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ON THE EXISTENCE OF SKYRMIONS IN PLANAR LIQUID CRYSTALS

Carlo Greco

Abstract. The study of topologically nontrivial field configurations is an important topic in many branches of physics and applied sciences. In this paper we are interested to the existence of such structures, the so-called skyrmions, in the context of liquid crystals. More precisely, we consider a two-dimensional nematic or cholesteric liquid crystal. In the nematic case we use a Bogomol'nyi type decomposition in order to get a topological lower bound on the configurations with a given degree for the full Oseen–Frank energy functional, and so we can find a global minimum of degree ± 1 for the energy. Then we consider the cholesteric case in presence of an electric field under the one constant approximation assumption, and, by using the concentration-compactness method, we prove the existence of a minimum again on the configurations of degree ± 1, for sufficiently large electric fields.

1. Introduction

Let us consider a thin infinity plate of a nematic or cholesteric liquid crystal, possibly in the presence of an orthogonal applied electric (or magnetic) field $\mathbf{E}$. In the Oseen–Frank model, the configurations of the liquid crystal are described by means of functions $u\colon \mathbb{R}^2 \to S^2$, where S^2 is the unit sphere of $\mathbb{R}^3$, and the unit length vector $u(x)$ is the the optical axis, namely the (average) direction of the molecules at the point $x \in \mathbb{R}^2$. The energy density is given by $2W = K_1 \operatorname{div}(u)^2 + K_2(u \cdot \operatorname{curl} u + \tau)^2 + K_3|u \times \operatorname{curl} u|^2 - \varepsilon_a(\mathbf{E} \cdot u)^2$, where K_1, K_2

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and K_3 are respectively the elastic splay, twist and bend constants, $\varepsilon_a > 0$ is the electric susceptibility. Moreover, we denote with $\cdot$ and $\times$ the scalar and vector product on $\mathbb{R}^3$ respectively, and with $|v|$ the norm on $\mathbb{R}^n$, $n \geq 1$.

The constant τ is $= 0$ for nematic liquid crystals, and is $\neq 0$ in the cholesteric case; in this second case the director field tends to form a helicoid with period $2\pi/|\tau|$ around a twist axis, and this helicoid is right-handed or left-handed depending on the sign of τ (see, for instance [22]).

In the first part of the paper, we consider the nematic case without electric field, so that the energy became:

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(K_1 \operatorname{div}(u)^2 + K_2(u \cdot \operatorname{curl} u)^2 + K_3|u \times \operatorname{curl} u|^2\right) dx.$$

More precisely, we consider functions $u \in L^\infty(\mathbb{R}^2, S^2)$ and $\nabla u \in L^2(\mathbb{R}^2, \mathbb{R}^6)$, so that $E(u) < +\infty$, and $u(x)$ goes to a constant vector at infinity, and we assume $u(\infty) = e_3 = (0,0,1)$ (the north pole of the sphere S^2). Then u can be identified with a function $u\colon S^2 \to S^2$ with a well defined topological degree given by

$$Q(u) = \frac{1}{4\pi}\int_{\mathbb{R}^2} u \cdot u_{x_1} \times u_{x_2}\, dx$$

(see, for instance, [6]). Topologically non trivial configuration that minimizes the energy $E(u)$ are sometimes called skyrmions for the analogy with the Skyrme model for mesons and baryons (see [20], [21], [5]). We are interested to find an energy lower bound on every topological sector $Q_q = \{u \mid Q(u) = q\}$, with $q \in \mathbb{Z}$, and to find if this lower bound is attained. Clearly, in the case of the one constant approximation, namely if $K_1 = K_2 = K_3 = K > 0$, the functional $E(u)$ reduces to Dirichlet integral, and we have the sigma model solved by [1], with the classical lower bound

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx \geq 4\pi|Q(u)|$$

attained by the rational functions (see, for instance, [24]). In the case of the functional $E(u)$, we define the function

$$j(k) = \begin{cases} 1 + \dfrac{k}{2\sqrt{1-k}} \log\left(\dfrac{1+\sqrt{1-k}}{1-\sqrt{1-k}}\right) & \text{if } 0 < k < 1, \\[2ex] 1 + \dfrac{k}{\sqrt{k-1}} \arcsin\left(\dfrac{\sqrt{k-1}}{\sqrt{k}}\right) & \text{if } k \geq 1, \end{cases}$$

and we have the following result.

THEOREM 1.1. *Let $u \in L^\infty(\mathbb{R}^2, S^2)$ and $\nabla u \in L^2(\mathbb{R}^2, \mathbb{R}^6)$. Then*

$$E(u) \geq 2\pi\alpha j\left(\frac{K_3}{\alpha}\right)|Q(u)|,$$

where $\alpha = \min(K_1, K_2)$.

Since $j(1) = 2$, in the case of one constant approximation we have the classical lower bound $E(u) \geq 4\pi K|Q(u)|$. On the topological sector Q_{-1} the previous result can be improved. In fact, we have the following theorem.

THEOREM 1.2. *Let* $\theta_i = \theta_i(r)$, $\theta_i \colon [0, +\infty[\to [0, \pi[$, $i = 1, 2$, *be two functions such that* $\theta_i(0) = \pi$, $\theta_i(+\infty) = 0$ *and*

$$r\theta_1' + \sin\theta_1 \sqrt{\cos^2\theta_1 + \frac{K_3}{K_2}\sin^2\theta_1} = 0, \quad r\theta_2' \sqrt{\cos^2\theta_2 + \frac{K_3}{K_1}\sin^2\theta_2} + \sin\theta_2 = 0.$$

Then, if we set

$$v_1(x) = \sin\theta_1(|x|)\widehat{e}_r^{\perp} + \cos\theta_1(|x|)e_3, \quad v_2(x) = \sin\theta_2(|x|)\widehat{e}_r^{\perp} + \cos\theta_2(|x|)e_3,$$

where $\widehat{e}_r = x/|x|$, *we have* $Q(v_1) = Q(v_2) = -1$, *and:*

$$\min_{u \in Q_{-1}} E(u) = \begin{cases} E(v_1) = 2\pi K_2\, j\left(\dfrac{K_3}{K_2}\right) & \text{if } K_2 < K_1, \\[2ex] E(v_2) = 2\pi K_1\, j\left(\dfrac{K_3}{K_1}\right) & \text{if } K_1 < K_2. \end{cases}$$

The explicit expression of θ_1 and θ_2 will be given later (see Remark 2.7). Since of course $E(u)$ is invariant under translations and dilatation, the minimum of $E(u)$ is attained, in fact, on a family of functions contained in Q_{-1}. Clearly, the same conclusion is valid for the topological sector Q_1 by changing u to $-u$; in the Oseen–Frank model, u and $-u$ are identified.

The existence of the axially symmetric skyrmion v_1 has been found in [5] and its stability has been studied. Moreover, in [5] has been introduced the function $j(k)$ as the energy value of v_1. Theorem 1.1 uses the same function to get a lower bound on every topological sector Q_q and Theorem 1.2 identifies a global minimum of $E(u)$ on Q_{-1} for every value of the constants K_1, K_2, K_3.

The lower bound in Theorem 1.1 derive from a Bogomol'nyi type decomposition for the energy or, more precisely, for the expression $|\nabla u|^2 + k|u \times \operatorname{curl} u|^2$, which turns out to be the sum of a topological term and a quadratic term. For the functions v_1 and v_2 the quadratic term is equal to zero. Notice that, from [5], we know that on topological sectors Q_q, with $|q| > 1$, there are not simple axially symmetric skyrmions like v_i, $i = 1, 2$, except in the case of the one constant approximation.

In the second part of the paper we study a liquid crystal plate under the action of the electric field $\mathbf{E} = (0, 0, E)$. In this case it is well know that critical points of the energy are not allowed for nematics ($\tau = 0$), since, for every u, the energy $E(u_\lambda)$ of the scaled functions $u_\lambda(x) = u(\lambda x)$ ($\lambda > 0$) is strictly decreasing. Then we consider the cholesteric case ($\tau \neq 0$). Moreover, we consider only the one constant approximation case.

