3.15), implies that, for every $\lambda\ge\lambda_0$,

$$
\Phi_\lambda(u)\ge\frac{1}{8}\rho^2:=\delta,\quad \text{for all } \|u\|_{E,\lambda}=\rho=\bigg(\frac{1}{8\lambda\mathcal{C}_1(\gamma_0\gamma_6)^2}\bigg)^{1/4}.
\tag{3.17}
$$

□

Lemma 3.3. *Assume that* (V1), (K1) *and* (G1) *hold. Then, for every* $\lambda \geq \lambda_0$, *there exists a sequence* $\{u_{\lambda,n}\} \subset E$ *such that*

$$\Phi_\lambda(u_{\lambda,n}) \to c_\lambda > 0, \qquad \Phi'_\lambda(u_{\lambda,n}) \to 0, \tag{3.18}$$

where

$$\begin{aligned} c_\lambda &= \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\lambda(\gamma(t)), \\ \Gamma &= \big\{\gamma \in \mathcal{C}([0,1], E) : \gamma(0) = 0,\ \gamma(1) = 5\theta_1^{-1/2} e_\lambda\big\}. \end{aligned} \tag{3.19}$$

Proof. Since $\Phi_\lambda(0) = 0$, from Lemmas 3.1 and 3.2, we then deduce the above conclusion. □

Lemma 3.4. *Assume that* (V1), (K1) *and* (G1) *hold. Then, for every* $\lambda \geq \lambda_0$,

$$c_\lambda \leq \sup_{t\geq 0} \Phi_\lambda(te_\lambda) \leq \frac{1}{126\sqrt{6K_\infty}(\gamma_0\gamma_6)^3}\lambda^{-1/2}.$$

Proof. For every $\lambda \geq \lambda_0$, by (3.14), (3.15) and the definition of c_λ, we have

$$\begin{aligned} c_\lambda &\leq \sup_{t\in[0,1]} \Phi_\lambda(5t\theta_1^{-1/2}e_\lambda) = \sup_{t\geq 0} \Phi_\lambda(te_\lambda) \\ &\leq \lambda^{-1/2}\frac{2\pi}{3h_0} \sup_{t\geq 0} \big[t^2(16 - \theta_1 t^2)\big] \\ &= \lambda^{-1/2}\frac{128\pi}{3h_0} \leq \frac{1}{126\sqrt{6K_\infty}(\gamma_0\gamma_6)^3}\lambda^{-1/2}. \end{aligned} \tag{3.20}$$ □

Lemma 3.5. *Assume that* (V1), (K1), (G1) *and* (G2) *hold. Then any sequence* $\{u_{\lambda,n}\} \subset E$ *satisfying* (3.18) *is bounded for every* $\lambda \geq \lambda_0$.

Proof. By (G2), (3.4) and (3.18), one has

$$\begin{aligned} c_\lambda + o(1) &= \Phi_\lambda(u_{\lambda,n}) - \frac{1}{4}\langle \Phi'(u_{\lambda,n}), u_{\lambda,n}\rangle \\ &= \frac{1}{4}\|u_{\lambda,n}\|^2_{E,\lambda} + \lambda \int_{\mathbb{R}^3} \bigg[\frac{1}{4}\, g(x, u_{\lambda,n})u_{\lambda,n} - G(x, u_{\lambda,n}) + \frac{1}{12}\, K(x)u^6_{\lambda,n}\bigg]\, dx \\ &\geq \frac{1}{4}\|u_{\lambda,n}\|^2_{E,\lambda}. \end{aligned} \tag{3.21}$$

This shows that $\{u_{\lambda,n}\}$ is bounded in E. □

Lemma 3.6. *Assume that* (V1), (K1) *and* (G1)–(G3) *hold. Then, for any* $\lambda \geq \lambda_0$, *problem* $(\mathcal{C})_\lambda$ *admits at least one nontrivial solution* $\widetilde{u}_\lambda \in E$ *such that*

$$0 < \Phi_\lambda(\widetilde{u}_\lambda) \leq \frac{1}{126\sqrt{6K_\infty}(\gamma_0\gamma_6)^3}\, \lambda^{-1/2}. \tag{3.22}$$

Proof. In view of Lemmas 3.3–3.5, for every $\lambda \geq \lambda_0$ there exists a bounded sequence $\{u_{\lambda,n}\}$ satisfying (3.18), for brevity, we denote it by $\{u_n\}$. Then there exists $\widetilde{u}_\lambda \in E$ such that up to a subsequence, $u_n \rightharpoonup \widetilde{u}_\lambda$ in E, $u_n \to \widetilde{u}_\lambda$ in $L^s_{\rm loc}(\mathbb{R}^3)$ for $2 \leq s < 6$ and $u_n \to \widetilde{u}_\lambda$ almost everywhere in $\mathbb{R}^3$. Next, we prove that $\widetilde{u}_\lambda \neq 0$. Arguing by contradiction, suppose that $\widetilde{u}_\lambda = 0$, i.e. $u_n \rightharpoonup 0$ in E. Then $u_n \to 0$ in $L^s_{\rm loc}(\mathbb{R}^3)$ for $2 \le s < 6$ and $u_n \to 0$ almost everywhere in $\mathbb{R}^3$. Since $\mathcal{V}_b$ is a set of finite measure and $u_n \rightharpoonup 0$ in E, then

$$\|u_n\|_2^2 \le \int_{\mathbb{R}^3\setminus\mathcal{V}_b} u_n^2\,dx + \int_{\mathcal{V}_b} u_n^2\,dx \le \frac{1}{\lambda b}\,\|u_n\|_{E,\lambda}^2 + o(1). \tag{3.23}$$

For $s \in (2,6)$, by (3.1), (3.2) and (3.23), one has

$$\begin{aligned}\|u_n\|_s^s &\le \|u_n\|_2^{(6-s)/2}\|u_n\|_6^{3(s-2)/2} \\ &\le (\gamma_0\gamma_6)^{3(s-2)/2}(\lambda b)^{-(6-s)/4}\|u_n\|_{E,\lambda}^s + o(1).\end{aligned} \tag{3.24}$$

Let

$$\Omega_n := \left\{x \in \mathbb{R}^3 : \left|\frac{g(x,u_n)}{u_n}\right| \le \frac{b}{3}\right\}.$$

By (3.23), one has

$$\lambda\int_{\Omega_n} |g(x,u_n)u_n|\,dx = \lambda\int_{\Omega_n}\left|\frac{g(x,u_n)}{u_n}\right|u_n^2\,dx \le \frac{\lambda b}{3}\,\|u_n\|_2^2 \le \frac{1}{3}\|u_n\|_{E,\lambda}^2. \tag{3.25}$$

By (G1), there exists $\varrho > 0$ small enough such that

$$|t| \le \varrho \;\Rightarrow\; \left|\frac{g(x,t)}{t}\right| \le \frac{b}{3},$$

which implies

$$\left|\frac{g(x,t)}{t}\right| > \frac{b}{3} \Rightarrow |t| > \varrho. \tag{3.26}$$

From (G1), (3.26) and the boundedness of $\{\|u_n\|_{E,\lambda}\}$, we deduce that

$$\begin{aligned}\frac{\lambda b}{3}\,\varrho^2|\mathbb{R}^3\setminus\Omega_n| &\le \lambda\int_{\mathbb{R}^3\setminus\Omega_n}\left|\frac{g(x,u_n)}{u_n}\right|u_n^2\,dx \\ &\le \lambda\int_{\mathbb{R}^3}|g(x,u_n)u_n|\,dx \le \lambda C_1\big(\|u_n\|_{E,\lambda}^2 + \|u_n\|_{E,\lambda}^6\big) \le \lambda C_2,\end{aligned} \tag{3.27}$$

which implies

$$\big|\mathbb{R}^3\setminus\Omega_n\big| \le \frac{3C_2}{b\varrho^2}. \tag{3.28}$$

By (G1), for every $\varepsilon > 0$ and some $q \in (2,6)$, there exists a constant $C_\varepsilon > 0$ such that

$$|g(x,t)t| \le \varepsilon\big(t^2 + t^6\big) + C_\varepsilon|t|^q, \quad \text{for all } (x,t) \in \mathbb{R}^3\times\mathbb{R}. \tag{3.29}$$