Then, the energy density reduce to $2W = K|\nabla u|^2 + 2K\tau u \cdot \operatorname{curl} u + K\tau^2 - (\varepsilon_a u \cdot \mathbf{E})^2 - K \operatorname{div}(u \cdot (\nabla u)^t - \operatorname{div}(u)u)$, and since the energy density of the trivial configurations $u(x) = (0,0,\pm 1)$ is $2W_0 = K\tau^2 - \varepsilon_a E^2$, this leads us to consider the energy density difference $W - W_0$, and then the functional

$$E(u) = \int_{\mathbb{R}^2} \left(\frac{1}{2}|\nabla u|^2 + \tau u \cdot \operatorname{curl} u + \frac{\varepsilon_a E^2}{2K}(1 - u_3^2) \right) dx,$$

where we have set $u(x) = (u_1(x), u_2(x), u_3(x))$. Since $|u| = 1$, we have $1 - u_3^2 = |u - e_3|^2 - 1/4|u - e_3|^4$, so that we can write the potential term associated with the external electric field into a slightly different form:

$$E(u) = \int_{\mathbb{R}^2} \left(\frac{1}{2}|\nabla u|^2 + \tau u \cdot \operatorname{curl} u + \frac{k}{2}\left(|u - e_3|^2 - \frac{1}{4}|u - e_3|^4 \right) \right) dx,$$

where $k = \varepsilon_a E^2/K$. Notice that $E(u) < \infty$ implies $u(\infty) = \pm e_3$. We consider $E(u)$ defined on $\mathcal{M} = \{u \colon \mathbb{R}^2 \to \mathbb{R}^3 \mid u - e_3 \in H^{1,2}(\mathbb{R}^2, \mathbb{R}^3),\ |u(x)| = 1$ almost everywhere$\}$ as in [17], and we prove the following theorem.

THEOREM 1.3. *If $\tau \neq 0$, and $k \geq 14\tau^2$, then the minimum of E on the topological sector $Q_{-1} = \{u \in \mathcal{M} \mid Q(u) = -1\}$ is attained.*

Since the topological sectors are not closed with respect to the weak convergence, in order to prove that a minimizing sequence $(u_n)_n \subset Q_{-1}$ converges (modulo subsequences) to some $u_* \in Q_{-1}$, we use the concentration-compactness method ([16]) as in the case of the Skyrme model ([9]–[11], [14], [15]).

Notice that chiral terms like $u \cdot \operatorname{curl} u$ also appear on the study of ferromagnetic materials in connection to the Dzyaloshinskiĭ–Moriya energy density, which is composed of the Lifshitz invariants $\mathcal{L}_{i,j}^k = u_i(\partial u_j/\partial x_k) - u_j(\partial u_i/\partial x_k)$. In fact, $u \cdot \operatorname{curl} u = \mathcal{L}_{3,2}^1 + \mathcal{L}_{1,3}^2$ (see [3], [4], [13]). However, in that case, the energy associated with an external magnetic field (Zeeman energy) is the L^2-norm of $u - e_3$ instead of $|u - e_3|^2 - 1/4|u - e_3|^4$ as in our case. A functional of the form Dirichlet integral + chiral term + Zeeman energy is studied in [17] with concentration-compactness method. In this case, the minimum of the energy on Q_{-1} is obtained under less restrictive assumptions on the coupling constants between chiral term and Zeeman energy (see [17], Theorem 1.1. See also [8], where the Zeeman energy is replaced by the more general expression $|u - e_3|^p$, with $2 \leq p \leq 4$).

In the liquid crystal case, the potential term has the form of the double-vacuum potential as in some Skyrme model (see [23]), according to the fact that the opposite orientations $u(x)$ and $-u(x)$ of the optical axis are indistinguishable, and the function $-u_*$ minimize $E(u)$ on $Q_1 = \{u \in \overline{\mathcal{M}} \mid Q(u) = 1\}$, where $\overline{\mathcal{M}} = \{u \colon \mathbb{R}^2 \to \mathbb{R}^3 \mid u + e_3 \in H^{1,2}(\mathbb{R}^2, \mathbb{R}^3), |u(x)| = 1$ almost everywhere$\}$.

Because of the chiral term, and the particular form of the potential, an important ingredient of the concentration-compactness method, namely a lower bound for the energy on the topological sectors $Q(u) \geq 1$, is obtained for "large" electric field, namely for $k \geq 14\tau^2$. For some other ingredients, as the coerciveness, the assumption $k > 4\tau^2$ is enough.

Finally, we remark that axially symmetric skyrmions in a thin layer of cholesteric liquid crystal are studied in [2], [12], [7], with analytic and numerical methods.

2. Proofs of Theorems 1.1 and 1.2

In order to prove Theorems 1.1 and 1.2, we state some lemmas. We set $x = (x_1, x_2)$, and $u = (u_1, u_2, u_3)$.

LEMMA 2.1. *Let $u \in L^\infty(\mathbb{R}^2, S^2)$ with $\nabla u \in L^2(\mathbb{R}^2, \mathbb{R}^6)$ and let $k > 0$. Then we have*:

$$\begin{aligned}|\nabla u|^2 &+ (k-1)|u \times \operatorname{curl} u|^2 \\ &= \big(1 + (k-1)u_1^2\big)|u_{x_1}|^2 + \big(1 + (k-1)u_2^2\big)|u_{x_2}|^2 + 2(k-1)u_1u_2u_{x_1} \cdot u_{x_2}.\end{aligned}$$

PROOF. First of all we claim that

$$|u \times \operatorname{curl} u|^2 = u_1^2|u_{x_1}|^2 + u_2^2|u_{x_2}|^2 + 2u_1u_2\, u_{x_1} \cdot u_{x_2}.$$

In fact, let us denote with $u_{i,j}$ the derivative of u_i with respect to the spatial variable x_j ($i = 1, 2, 3$, $j = 1, 2$), and let $(e_i)_i$ the canonical basis of $\mathbb{R}^3$. We have

$$u \times \operatorname{curl} u = e_i \varepsilon_{k,i,j} \varepsilon_{k,p,q} u_j u_{q,p} = e_i(\delta_{i,p}\delta_{j,q} - \delta_{i,q}\delta_{j,p})u_j u_{q,p},$$

but, since $|u| = 1$, we get $\delta_{j,q}u_j u_{q,p} = 0$ for $p = 1, 2, 3$, so that

$$u \times \operatorname{curl} u = -e_i \delta_{i,q}\delta_{j,p} u_j u_{q,p} = -e_i u_j u_{i,j}.$$

Then $|u \times \operatorname{curl} u|^2 = u_j u_p u_{i,j} u_{i,p} = u_1^2 u_{i,1}u_{i,1} + u_2^2 u_{i,2}u_{i,2} + 2u_1u_2u_{i,1}u_{i,2} = u_1^2|u_{x_1}|^2 + u_2^2|u_{x_2}|^2 + 2u_1u_2u_{x_1} \cdot u_{x_2}$, and the claim is proved. Since $|\nabla u|^2 = |u_{x_1}|^2 + |u_{x_2}|^2$, the lemma follows immediately. □

LEMMA 2.2. *Let $u \in L^\infty(\mathbb{R}^2, S^2)$ with $\nabla u \in L^2(\mathbb{R}^2, \mathbb{R}^6)$, and let $k > 0$. Let $a = a(x)$, $b = b(x)$, $c = c(x)$, $d = d(x)$ be functions on $\mathbb{R}^2$ such that*:

$$\begin{cases} a^2 + d^2 = 1 + (k-1)u_1^2, \\ b^2 + c^2 = 1 + (k-1)u_2^2, \\ ab + cd = (k-1)u_1u_2. \end{cases} \tag{2.1}$$

Then we have

$$\begin{aligned}|\nabla u|^2 &+ (k-1)|u \times \operatorname{curl} u|^2 \\ &= 2(ac - bd)u \cdot u_{x_1} \times u_{x_2} + |au_{x_1} + bu_{x_2} + cu \times u_{x_2} + du \times u_{x_1}|^2,\end{aligned} \tag{2.2}$$

$$\begin{aligned}|\nabla u|^2 &+ (k-1)|u \times \operatorname{curl} u|^2 \\ &= -2(ac - bd)u \cdot u_{x_1} \times u_{x_2} + |au_{x_1} + bu_{x_2} - cu \times u_{x_2} - du \times u_{x_1}|^2.\end{aligned} \tag{2.3}$$

PROOF. Expanding the quadratic term in (2.2), and using some algebraic identity such as $|u \times u_{x_i}| = |u_{x_i}|$, $u \times u_{x_1} \cdot u \times u_{x_2} = u_{x_1} \cdot u_{x_2}$, we have

$$\begin{aligned}&|au_{x_1} + bu_{x_2} + cu \times u_{x_2} + du \times u_{x_1}|^2\\ &= -2(ac-bd)u\cdot u_{x_1} \times u_{x_2} + (a^2+d^2)|u_{x_1}|^2 + (b^2+c^2)|u_{x_2}|^2 + 2(ab+cd)u_{x_1}\cdot u_{x_2}.\end{aligned}$$

Since the functions $a,\dots,d$ satisfy (2.1), using Lemma 2.1 we have (2.2). The proof of (2.3) is similar. □

REMARK 2.3. Conditions (2.1) implies $ac - bd = \pm\sqrt{1 + (k-1)(u_1^2 + u_2^2)} = \pm\sqrt{k + (1-k)u_3^2}$. A possible choice for the functions $a,\dots,d$ is

$$\begin{cases} a = \sqrt{1+(k-1)u_1^2},\\ b = \dfrac{(k-1)u_1u_2}{\sqrt{1+(k-1)u_1^2}},\\ c = \dfrac{\sqrt{1+(k-1)(u_1^2+u_2^2)}}{\sqrt{1+(k-1)u_1^2}},\\ d = 0. \end{cases} \tag{2.4}$$

Since $0 \le u_1^2 \le 1$, we have $1+(k-1)u_1^2 \ge \min(1,k) > 0$, and then the functions in (2.4) are well defined on $\mathbb{R}^2$, and they belong to $L^\infty(\mathbb{R}^2,\mathbb{R})$. Moreover, from (2.4) we have $ac - bd = \sqrt{k+(1-k)u_3^2}$.

REMARK 2.4. We observe that, by an exchange of variables x_1 and x_2, or arguing as in the previous Lemma, we have that the conditions

$$\begin{cases} a^2 + d^2 = 1 + (k-1)u_2^2,\\ b^2 + c^2 = 1 + (k-1)u_1^2,\\ ab + cd = (k-1)u_1u_2, \end{cases}$$

implies that $|\nabla u|^2 + (k-1)|u \times \operatorname{curl} u|^2$ is equal to $2(ac-bd)u \cdot u_{x_1} \times u_{x_2} + |au_{x_2} + bu_{x_1} - cu \times u_{x_1} - du \times u_{x_2}|^2$ and to $-2(ac-bd)u \cdot u_{x_1} \times u_{x_2} + |au_{x_2} + bu_{x_1} + cu \times u_{x_2} + du \times u_{x_1}|^2$.

We can now prove Theorems 1.1 and 1.2.

PROOF OF THEOREM 1.1. Let $u \in L^\infty(\mathbb{R}^2, S^2)$ with $\nabla u \in L^2(\mathbb{R}^2,\mathbb{R}^6)$, and let $\alpha = \min(K_1, K_2)$, $k = K_3/\alpha$. Then, from the well known relations

$$\begin{aligned} (u\cdot \operatorname{curl} u)^2 &= |\operatorname{curl} u|^2 - |u \times \operatorname{curl} u|^2\\ \operatorname{div}(u)^2 + |\operatorname{curl} u|^2 &= |\nabla u|^2 - \operatorname{div}\big(u\cdot(\nabla u)^t - \operatorname{div}(u)u\big) \end{aligned} \tag{2.5}$$

we have $K_1 \operatorname{div}(u)^2 + K_2(u\cdot\operatorname{curl} u)^2 + K_3|u\times\operatorname{curl} u|^2 \ge \alpha(\operatorname{div}(u)^2 + (u\cdot\operatorname{curl} u)^2) + K_3|u\times\operatorname{curl} u|^2 = \alpha(|\nabla u|^2 + (k-1)|u\times\operatorname{curl} u|^2) - \alpha\operatorname{div}(u\cdot(\nabla u)^t - \operatorname{div}(u)u)$.

Integrating on $\mathbb{R}^2$, and recalling that u is constant at infinity, we get

$$E(u) \geq \frac{\alpha}{2} \int_{\mathbb{R}^2} \left(|\nabla u|^2 + (k-1)|u \times \operatorname{curl} u|^2\right) dx \tag{2.6}$$

and, from Lemma 2.2,

$$E(u) \geq \pm\alpha \int_{\mathbb{R}^2} (ac - bd) u \cdot u_{x_1} \times u_{x_2} \, dx.$$

Choosing the functions $a, \ldots, d$ as in (2.4), we get

$$E(u) \geq \pm\alpha \int_{\mathbb{R}^2} \sqrt{k + (1-k)u_3^2}\, u \cdot u_{x_1} \times u_{x_2} \, dx.$$

In order to evaluate the last integral, we consider the volume form $\nu = y_1 dy_2 \wedge dy_3 + y_2 dy_3 \wedge dy_1 + y_3 dy_1 \wedge dy_2$ on S^2 and the two-form $\mu = \sqrt{k + (1-k)y_3^2}\,\nu$. Moreover, by the density result of [19, Part 4], (see also [6]) we can assume u smooth. Then, by the generalized integral formula of the degree (see, for instance [18]), we have:

$$\begin{aligned}\int_{\mathbb{R}^2} &\sqrt{k + (1-k)u_3^2}\, u \cdot u_{x_1} \times u_{x_2} \, dx \\ &= \int_{\mathbb{R}^2} \mu \circ u \, dx = Q(u) \int_{S^2} \mu \, dy = Q(u) \int_B \operatorname{div}(F) \, dy_1 \, dy_2 \, dy_3,\end{aligned}$$

where F is the vector field $F = \sqrt{k + (1-k)y_3^2}\,(y_1, y_2, y_3)$, and B is the unit ball of $\mathbb{R}^3$. A this point, by a direct calculation, we get:

$$\int_B \operatorname{div}(F) \, dy_1 \, dy_2 \, dy_3 = \int_B \frac{3k + 4(1-k)y_3^2}{\sqrt{k + (1-k)y_3^2}} \, dy_1 \, dy_2 \, dy_3 = 2\pi j(k),$$

where $j(k)$ is defined in the Introduction. Then $E(u) \geq \pm 2\pi\alpha j(k) Q(u)$, namely $E(u) \geq 2\pi\alpha j(K_3/\alpha)|Q(u)|$, and the theorem is proved. □

Let $u \in L^\infty(\mathbb{R}^2, S^2)$ with $\nabla u \in L^2(\mathbb{R}^2, \mathbb{R}^6)$, let $k > 0$, and let us consider the functional

$$F_k(u) = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + (k-1)|u \times \operatorname{curl} u|^2\right) dx.$$

From Lemma 2.2, and from the calculations above it follows that, on the topological sector Q_{-1}, namely if $Q(u) = -1$, we have

$$F_k(u) = 4\pi j(k) + \int_{\mathbb{R}^2} |a u_{x_1} + b u_{x_2} - c u \times u_{x_2}|^2 \, dx,$$

where the functions a, b, c, satisfy (2.4), and the minimum value $4\pi j(k)$ of F_k is attained on Q_{-1} if and only if $a u_{x_1} + b u_{x_2} - c u \times u_{x_2} = 0$, namely if and only if

$$|\nabla u|^2 + (k-1)|u \times \operatorname{curl} u|^2 + 2\sqrt{k + (1-k)u_3^2}\, u \cdot u_{x_1} \times u_{x_2} = 0. \tag{2.7}$$

We have the following proposition.

PROPOSITION 2.5. *Let $k > 0$, and let $\theta_i = \theta_i(r)$, $\theta_i \colon [0, +\infty[\to [0, \pi[$ be two functions such that $\theta_i(0) = \pi$, $\theta_i(+\infty) = 0$, and*

$$\sin\theta_1 \sqrt{\cos^2\theta_1 + k\sin^2\theta_1} + r\theta_1' = 0, \qquad \sin\theta_2 + \sqrt{\cos^2\theta_2 + k\sin^2\theta_2}\, r\theta_2' = 0,$$

and set

$$v_1(x) = \sin\theta_1(|x|)\widehat{e}_r^{\perp} + \cos\theta_1(|x|)e_3, \qquad v_2(x) = \sin\theta_2(|x|)\widehat{e}_r^{\perp} + \cos\theta_2(|x|)e_3,$$

where $\widehat{e}_r = x/|x|$. Then, the minimum $4\pi j(k)$ of the functional F_k (defined above) on Q_{-1} is attained on the functions v_1 and v_2.

PROOF. Clearly $v_i \in Q_{-1}$, $i = 1, 2$, so that it will suffice to prove (2.7) for v_i. In fact, it is easy to check that the l.h.t. of (2.7), calculated for v_1 and for v_2 is equal respectively, to

$$\frac{(\sin\theta_1 \sqrt{\cos^2\theta_1 + k\sin^2\theta_1} + r\theta_1')^2}{r^2}, \qquad \frac{(\sin\theta_2 + \sqrt{\cos^2\theta_2 + k\sin^2\theta_2}\, r\theta_2')^2}{r^2}$$

where $r = |x|$, and the result follows immediately. □

PROOF OF THEOREM 1.2. Let us suppose that the functions θ_i and v_i, $i = 1, 2$ are as in Theorem 1.2. Then v_1 and v_2 minimize on Q_{-1}, respectively, the functionals F_{K_3/K_2} and F_{K_3/K_1}, and we have $F_{K_3/K_2} = 4\pi j(K_3/K_2)$, and $F_{K_3/K_1} = 4\pi j(K_3/K_1)$. Moreover, since $\operatorname{div} v_1 = 0$ and $v_2 \cdot \operatorname{curl} v_2 = 0$, using again (2.5), we have $E(v_1) = (K_2/2)F_{K_3/K_2}(v_1)$, and $E(v_2) = (K_1/2)F_{K_3/K_1}(v_2)$. Then, from (2.6) we get, for every $u \in Q_{-1}$,

$$E(u) \geq \frac{K_2}{2} F_{K_3/K_2}(u) \geq \frac{K_2}{2} F_{K_3/K_2}(v_1) = E(v_1) = 2\pi K_2 j\left(\frac{K_3}{K_2}\right)$$

in the case $K_2 < K_1$, and

$$E(u) \geq \frac{K_1}{2} F_{K_3/K_1}(u) \geq \frac{K_1}{2} F_{K_3/K_1}(v_2) = E(v_2) = 2\pi K_1 j\left(\frac{K_3}{K_1}\right)$$

in the case $K_1 < K_2$, and the theorem is proved. □

REMARK 2.6. Since the function $t \to tj(K_3/t)$ is strictly increasing, we have that $K_2 \lesseqgtr K_1$ implies $E(v_1) \lesseqgtr E(v_2)$.