Since $u_n \to 0$ in $L^s_{\rm loc}(\mathbb{R}^3)$ for $s \in (2,6)$, then it follows from (3.28) and (3.29) that for any $\varepsilon > 0$,

$$\begin{aligned}\int_{\mathbb{R}^3\setminus\Omega_n}|g(x,u_n)u_n|\,dx &\le \varepsilon\big(\|u_n\|_2^2 + \|u_n\|_6^6\big) + \int_{\mathbb{R}^3\setminus\Omega_n}|u_n|^q\,dx \\ &= \varepsilon\big(\|u_n\|_2^2 + \|u_n\|_6^6\big) + o(1).\end{aligned} \tag{3.30}$$

By (G3), (3.18), (3.25) and (3.30), we have

$$\begin{aligned}c_\lambda + o(1) &= \Phi_\lambda(u_n) - \frac{1}{6}\langle \Phi'_\lambda(u_n), u_n\rangle \\ &= \frac{1}{3}\|u_n\|^2_{E,\lambda} + \frac{\lambda}{12}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\, dx + \lambda \int_{\mathbb{R}^3} \left[\frac{1}{2} g(x,u_n)u_n - G(x,u_n)\right] dx \\ &\quad - \frac{\lambda}{3}\int_{\Omega_n} g(x,u_n)u_n\, dx - \frac{\lambda}{3}\int_{\mathbb{R}^3\setminus\Omega_n} g(x,u_n)u_n\, dx \\ &\geq \frac{2}{9}\|u_n\|^2_{E,\lambda} + \frac{\lambda}{12}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\, dx - \frac{\lambda\varepsilon}{3}\big(\|u_n\|_2^2 + \|u_n\|_6^6\big) + o(1),\end{aligned}$$

which, together with the fact that ε is arbitrary, yields

$$\lambda \int_{\mathbb{R}^3} \phi_{u_n} u_n^2\, dx \leq 12 c_\lambda + o(1). \tag{3.31}$$

Note that

$$\begin{aligned}c_\lambda + o(1) &= \Phi_\lambda(u_n) - \frac{1}{2}\langle \Phi'_\lambda(u_n), u_n\rangle \\ &= -\frac{\lambda}{2}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\, dx + \lambda \int_{\mathbb{R}^3} \left[\frac{1}{2} g(x,u_n)u_n - G(x,u_n)\right] dx + \frac{\lambda}{3}\int_{\mathbb{R}^3} K(x)u_n^6\, dx.\end{aligned} \tag{3.32}$$

By (G3), (3.31) and (3.32), one has

$$\lambda \int_{\mathbb{R}^3} K(x)u_n^6\, dx \leq 21 c_\lambda + o(1). \tag{3.33}$$

Thus, it follows from (3.13), (3.24), (3.33), the Hölder inequality and Lemma 3.4 that

$$\begin{aligned}\lambda \int_{\mathbb{R}^3} K(x)u_n^6\, dx &= \lambda\bigg(\int_{\mathbb{R}^3} K(x)u_n^6\, dx\bigg)^{1/3}\bigg(\int_{\mathbb{R}^3} K(x)u_n^6\, dx\bigg)^{2/3} \\ &\leq \lambda K_\infty^{1/3}\|u_n\|_6^2\bigg(\frac{21c_\lambda}{\lambda}\bigg)^{2/3} + o(1) \\ &\leq K_\infty^{1/3}\gamma_6^2\,\gamma_0^2(21)^{2/3}(\lambda^{1/2}c_\lambda)^{2/3}\|u_n\|^2_{E,\lambda} + o(1) \\ &\leq \frac{1}{6}\|u_n\|^2_{E,\lambda} + o(1).\end{aligned} \tag{3.34}$$

Combining (3.18), (3.25), (3.26), (3.29) and (3.34), we have

$$\begin{aligned}o(1) &\geq \frac{5}{6}\|u_n\|^2_{E,\lambda} - \lambda\int_{\Omega_n} |g(x,u_n)u_n|\, dx - \lambda\int_{\mathbb{R}^3} K(x)u_n^6\, dx \\ &\geq \frac{1}{6}\|u_n\|^2_{E,\lambda} + o(1),\end{aligned} \tag{3.35}$$

which implies that $\|u_n\|_{E,\lambda} \to 0$ and so $c_\lambda = 0$. The contradiction shows $\widetilde{u}_\lambda \neq 0$. By a standard argument, we have $\Phi'_\lambda(\widetilde{u}_\lambda) = 0$. Then it follows from (3.18), the weak semicontinuity of norm and Fatou's Lemma that