REMARK 2.7. Solving the first equation in Proposition 2.5, we get

$$\theta_1(r) = \frac{\pi}{2} + \arctan\left(\frac{K_2 - K_3c^2r^2}{2K_2c\,r}\right)$$

with $c > 0$, the solution obtained in [5] (in our case it is decreasing because of the reversed boundary conditions). Solving the second equation in Proposition 2.5 we have

$$\theta_2(r) = \frac{\pi}{2} - \arctan\left(\frac{\eta^{-1}_{K_3/K_1}(\log(cr))}{\sqrt{1 - (\eta^{-1}_{K_3/K_1}(\log(cr)))^2}}\right),$$

where $\eta^{-1}_{K_3/K_1}$ is the inverse of the function

$$\eta_k(x) = \begin{cases} -\log\left(\dfrac{\sqrt{1-x^2}(\sqrt{1-k}\,x + \sqrt{k+(1-k)x^2})^{\sqrt{1-k}}}{x+\sqrt{k+(1-k)x^2}}\right) & \text{if } 0<k<1, \\ -\log\left(\dfrac{\sqrt{1-x^2}}{x+\sqrt{k+(1-k)x^2}}\right) & \\ \qquad +\sqrt{k-1}\arctan\left(\dfrac{\sqrt{k-1}\,x}{\sqrt{k+(1-k)x^2}}\right) & \text{if } k \geq 1. \end{cases}$$

In fact, we have $\eta_k'(x) = \sqrt{k+(1-k)x^2}/(1-x^2)$, so that $(\sin\theta_2)\partial_r \eta_k(\cos\theta_2) + \sqrt{\cos^2\theta_2 + k\sin^2\theta_2}\,\theta_2' = 0$.

Let us consider now the quadratic terms in (2.2) and (2.3). As in the classic sigma model, the stereographic projection reduce a vector equation to a system of first order partial differential equations. In fact, let us denote with $w_1 = u_1/(1+u_3)$, $w_2 = u_2/(1+u_3)$ the stereographic projection of the point $u = (u_1, u_2, u_3) \in S^2$ in $\mathbb{R}^2$. Then, we have the following proposition.

PROPOSITION 2.8. *We have* $au_{x_1} + bu_{x_2} \pm cu \times u_{x_2} \pm du \times u_{x_1} = 0$ *if and only if the functions* w_1 *and* w_2 *satisfy the system*

$$\begin{cases} aw_{1,x_1} \mp cw_{2,x_2} + bw_{1,x_2} \mp dw_{2,x_1} = 0, \\ aw_{2,x_1} \pm cw_{1,x_2} + bw_{2,x_2} \pm dw_{1,x_1} = 0. \end{cases} \tag{2.8}$$

PROOF. Set, for brevity, $s = (s_1, s_2, s_3) = au_{x_1} + bu_{x_2} + cu \times u_{x_2} + du \times u_{x_1}$. A direct calculation shows that

$$\begin{aligned} aw_{1,x_1} - cw_{2,x_2} + bw_{1,x_2} - dw_{2,x_1} &= (s_1 - w_1 s_3)/(1+u_3), \\ aw_{2,x_1} + cw_{1,x_2} + bw_{2,x_2} + dw_{1,x_1} &= (s_2 - w_2 s_3)/(1+u_3). \end{aligned}$$

Now, if $s = 0$, clearly the system (2.8) is satisfied. Vice versa, (2.8) implies $s = (w_1, w_2, 1)s_3$, and since $u \cdot s = 0$, and $u \cdot (w_1, w_2, 1) = 1$, we get $0 = u \cdot s = u \cdot (w_1, w_2, 1)s_3 = s_3$, and then $s = 0$. The proof for the quadratic term in (2.3) is similar. □

REMARK 2.9. Clearly, from (2.4) we see that, in the case $k = 1$, namely in the classical sigma model, the quadratic terms in (2.2), (2.3) becames $u_{x_1} \pm u \times u_{x_2}$, and (2.8) are reduced to the Cauchy–Riemann equations $w_{1,x_1} \mp w_{2,x_2} = 0$, $w_{2,x_1} \pm w_{1,x_2} = 0$; for $k \neq 1$ the system (2.8) can not be solved explicitly.

3. Proof of Theorem 1.3

In order to minimize the functional

$$E(u) = \int_{\mathbb{R}^2} \left(\frac{1}{2}|\nabla u|^2 + \tau u \cdot \operatorname{curl} u + \frac{k}{2}\left(|u-e_3|^2 - \frac{1}{4}|u-e_3|^4\right)\right) dx$$

under the constraint $Q(u) = -1$, we assume the functional framework of [17], namely we consider $\mathcal{M} = \{u\colon \mathbb{R}^2 \to \mathbb{R}^3 \mid u - e_3 \in H^{1,2}(\mathbb{R}^2, \mathbb{R}^3),\ |u(x)| = 1$ almost everywhere$\}$ (e_i, $i = 1,2,3$ is the canonical basis on $\mathbb{R}^3$), which is a complete metric space for the distance $d(u,v) = \|u - v\|_{H^{1,2}}$. The functionals $E(u)$ and $Q(u)$ are well defined and continuous on $\mathcal{M}$. The set $\mathcal{M}_0 = \{u\colon \mathbb{R}^2 \to \mathbb{R}^3 \mid u - e_3 \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^3),\ |u(x)| = 1\}$ is dense in $\mathcal{M}$. If $u = (u_1, u_2, u_3) \in \mathcal{M}_0$, then $u_1 = u_2 = 0$, $u_3 = 1$ outside a compact subset of $\mathbb{R}^2$.

Lemma 3.1. *For every $u \in \mathcal{M}$ and every $\lambda > 0$ we have*

$$\left| \int_{\mathbb{R}^2} u \cdot \operatorname{curl} u \, dx \right| \le \lambda \int_{\mathbb{R}^2} \big(1 - u_3^2\big) \, dx + \frac{1}{\lambda} \int_{\mathbb{R}^2} |\nabla u_3|^2 \, dx.$$

Proof. Clearly we can assume $u = (u_1, u_2, u_3) \in \mathcal{M}_0$. Since $u \cdot \operatorname{curl} u = u_1 u_{3,x_2} - u_2 u_{3,x_1} + u_3(u_{2,x_1} - u_{1,x_2})$, integrating by parts, and using the inequality $2ab \le \lambda a^2 + (1/\lambda) b^2$, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^2} u \cdot \operatorname{curl} u \, dx \right| &= 2 \left| \int_{\mathbb{R}^2} (u_1 u_{3,x_2} - u_2 u_{3,x_1}) \, dx \right| \\ &\le \lambda \int_{\mathbb{R}^2} \big(u_1^2 + u_2^2\big) \, dx + \frac{1}{\lambda} \int_{\mathbb{R}^2} |\nabla u_3|^2 \, dx, \end{aligned}$$

and since $u_1^2 + u_2^2 = 1 - u_3^2$, the lemma is proved. □

Lemma 3.2. *Let us suppose $k > 4\tau^2$. Then, there exists $c_1 > 0$ such that, for every $u \in \mathcal{M}$, we have*

$$\int_{\mathbb{R}^2} \big(|\nabla u|^2 + (1 - u_3^2)\big) \, dx \le c_1 E(u).$$

Proof. Because of the previous lemma, we have

$$\begin{aligned} E(u) &\ge \int_{\mathbb{R}^2} \left(\frac{1}{2} |\nabla u|^2 - |\tau| \left(\lambda (1 - u_3^2) + \frac{1}{\lambda} |\nabla u_3|^2 \right) + \frac{k}{2} \big(1 - u_3^2\big) \right) dx \\ &= \int \left(\frac{1}{2} \big(|\nabla u_1|^2 + |\nabla u_2|^2\big) + \left(\frac{1}{2} - \frac{|\tau|}{\lambda} \right) |\nabla u_3|^2 + \left(\frac{k}{2} - |\tau| \lambda \right) \big(1 - u_3^2\big) \right) dx. \end{aligned}$$

Since $k > 4\tau^2$, we can assume $2|\tau| < \lambda < k/2|\tau|$, so that $1/2 - |\tau|/\lambda > 0$ and $k/2 - |\tau|\lambda > 0$, and the claim follows. □

Remark 3.3. With the choice $\lambda = k/2|\tau|$ in the previous lemma, we have

$$E(u) \ge \int_{\mathbb{R}^2} \left(\frac{1}{2} \big(|\nabla u_1|^2 + |\nabla u_2|^2\big) + \frac{1}{2} \left(1 - \frac{4\tau^2}{k} \right) |\nabla u_3|^2 \right) dx,$$

so that

$$E(u) \ge 4\pi \left(1 - \frac{4\tau^2}{k} \right) |Q(u)|, \tag{3.1}$$

$$\int_{\mathbb{R}^2} |\nabla u_3|^2 \, dx \le \frac{2k}{k - 4\tau^2} E(u). \tag{3.2}$$

We want to show now the coercivity of E. For this we need the following lemma.

LEMMA 3.4. *Let α, β be such that $0 < \alpha < \beta < 1$. Then, for every $u = (u_1, u_2, u_3) \in \mathcal{M}_0$, we have*

$$|A| \leq \frac{1}{16\pi\alpha^2(\beta - \alpha)^2} \|v\|^4_{H^{1,2}}$$

where $|A|$ is the Lebesgue measure of the level set $A = \{x \in \mathbb{R}^2 \mid u_3(x) < \alpha\}$, and $v = (u_1, u_2, 0)$ is the planar component of u.

PROOF. We observe firstly that

$$\int_{\mathbb{R}^2} |u_3 \nabla u_3|\, dx \leq \frac{1}{2}\|v\|^2_{H^{1,2}},$$

in fact we have $u \cdot \nabla u = v \cdot \nabla v + u_3 \nabla u_3$, and since $u \cdot \nabla u = 0$ because of $|u(x)| = 1$, we get $|u_3 \cdot \nabla u_3| \leq |v||\nabla v| \leq \big(|v|^2 + |\nabla v|^2\big)/2$, and, integrating on $\mathbb{R}^2$, we have the previous the inequality. We set now $B = \{x \in \mathbb{R}^2 \mid \alpha < u_3(x) < \beta\}$. Then, from the coarea formula,

$$\int_\alpha^\beta \mathcal{H}^1(\Gamma_y)\, dy = \int_B |\nabla u_3|\, dx \leq \frac{1}{\alpha} \int_{\mathbb{R}^2} |u_3 \nabla u_3|\, dx \leq \frac{1}{2\alpha}\|v\|^2_{H^{1,2}},$$

where $\Gamma_y = \{x \in \mathbb{R}^2 \mid u_3(x) = y\}$. Then we have $\mathcal{H}^1(\Gamma_{y_0}) \leq \|v\|^2_{H^{1,2}}/(2\alpha(\beta - \alpha))$ for some suitable $y_0 \in [\alpha, \beta]$. Set $C = \{x \in \mathbb{R}^2 \mid u_3(x) < y_0\}$. Since on C we have $1 - u_3(x) \geq 1 - \beta > 0$, C is contained in the support of $1 - u_3$ so it is open and bounded, and, for the isoperimetric inequality,

$$|C| \leq \frac{1}{4\pi}\mathcal{H}^1(\Gamma_{y_0})^2 \leq \frac{1}{16\pi\alpha^2(\beta - \alpha)^2}\|v\|^4_{H^{1,2}}.$$

Since $A \subset C$, the lemma is proved. □

REMARK 3.5. Notice that, with $\alpha = 1/2$ and $\beta = 3/4$, the previous lemma implies that, for every $u \in \mathcal{M}_0$, we have $\big|\{x \in \mathbb{R}^2 \mid u_3(x) < 0\}\big| \leq \big|\{x \in \mathbb{R}^2 \mid u_3(x) < 1/2\}\big| \leq 4\|v\|^4_{H^{1,2}}/\pi$.

PROPOSITION 3.6. *Let us suppose $k > 4\tau^2$. Then for every $u \in \mathcal{M}$, we have*

$$\int_{\mathbb{R}^2} \big(|\nabla u|^2 + |u - e_3|^2\big)\, dx \leq \frac{16c_1^2}{\pi} E(u)^2 + 3c_1 E(u)$$

(c_1 is the constant in Lemma 3.2), so that $E(u)$ is coercive on $\mathcal{M}$.

PROOF. Clearly, because of the density of $\mathcal{M}_0$ in $\mathcal{M}$, we can limit us to prove the lemma for every $u \in \mathcal{M}_0$. In fact, let $u = (u_1, u_2, u_3) \in \mathcal{M}_0$. From Lemma 3.2 we have

$$\int_{\mathbb{R}^2} |\nabla u|^2\, dx \leq c_1 E(u). \tag{3.3}$$

Set now $A = \{x \in \mathbb{R}^2 \mid u_3(x) < 0\}$. Clearly

$$\int_{\mathbb{R}^2} |u - e_3|^2 \, dx = 2 \int_{\mathbb{R}^2} (1 - u_3) \, dx = 2 \biggl(\int_A (1 - u_3) \, dx + \int_{\mathbb{R}^2 \setminus A} (1 - u_3) \, dx \biggr).$$

From Remark 3.5 and from Lemma 3.2 we have:

$$|A| \le \frac{4}{\pi} \|v\|^4_{H^{1,2}} = \frac{4}{\pi} \biggl(\int_{\mathbb{R}^2} \bigl(|\nabla u_1|^2 + |\nabla u_2|^2 + 1 - u_3^2 \bigr) \, dx \biggr)^2 \le \frac{4c_1^2}{\pi} E(u)^2,$$

so that

$$\int_A (1 - u_3) \, dx \le 2|A| \le \frac{8c_1^2}{\pi} \, E(u)^2.$$

On the other hand, using again Lemma 3.2,

$$\int_{\mathbb{R}^2 \setminus A} (1 - u_3) \, dx \le \int_{\mathbb{R}^2 \setminus A} (1 + u_3)(1 - u_3) \, dx \le \int_{\mathbb{R}^2} (1 - u_3^2) \, dx \le c_1 E(u)$$

therefore

$$\int_{\mathbb{R}^2} |u - e_3|^2 \, dx \le \frac{16c_1^2}{\pi} \, E(u)^2 + 2c_1 E(u). \tag{3.4}$$

From (3.3) and (3.4) we have the result. □

Let us consider now a minimizing sequence $(u_n)_n \subset \mathcal{M} \cap Q_{-1}$:

$$\lim_{n \to +\infty} E(u_n) = I_{-1} = \inf_{Q_{-1}} E(u).$$

For the coerciveness of E (Proposition 3.6), $u_n \to u_*$ weakly (modulo subsequences) in $\mathcal{M}$, that is $u_n - e_3 \to u_* - e_3$ weakly in $H^{1,2}(\mathbb{R}^2, \mathbb{R}^3)$. In order to prove Theorem 1.3, we set, as in [17], $\rho_n = |\nabla u_n|^2 + |u_n - e_3|^2$, so that $(\rho_n)_n$ is a sequence of positive functions in $L^1(\mathbb{R}^2, \mathbb{R})$ with $(\|\rho_n\|_{L^1})_n$ bounded, and $\|\rho_n\|_{L^1} \ge 8\pi$ for the classical topological lower bound. Then, passing eventually to a subsequence, and modulo translations, the following cases may occur ([16]):

(1) *compactness*: for every $\varepsilon > 0$, there exists $R > 0$, such that

$$\int_{|x|>R} \rho_n \, dx < \varepsilon;$$

(2) *vanishing*: for every $R > 0$

$$\lim_{n \to +\infty} \sup_{x \in \mathbb{R}^2} \int_{D_R(x)} \rho_n \, dx = 0;$$

(here, and in the following, $D_R(x)$ is the disk $\{y \in \mathbb{R}^2 \mid |x - y| < R\}$, and $D_R = D_R(0)$);

(3) *dichotomy*: there exists t, with $0 < t < 1$ such that, for every $\varepsilon > 0$ there exist $R > 0$, $(R_n)_n \subset \mathbb{R}$ with $R_n \to +\infty$, and two sequences $(\rho_n^1)_n$, $(\rho_n^2)_n \subset L^1(\mathbb{R}^2, \mathbb{R})$ such that: $0 \le \rho_n^1 + \rho_n^2 \le \rho_n$, $\mathrm{Supp}(\rho_n^1) \subset D_R$, $\mathrm{Supp}(\rho_n^2) \subset \mathbb{R}^2 \setminus D_{R_n}$, and

$$\biggl| \int_{\mathbb{R}^2} \rho_n^1 \, dx - t\|\rho_n\|_{L^1} \biggr| < \varepsilon, \qquad \biggl| \int_{\mathbb{R}^2} \rho_n^2 \, dx - (1 - t)\|\rho_n\|_{L^1} \biggr| < \varepsilon.$$

The proof of Theorem 1.3 consists, as usual, in showing that in the compactness case we have actually $u_* \in Q_{-1}$, so that the minimum of E on Q_{-1} is achieved, and that vanishing and dichotomy do not occur.