$$\begin{aligned}c_\lambda &= \lim_{n\to\infty}\bigg[\Phi_\lambda(u_n) - \frac{1}{4}\langle\Phi'(u_n),u_n\rangle\bigg]\\ &= \lim_{n\to\infty}\bigg\{\frac{1}{4}\|u_n\|^2_{E,\lambda} + \lambda\int_{\mathbb{R}^3}\bigg[\frac{1}{4}g(x,u_n)u_n - G(x,u_n) + \frac{1}{12}K(x)u_n^6\bigg]\,dx\bigg\}\\ &\geq \frac{1}{4}\|\widetilde{u}_\lambda\|^2_{E,\lambda} + \lambda\int_{\mathbb{R}^3}\bigg[\frac{1}{4}\,g(x,\widetilde{u}_\lambda)\widetilde{u}_\lambda - G(x,\widetilde{u}_\lambda) + \frac{1}{12}\,K(x)\widetilde{u}_\lambda^6\bigg]\,dx\\ &= \Phi_\lambda(\widetilde{u}_\lambda) - \frac{1}{4}\langle\Phi'(\widetilde{u}_\lambda),\widetilde{u}_\lambda\rangle = \Phi_\lambda(\widetilde{u}_\lambda) > 0,\end{aligned}$$

which, together with Lemma 3.4, implies that (3.22) holds. □

Proof of Theorem 1.2. Since $\varepsilon = \lambda^{-1/2}$, we deduce from Lemma 3.6 that the conclusion of Theorem 1.2 holds. □

4. Proof of Theorem 1.3

In this section, we consider the existence of semiclassical states for $(\mathcal{CP})_\varepsilon$ satisfying (V2), and give the proof of Theorem 1.3. Since (V2) implies (V1), the conclusions in Lemmas 3.2–3.4 of Section 3 still hold. Moreover, under (V2), we have the following compactness lemma.

Lemma 4.1 (Lemma 3.1 [4], [42]). *Assume that* (V2) *holds. Then the embedding* $E \hookrightarrow L^s(\mathbb{R}^3)$ *is compact for* $2 \leq s < 6$.

Lemma 4.2. *Assume that* (V2), (K1), (G1) *and* (G2′) *hold. Then any sequence* $\{u_{\lambda,n}\} \subset E$ *satisfying* (3.18) *is bounded for every* $\lambda \geq \lambda_0$.

Proof. Arguing by contradiction, suppose that $\|u_{\lambda,n}\|_{E,\lambda} \to \infty$. For every $\lambda \geq \lambda_0$, let $w_n = u_{\lambda,n}/\|u_{\lambda,n}\|_{E,\lambda}$. Then $\|w_n\|_{E,\lambda} = 1$. Passing to a subsequence, we may assume that $w_n \rightharpoonup w$ in E, then by Lemma 4.1, we have $w_n \to w$ in $L^s(\mathbb{R}^3)$ for $2 \leq s < 6$, and $w_n \to w$ a.e. on $\mathbb{R}^3$. Note that

$$\begin{aligned}c_\lambda + o(1) &= \Phi_\lambda(u_{\lambda,n}) - \frac{1}{4}\langle\Phi'(u_{\lambda,n}),u_{\lambda,n}\rangle\\ &= \frac{1}{4}\|u_{\lambda,n}\|^2_{E,\lambda}\\ &\quad + \lambda\int_{\mathbb{R}^3}\bigg[\frac{1}{4}\,g(x,u_{\lambda,n})u_{\lambda,n} - G(x,u_{\lambda,n}) + \frac{1}{12}\,K(x)|u_{\lambda,n}|^6\bigg]\,dx\\ &\geq \frac{1}{4}\|u_{\lambda,n}\|^2_{E,\lambda} - \lambda\theta\|u_{\lambda,n}\|_2^2.\end{aligned}\tag{4.1}$$

Multiplying (4.1) by $1/\|u_{\lambda,n}\|^2_{E,\lambda}$, we deduce from $\theta > 0$ and $w_n \to w$ in $L^2(\mathbb{R}^3)$ that

$$\lambda\theta\|w\|_2^2 = \lambda\theta\lim_{n\to\infty}\|w_n\|_2^2 \geq \frac{1}{4} > 0.\tag{4.2}$$

This shows that $w \neq 0$. Let $\Omega := \{y \in \mathbb{R}^3 : w(y) \neq 0\}$. Since

$$w_n(x) = \frac{u_{\lambda,n}(x)}{\|u_{\lambda,n}\|_{E,\lambda}} \to w(x)$$

for almost every $x \in \Omega$ and $\|u_{\lambda,n}\|_{E,\lambda} \to \infty$, then we have $\lim_{n\to\infty} |u_{\lambda,n}(x)| = \infty$ for almost every $x \in \Omega$. Thus, it follows from (2.1), (3.11), (3.12), Lebesgue dominated convergence theorem and Fatou's Lemma that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c_\lambda + o(1)}{\|u_{\lambda,n}\|_{E,\lambda}^4} = \lim_{n\to\infty} \frac{\Phi_\lambda(u_{\lambda,n})}{\|u_{\lambda,n}\|_{E,\lambda}^4} \\
&\leq \frac{\lambda}{4} \limsup_{n\to\infty} \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 \, dx \\
&\quad - \frac{\lambda}{6} \liminf_{n\to\infty} \int_{\mathbb{R}^3} \frac{6G(x, u_{\lambda,n}) + K(x) u_{\lambda,n}^6 + 6\theta_2 u_{\lambda,n}^2}{u_{\lambda,n}^4} w_n^4 \, dx \\
&\leq \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_w w^2 \, dx \\
&\quad - \frac{\lambda}{6} \int_{\Omega} \liminf_{n\to\infty} \frac{6G(x, u_{\lambda,n}) + K(x) u_{\lambda,n}^6 + 6\theta_2 u_{\lambda,n}^2}{u_{\lambda,n}^4} w_n^4 \, dx = -\infty.
\end{aligned} \tag{4.3}$$

The contradiction shows that $\{u_{\lambda,n}\}$ is bounded in E. □

Lemma 4.3. *Assume that* (V2), (K1), (G1) *and* (G2′) *hold. Then, for any* $\lambda \geq \lambda_0$, *problem* $(\mathcal{C})_\lambda$ *admits at least one nontrivial solution* $\widetilde{u}_\lambda \in E$ *such that* (3.22) *holds.*

Proof. In view of Lemmas 3.3, 3.4 and 4.2, for every $\lambda \geq \lambda_0$ there exists a bounded sequence $\{u_{\lambda,n}\}$ satisfying (3.18). Then there exists $\widetilde{u}_\lambda \in E$ such that up to a subsequence, $u_{\lambda,n} \rightharpoonup \widetilde{u}_\lambda$ in E. By Lemma 4.1, one has $u_{\lambda,n} \to \widetilde{u}_\lambda$ in $L^s(\mathbb{R}^3)$ for $2 \leq s < 6$. If $\widetilde{u}_\lambda = 0$, then by (3.32) and the Lebesgue dominated convergence theorem, one has

$$\lambda \int_{\mathbb{R}^3} K(x) |u_{\lambda,n}|^6 \, dx \leq 3c_\lambda + o(1),$$

which implies that (3.34) holds. Thus it follows from (3.18), (3.29), (3.34) and the Lebesgue dominated convergence theorem that

$$\begin{aligned}
o(1) &= \langle \Phi'(u_{\lambda,n}), u_{\lambda,n} \rangle \\
&\geq \frac{5}{6} \|u_{\lambda,n}\|_{E,\lambda}^2 - \lambda \int_{\mathbb{R}^3} K(x) |u_{\lambda,n}|^6 \, dx + o(1) \geq \frac{2}{3} \|u_{\lambda,n}\|_{E,\lambda}^2 + o(1),
\end{aligned}$$

which implies that $\|u_{\lambda,n}\|_{E,\lambda} \to 0$ and so $c_\lambda = 0$. This contradiction shows $\widetilde{u}_\lambda \neq 0$. A standard argument shows that $\Phi'(\widetilde{u}_\lambda) = 0$. From (3.18), Lemma 4.1 and Lebesgue dominated convergence theorem, one can deduce easily that $u_{\lambda,n} \to \widetilde{u}_\lambda$ in E, and so (3.22) holds. □

PROOF OF THEOREM 1.3. Since $\varepsilon = \lambda^{-1/2}$, we deduce from Lemma 4.3 that the conclusion 1.3 holds. □

Acknowledgements. The authors thank the anonymous referees for their valuable suggestions and comments.

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Manuscript received September 2, 2018

accepted November 2, 2018

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