In the following we set

$$E(u) = E_0(u) - \frac{k}{8}\int_{\mathbb{R}^2} |u - e_3|^4 \, dx$$

where

$$E_0(u) = \int_{\mathbb{R}^2} \left(\frac{1}{2}|\nabla u|^2 + \tau u \cdot \operatorname{curl} u + \frac{k}{2}|u - e_3|^2\right) dx.$$

Since $\int_{\mathbb{R}^2} e_3 \cdot \operatorname{curl} u \, dx = 0$ on $\mathcal{M}$, the functional E_0 has the same form as in [17], and we will take advantage of its results.

Let us suppose that the compactness case occurs. From [17, Lemma 4.1], we have that $E_0(u_*) \pm 4\pi Q(u_*) \le \liminf_{n\to+\infty}(E_0(u_n) \pm 4\pi Q(u_n))$, and since, by the compactness assumption,

$$\int_{\mathbb{R}^2} |u_n - e_3|^4 \, dx \to \int_{\mathbb{R}^2} |u_* - e_3|^4 \, dx$$

along subsequences, we have also $E(u_*) \pm 4\pi Q(u_*) \le \liminf_{n\to+\infty}(E(u_n) \pm 4\pi Q(u_n))$, namely $E(u_*) + 4\pi|1 + Q(u_*)| \le I_{-1}$, where, as before, $I_{-1} = \inf_{Q_{-1}} E(u)$. From [17, Lemma 3.1], we know that $\inf_{Q_{-1}} E_0(u) < 4\pi$, and since $E(u) \le E_0(u)$ we have also $I_{-1} < 4\pi$. Then $Q(u_*) = -1$. For if not, we would have $|1 + Q(u_*)| \ge 1$, and then $4\pi \le E(u_*) + 4\pi|1 + Q(u_*)| \le I_{-1}$, which is not possible. So, in the compactness case, the minimum of $E(u)$ on Q_{-1} is achieved.

In the vanishing case we would have that $\liminf_{n\to+\infty} E_0(u_n) \ge 4\pi$ (see [17, Lemma 4.2]) and also, by the covering argument, that

$$\int_{\mathbb{R}^2} |u_n - e_3|^4 \, dx \to 0,$$

so that $I_{-1} = \lim_{n\to+\infty} E(u_n) \ge 4\pi$, whereas $I_{-1} < 4\pi$, so this case do not occur.

In order to ruling out dichotomy, we need, first of all, a topological lower bound for the energy. We start with a lemma.

LEMMA 3.7. *Let $u = (u_1, u_2, u_3) \in \mathcal{M}_0$, and let α, β be such that $-1 < \alpha < \beta < 1$. Then we have*

$$|A| \le \frac{|B|}{4\pi(\beta - \alpha)^2} \int_B |\nabla u_3|^2 \, dx,$$

where $A = \{x \in \mathbb{R}^2 \mid u_3(x) < \alpha\}$, and $B = \{x \in \mathbb{R}^2 \mid \alpha < u_3(x) < \beta\}$.

PROOF. Clearly $|A| < +\infty$ and $|B| < +\infty$. For the coarea formula and Cauchy–Schwarz inequality

$$\int_\alpha^\beta \mathcal{H}^1(\Gamma_y) \, dy = \int_B |\nabla u_3| \, dx \le \sqrt{|B|}\sqrt{\int_B |\nabla u_3|^2 \, dx},$$

where $\Gamma_y = \{x \in \mathbb{R}^2 \mid u_3(x) = y\}$, so that

$$\mathcal{H}^1(\Gamma_{y_0}) \leq \frac{\sqrt{|B|}}{\beta - \alpha}\sqrt{\int_B |\nabla u_3|^2\,dx}$$

for some $y_0 \in [\alpha, \beta]$. From the isoperimetric inequality we have

$$\big|\{x \in \mathbb{R}^2 \mid u_3(x) < y_0\}\big| \leq \frac{1}{4\pi}\mathcal{H}^1(\Gamma_{y_0})^2 \leq \frac{|B|}{4\pi(\beta-\alpha)^2}\int_B |\nabla u_3|^2\,dx,$$

and since $A \subset \{x \in \mathbb{R}^2 \mid u_3(x) < y_0\}$, the lemma is proved. □

We observe now that, for every $u \in \mathcal{M}$ we have ([17, Lemma 3.2]):

$$\frac{1}{2}|\nabla u|^2 + \tau u \cdot \operatorname{curl} u = u \cdot u_{x_1} \times u_{x_2} - \frac{\tau^2}{2}(1-u_3)^2 + \tau(u_{2,x_1} - u_{1,x_2}) + \frac{1}{2}|u_{x_1} - \tau e_1 \times u + u \times (u_{x_2} - \tau e_2 \times u)|^2,$$

so that

$$\frac{1}{2}|\nabla u|^2 + \tau u \cdot \operatorname{curl} u + \frac{k}{2}(1-u_3^2) = u \cdot u_{x_1} \times u_{x_2} + f(u_3) + \tau(u_{2,x_1} - u_{1,x_2}) + \frac{1}{2}|u_{x_1} - \tau e_1 \times u + u \times (u_{x_2} - \tau e_2 \times u)|^2, \tag{3.5}$$

were we have set, for brevity,

$$f(u_3) = \frac{k+\tau^2}{2}(1-u_3)\bigg(\frac{k-\tau^2}{k+\tau^2} + u_3\bigg).$$

Since $f(u_3) < 0$ for $-1 \leq u_3 < -(k-\tau^2)/(k+\tau^2)$, an inequality like $|\nabla u|^2/2 + \tau u \cdot \operatorname{curl} u + k(1-u_3^2)/2 \geq u \cdot u_{x_1} \times u_{x_2} + \tau(u_{2,x_1} - u_{1,x_2})$ (which leads us to the lower bound $E(u) \geq 4\pi Q(u)$) does not hold for every $x \in \mathbb{R}^2$. However, from (3.5) we have

$$E(u) \geq 4\pi Q(u) + \int_{\mathbb{R}^2} f(u_3(x))\,dx \tag{3.6}$$

and the last integral is positive for k large enough. More precisely, we have the following proposition.

PROPOSITION 3.8. *Let us suppose $k \geq 14\tau^2$. Then, for every $u = (u_1, u_2, u_3)$ in $\mathcal{M}$, with $Q(u) > 0$, we have $E(u) \geq 4\pi$.*

PROOF. If $Q(u) \geq 2$ the claim follows immediately from (3.1). In the case $Q(u) = 1$ we argue by contradiction, and suppose that $E(u) < 4\pi$ for some $u \in \mathcal{M}$ with $Q(u) = 1$. Without loss of generality we can assume $u \in \mathcal{M}_0$. Set

$$\alpha = \frac{\tau^2 - \sqrt{k^2 - 7\tau^2(k+\tau^2)}}{k+\tau^2}, \qquad \beta = \frac{\tau^2 + \sqrt{k^2 - 7\tau^2(k+\tau^2)}}{k+\tau^2}.$$

Then $-2\tau^2 \leq f(u_3) < 0$ for $-1 \leq u_3 < -(k-\tau^2)/(k+\tau^2)$, and $f(u_3) \geq 7\tau^2/2$ for $\alpha \leq u_3 \leq \beta$, where $f(u_3)$ is the function defined above.

If we set $A = \{x \in \mathbb{R}^2 \mid u_3(x) < \alpha\}$ and $B = \{x \in \mathbb{R}^2 \mid \alpha < u_3(x) < \beta\}$, then, from Lemma 3.7 and (3.2), we have:

$$|A| \le \frac{|B|}{4\pi(\beta-\alpha)^2}\int_B |\nabla u_3|^2\,dx \le \frac{2k|B|E(u)}{4\pi(\beta-\alpha)^2(k-4\tau^2)}$$
$$< \frac{2k|B|}{(\beta-\alpha)^2(k-4\tau^2)} = \frac{k(k+\tau^2)^2|B|}{2(k^2-7k\tau^2-7\tau^4)(k-4\tau^2)}.$$

We set now $A_0 = \{x \in \mathbb{R}^2 \mid -1 \le u_3(x) < -(k-\tau^2)/(k+\tau^2)\}$. Since $A_0 \subset A$, we get

$$\int_{\mathbb{R}^2} f(u_3(x))\,dx \ge \int_{A_0} f(u_3(x))\,dx + \int_B f(u_3(x))\,dx$$
$$\ge -2\tau^2|A| + \frac{7\tau^2}{2}|B| \ge \left(\frac{-\tau^2 k(k+\tau^2)^2}{(k^2-7k\tau^2-7\tau^4)(k-4\tau^2)} + \frac{7\tau^2}{2}\right)|B|.$$

For $k \ge 14\tau^2$ the last term in the previous inequality is $\ge \tau^2|B|/26 > 0$ and, from (3.6) follows $E(u) \ge 4\pi$, a contradiction. □

The next lemma is a simple version of a truncation lemma that will be enough in our case (see also [15], [14], [8]).

LEMMA 3.9. *Let $u \in \mathcal{M}_0$, $0 < \varepsilon < 1/8$, $a, b \in \mathbb{R}$, be such that $1 \le a < b$, $b - a \ge 1$ and*

$$\int_{a<|x|<b} \left(|\nabla u|^2 + |u - e_3|^2\right) dx < \varepsilon.$$

Then, there exist $\sigma \in [a, b]$ and functions $u^1, u^2 \in \mathcal{M}$, such that:

$$u^1(x) = \begin{cases} u(x) & \text{if } |x| \le \sigma, \\ e_3 & \text{if } |x| \ge \sigma + 1, \end{cases} \qquad u^2(x) = \begin{cases} e_3 & \text{if } |x| \le \sigma, \\ u(x) & \text{if } |x| \ge \sigma + 1, \end{cases}$$

and

$$\int_{\sigma<|x|<\sigma+1} \left(|\nabla u^i|^2 + |u^i - e_3|^2\right) dx < 42\varepsilon, \quad i = 1, 2.$$

PROOF. We sketch the proof for the function u^1, because the construction of u^2 is similar. Set $v(x) = |\nabla u(x)|^2 + |u(x) - e_3|^2$, and, using polar coordinates, set $\widehat{u}(r,\theta) = u(r\cos\theta, r\sin\theta)$, $\widehat{v}(r,\theta) = v(r\cos\theta, r\sin\theta)$. Since

$$\int_{a<|x|<b} v(x)\,dx = \int_a^b \int_0^{2\pi} \widehat{v}(r,\theta) r\,d\theta\,dr < \varepsilon$$

and since $b - a \ge 1$, we have

$$\int_0^{2\pi} \widehat{v}(\sigma,\theta)\sigma\,d\theta < \varepsilon \tag{3.7}$$

for some $\sigma \in [a, b]$. Since $\sigma \ge 1$, we have also $\widehat{v}(\sigma,\theta_0) \le \varepsilon/2\pi < \varepsilon$ for some $\theta_0 \in [0, 2\pi]$. Let $g(\theta) = \widehat{u}(\sigma,\theta) - e_3$. We claim that $|g(\theta)|^2 < 2\varepsilon$ for all $\theta \in [0, 2\pi]$. In fact, we have $|g(\theta_0)|^2 \le \widehat{v}(\sigma,\theta_0) < \varepsilon$, and moreover, for the derivative $\partial_\theta |g(\theta)|^2$

we have (setting $\nabla u = \nabla u(\sigma\cos\theta, \sigma\sin\theta)$ for brevity) $|\partial_\theta |g(\theta)|^2| = |2g(\theta)\cdot(\nabla u \cdot (-\sigma\sin\theta, \sigma\cos\theta, 0))| \le 2\sigma|g(\theta)||\nabla u| \le (|\nabla u|^2 + |g(\theta)|^2)\sigma = \widehat{v}(\sigma,\theta)\sigma$. Then, from (3.7) and $|g(\theta_0)|^2 < \varepsilon$, we get $|g(\theta)|^2 < 2\varepsilon$, as claimed. In other words, the curve $\theta \to u(\sigma\cos\theta, \sigma\sin\theta)$ is near to the north pole of the sphere S^2.

We set now

$$w(x) = \begin{cases} u(x) & \text{if } |x| \le \sigma, \\ (\sigma + 1 - |x|)u(\sigma x/|x|) + (|x| - \sigma)e_3 & \text{if } \sigma < |x| < \sigma + 1, \\ e_3 & \text{if } |x| \ge \sigma + 1. \end{cases}$$

Clearly $|w(x)| \le 1$. Moreover, for $\sigma < |x| < \sigma + 1$, we have $|w(x) - e_3| = (\sigma + 1 - |x|)|u(\sigma x/|x|) - e_3| \le |u(\sigma x/|x|) - e_3|$ and $1 - |w(x)| = ||e_3| - |w(x)|| \le |e_3 - w(x)| \le |u(\sigma x/|x|) - e_3|$, so that $|w(x)| \ge 1 - \max|g(\theta)| > 1 - \sqrt{2\varepsilon} > 1/2$ (because of $\varepsilon < 1/8$). Then we can normalize w by setting $u^1(x) = w(x)/|w(x)|$. Clearly $u^1 \in \mathcal{M}$.

We now proceed to estimate $|u^1(x) - e_3|$. For $\sigma < |x| < \sigma + 1$, we have $|u^1(x) - e_3| \le |w(x)/|w(x)| - w(x)| + |w(x) - e_3| = (1 - |w(x)|) + |w(x) - e_3| \le 2|u(\sigma x/|x|) - e_3|$, so that

$$\begin{aligned} (3.8)\quad \int_{\sigma<|x|<\sigma+1} |u^1(x) - e_3|^2\,dx &\le 4\int_0^{2\pi}\int_\sigma^{\sigma+1} |\widehat{u}(\sigma,\theta) - e_3|^2\,r\,dr\,d\theta \\ &= 4\Big(\frac{1}{2\sigma} + 1\Big)\int_0^{2\pi} |\widehat{u}(\sigma,\theta) - e_3|^2\sigma\,d\theta < 4\Big(\frac{1}{2\sigma} + 1\Big)\varepsilon < 6\varepsilon. \end{aligned}$$

We estimate now $|\nabla u^1|^2$. For $\sigma < |x| < \sigma + 1$, we have $|\nabla u^1|^2 = (|\nabla w|^2|w|^2 - (w.\nabla w)^2)/|w|^4 \le |\nabla w|^2/|w|^2$, and since $1/4 \le |w|^2$, $|\nabla u^1|^2 \le 4|\nabla w|^2$. It is easy to check that, for $\sigma < |x| < \sigma + 1$ we have

$$\begin{aligned} w_{x_1}(x) = &-\frac{x_1}{|x|}\Big(u\Big(\frac{\sigma x}{|x|}\Big) - e_3\Big) \\ &+ \sigma\frac{\sigma + 1 - |x|}{|x|}\Big(u_{x_1}\Big(\frac{\sigma x}{|x|}\Big)\frac{x_2^2}{|x|^2} - u_{x_2}\Big(\frac{\sigma x}{|x|}\Big)\frac{x_1 x_2}{|x|^2}\Big). \end{aligned}$$

Since $\sigma(\sigma + 1 - |x|)/|x| \le 1$, using the fact that $(a + b + c)^2 \le 3(a^2 + b^2 + c^2)$, we get

$$|w_{x_1}(x)|^2 \le 3\bigg(\Big|u\Big(\frac{\sigma x}{|x|}\Big) - e_3\Big|^2 + \Big|u_{x_1}\Big(\frac{\sigma x}{|x|}\Big)\Big|^2 + \Big|u_{x_2}\Big(\frac{\sigma x}{|x|}\Big)\Big|^2\bigg),$$

and a similar result holds for $w_{x_2}(x)$, so that, for $\sigma < |x| < \sigma + 1$, we have

$$|\nabla u^1(x)|^2 \le 4|\nabla w(x)|^2 \le 24\bigg(\Big|u\Big(\frac{\sigma x}{|x|}\Big) - e_3\Big|^2 + \Big|\nabla u\Big(\frac{\sigma x}{|x|}\Big)\Big|^2\bigg).$$

Then, integrating in polar coordinates, we have

$$\begin{aligned}(3.9)\qquad \int_{a<|x|<b}|\nabla u^1(x)|^2\,dx &\le 24\int_{\sigma}^{\sigma+1}\int_0^{2\pi}\widehat{v}(\sigma,\theta)r\,dr\,d\theta\\ &= 24\biggl(\frac{1}{2\sigma}+1\biggr)\int_0^{2\pi}\widehat{v}(\sigma,\theta)\sigma\,d\theta < 24\biggl(\frac{1}{2\sigma}+1\biggr)\varepsilon \le 36\varepsilon.\end{aligned}$$

From (3.8) and (3.9) we get

$$\int_{a<|x|<b}(|\nabla u^1|^2+|u^1-e_3|^2)\,dx < 42\varepsilon,$$

and the lemma is proved for the function u^1. □

LEMMA 3.10. *There exists $c_2>0$ such that, if $0<a<b$, $\varepsilon>0$ and $u\in\mathcal{M}$, $u^i\in\mathcal{M}$, $i=1,2$ are such that $u^1=u$ for $|x|\le a$, $u^2=u$ for $|x|\ge b$, and*

$$\int_{a<|x|<b}\bigl(|\nabla u|^2+|u-e_3|^2\bigr)\,dx<\varepsilon$$

$$\int_{|x|>a}\bigl(|\nabla u^1|^2+|u^1-e_3|^2\bigr)\,dx<\varepsilon,\qquad \int_{|x|<b}\bigl(|\nabla u^2|^2+|u^2-e_3|^2\bigr)\,dx<\varepsilon,$$

then we have

$$|E(u)-E(u^1)-E(u^2)|<3c_2\varepsilon,\qquad |Q(u)-Q(u^1)-Q(u^2)|<3\varepsilon.$$

PROOF. We set

$$F(u,\Omega)=\int_\Omega\biggl(\frac12|\nabla u|^2+\tau(u-e_3)\cdot\operatorname{curl}u+\frac{k}{2}(1-u_3^2)\biggr)\,dx.$$

Clearly, since for every $u\in\mathcal{M}$, $\int_{\mathbb{R}^2}e_3\cdot\operatorname{curl}u\,dx=0$, we have $F(u,\mathbb{R}^2)=E(u)$. Moreover, from $|(u-e_3)\cdot\operatorname{curl}u|\le(|u-e_3|^2+|\nabla u|^2)/2$, and $|u-e_3|^4\le 4|u-e_3|^2$, we get, for every $u\in\mathcal{M}$ and $\Omega\subset\mathbb{R}^2$:

$$|F(u,\Omega)|\le c_2\int_\Omega\bigl(|\nabla u|^2+|u-e_3|^2\bigr)\,dx,$$

where $c_2=|\tau|/2+\max(1/2,k)$. Then

$$\begin{aligned}&|E(u)-F(u^1,|x|\le a)-F(u^2,|x|\ge b)|\\ &\qquad=|F(u,\mathbb{R}^2)-F(u,|x|\le a)-F(u,|x|\ge b)|\le c_2\varepsilon.\end{aligned}$$

Likewise we get $|E(u^1)-F(u^1,|x|\le a)|\le c_2\varepsilon$ and $|E(u^2)-F(u^2,|x|\ge b)|\le c_2\varepsilon$, so that

$$\begin{aligned}\bigl|E(u)-E(u^1)-E(u^2)\bigr|&\le\bigl|E(u)-F(u^1,|x|\le a)-F(u^2,|x|\ge b)\bigr|\\ &\quad+\bigl|E(u^1)-F(u^1,|x|\le a)\bigr|+\bigl|E(u^2)-F(u^2,|x|\ge b)\bigr|\le 3c_2\varepsilon,\end{aligned}$$

and the first part of the lemma is proved. Moreover, using $|u\cdot u_{x_1}\times u_{x_2}|\le|\nabla u|^2/2$, we can argue in a similar way for $Q(u)$. □

Let us suppose now that a minimizing sequence $(u_n)_n \subset \mathcal{M} \cap Q_{-1}$ verifies the dichotomy condition. Because of the density of $\mathcal{M}_0$, we can assume, without loss of generality, that $(u_n)_n \subset \mathcal{M}_0 \cap Q_{-1}$. Clearly we can assume also $R \geq 1$, $R_n \geq R + 4$.

The dichotomy condition implies

$$\int_{R<|x|<R_n} \rho_n \, dx < 2\varepsilon$$

so that, for the truncation lemma with $[a, b] = [R, R+1]$ and $u = u_n$, there exist $\sigma_n \in [R, R+1]$ and a function $u_n^1 \in \mathcal{M}$ such that

$$u_n^1(x) = \begin{cases} u_n(x) & \text{if } |x| \leq \sigma_n, \\ e_3 & \text{if } |x| \geq \sigma_n + 1, \end{cases} \qquad \int_{\sigma_n<|x|<\sigma_n+1} \big(|\nabla u_n^1|^2 + |u_n^1 - e_3|^2\big) \, dx < 84\varepsilon.$$

For the same lemma, with $[a, b] = [R_n - 2, R_n - 1]$ and again $u = u_n$, there exist $\delta_n \in [R_n - 2, R_n - 1]$ and a function $u_n^2 \in \mathcal{M}$ such that

$$u_n^2(x) = \begin{cases} e_3 & \text{if } |x| \leq \delta_n, \\ u_n(x) & \text{if } |x| \geq \delta_n + 1, \end{cases} \qquad \int_{\delta_n<|x|<\delta_n+1} \big(|\nabla u_n^2|^2 + |u_n^2 - e_3|^2\big) \, dx < 84\varepsilon.$$

Clearly $u_n^1(x) = u_n(x)$ for $|x| \leq R$, $u_n^2(x) = u_n(x)$ for $|x| > R_n$, and we have

$$\begin{aligned} \int_{|x|>R} \big(|\nabla u_n^1|^2 + |u_n^1 - e_3|^2\big) \, dx &= \int_{R<|x|<\sigma_n} \big(|\nabla u_n^1|^2 + |u_n^1 - e_3|^2\big) \, dx \\ &\quad + \int_{\sigma_n<|x|<\sigma_n+1} \big(|\nabla u_n^1|^2 + |u_n^1 - e_3|^2\big) \, dx < 86\varepsilon, \end{aligned}$$

and, in the same way,

$$\int_{|x|<R_n} \big(|\nabla u_n^2|^2 + |u_n^2 - e_3|^2\big) \, dx < 86\varepsilon.$$

From Lemma 3.10 applied to $[a, b] = [R, R_n]$ and to the functions u_n, u_n^i, $i = 1, 2$, there exists a constant $c > 0$ such that $|E(u_n) - E(u_n^1) - E(u_n^2)| < c\varepsilon$ and $|Q(u_n) - Q(u_n^1) - Q(u_n^2)| < c\varepsilon$ for every n.

Then, for ε small enough, we have $Q(u_n) = Q(u_n^1) + Q(u_n^2)$, so that $Q(u_n^1) + Q(u_n^2) = -1$. Set, for brevity, $Q(u_n^1) = q_n$. Then $Q(u_n^2) = -(1 + q_n)$. We claim that $q_n \in \{-1, 0\}$ for n large enough. For if not, there exists a subsequence $(q_{n_k})_k$ such that, for every $k \in \mathbb{N}$, $q_{n_k} \geq 1$ or $-(1 + q_{n_k}) \geq 1$, then, for the Proposition 3.8, we have $E(u_{n_k}^1) \geq 4\pi$ or $E(u_{n_k}^2) \geq 4\pi$. In both cases $4\pi \leq E(u_{n_k}^1) + E(u_{n_k}^2)$, so that $4\pi \leq E(u_{n_k}^1) + E(u_{n_k}^2) \leq E(u_{n_k}) + c\varepsilon$, namely $4\pi \leq E(u_{n_k}) + c\varepsilon$. Passing to the limit we get $4\pi \leq I_{-1} + c\varepsilon$, and since ε is arbitrarily small, we get $4\pi \leq I_{-1}$, a contradiction because we know that $I_{-1} < 4\pi$.

Then we can assume (modulo subsequences) $q_n = -1$. We claim now that there exists $\delta > 0$ such that, for every $n \in \mathbb{N}$, we have $E(u_n^2) \geq \delta$. In fact, from

$\rho_n^2 \le \rho_n^1 + \rho_n^2 \le \rho_n$, the definition of ρ_n and Proposition 3.6, we have

$$\int_{|x|>R_n} \rho_n^2\,dx \le \int_{|x|>R_n} \rho_n\,dx = \int_{|x|>R_n} \big(|\nabla u_n^2|^2 + |u_n^2 - e_3|^2\big)\,dx$$
$$\le \int_{\mathbb{R}^2} \big(|\nabla u_n^2|^2 + |u_n^2 - e_3|^2\big)\,dx \le \frac{16c_1^2}{\pi}\,E(u_n^2)^2 + 3c_1 E(u_n^2).$$

Moreover,

$$\int_{R<|x|<R_n} \rho_n^2\,dx \le \int_{R<|x|<R_n} \rho_n\,dx < 2\varepsilon,$$

so that

$$\int_{\mathbb{R}^2} \rho_n^2\,dx < 2\varepsilon + \frac{16c_1^2}{\pi}\,E(u_n^2)^2 + 3c_1 E(u_n^2).$$

On the other hand, from $|\int_{\mathbb{R}^2} \rho_n^2\,dx - (1-t)\|\rho_n\|_{L^1}| < \varepsilon$, and from $\|\rho_n\|_{L^1} \ge 8\pi$, we get

$$8\pi(1-t) \le (1-t)\|\rho_n\|_{L^1} < \int_{\mathbb{R}^2} \rho_n^2\,dx + \varepsilon < 3\varepsilon + \frac{16c_1^2}{\pi}\,E(u_n^2)^2 + 3c_1 E(u_n^2).$$

Since we can assume $8\pi(1-t) - 3\varepsilon > 0$, the previous inequality shows that $E(u_n^2)$ is bounded away from zero by some $\delta > 0$ (that does not depend on ε), as claimed.

Finally, we are in the position to complete the proof. In fact, from $|E(u_n) - E(u_n^1) - E(u_n^2)| < c\varepsilon$, we get $I_{-1} \le E(u_n^1) \le E(u_n) - E(u_n^2) + c\varepsilon \le E(u_n) - \delta + c\varepsilon$, namely $I_{-1} \le E(u_n) - \delta + c\varepsilon$, and, passing to the limit, $I_{-1} \le I_{-1} - \delta + c\varepsilon$, so that $I_{-1} \le I_{-1} - \delta$. This is impossible, and dichotomy can not occur.

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Carlo Greco
Department of Mechanics, Mathematics and Management
Politecnico di Bari
Via Orabona, 4
Bari, 70125, ITALY

E-mail address: carlo.greco@poliba.it