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DECAY RATES FOR A VISCOELASTIC WAVE EQUATION WITH BALAKRISHNAN–TAYLOR AND FRICTIONAL DAMPINGS

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Abstract. In this paper we are concerned with a viscoelastic wave equation with Balakrishnan–Taylor damping and frictional damping. By using the multiplier method and some properties of convex functions, we establish general energy decay rates of the equation without imposing any growth assumption near the origin on the frictional term and strongly weakening the usual assumptions on the relaxation term. Our stability result generalizes the earlier related results.

1. Introduction

This paper is concerned with the following viscoelastic wave equation with Balakrishnan–Taylor and frictional dampings in $\Omega \times \mathbb{R}^+$,

$$(1.1)\quad u_{tt} - \big(\xi_1 + \xi_2\|\nabla u\|^2 + \sigma(\nabla u, \nabla u_t)\big)\Delta u + \int_0^t \operatorname{div}[a(x)g(t-s)\nabla u(s)]\,ds + \eta(t)b(x)h(u_t) = |u|^\rho u,$$

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together with the boundary condition

$$u(t) = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{1.2}$$

and the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.3}$$

where $\Omega \subseteq \mathbb{R}^n (n \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$. ξ_1,ξ_2 and σ are positive constants. a and b are functions of the space variable. The integral term is the memory responsible for the viscoelastic damping. The relaxation function $g(t)$ is a real function. The function $\eta(t)$ is a time-dependent coefficient of the frictional damping term and h is a specific function.

To motivate our work, we recall some results related to viscoelastic wave equation. In the absence of Balakrishnan–Taylor damping, the general form of viscoelastic wave equation reads,

$$u_{tt} - \varrho\Delta u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + \eta(t)h(u_t) = \mathcal{F}(u). \tag{1.4}$$

If $h = 0$ and g decays exponentially (polynomially), then it is well-known that so does the energy. Messaoudi [22], [23] considered (1.4) with $\varrho = 0$, by taking $\mathcal{F} = 0$ and $\mathcal{F} = |u|^\gamma u$, $\gamma > 0$ and established a more general decay for a wider class of relaxation function by assuming $g'(t) \leq -\xi(t)g(t)$, where $\xi(t)$ is a nonincreasing differentiable function. Even since many authors used this assumption on g to get general decay of energy for problems related to (1.4). See, for example, Messaoudi and Mustafa [24], Mustafa [25], [26] and Park and Park [34]. In Alabau-Boussouira and Cannarsa [1], they first introduced a general assumption on relaxation function by

$$g'(t) \leq -H(g(t)), \tag{1.5}$$

where H is a positive, strictly increasing and strictly convex function satisfying $H(0) = H'(0) = 0$. They established general decay rates for a semilinear abstract second order equation with a memory. After that many results appeared by the use of the condition (1.5). We refer a reader to Cavalcanti et al. [7], [6], Lasiecka et al. [15], [17], Mustafa [27], Mustafa and Messaoudi [31] and Xiao and Liang [38]. In addition, the energy decay rate first established by Lasiecka and Wang [17] constitute not only general but also optimal results in which the decay rates are characterized by an ODE of the same type as the one generated by the inequality (1.5) satisfied by g. Very recently, in [28], [29], Mustafa considered two classes of wave equation (1.4) under the assumption on relaxation function $g'(t) \leq -\xi(t)H(g(t))$, where $\xi(t)$ is a nonincreasing differentiable function, and proved the optimal decay of energy. If $g = 0$, one can find some decay results for damped wave equations in Lasiecka and Tataru [16], Liu and Zuazua [18], and Martinez [19], [20]. In the presence of the time-dependent coefficient $\eta(t)$, Mustafa and Messaoudi [32] and Mustafa and Abusharkh [30] established a general energy decay result, which depends on both h and η, for wave equation and plate equation, respectively. Here we also would like to mention the contribution of Cavalcanti and Oquendo [8]. In [8], the authors considered a wave equation with viscoelastic and frictional dampings of the form

$$u_{tt} - \Delta u + \int_0^t \operatorname{div}\big[a(x)g(t-s)\nabla u(s)\big]\,ds + b(x)h(u_t) + f(u) = 0,$$

and established exponential stability of energy for g decaying exponentially and h linear. In addition, they also obtained polynomial stability of energy for g decaying polynomially and h having a polynomial growth near zero.

Balakrishnan–Taylor damping $\sigma(\nabla u(t), \nabla u_t(x,t))$ was firstly proposed by Balakrishnan–Taylor [3] and Bass and Zes [4]. It is related to the panel flutter equation and to the spillover problem. The system is reduced to well-known wave equation with $\xi_2 = \sigma = 0$ and Kirchhoff type wave equation with $\sigma = 0$ and have been extensively studied. For results on viscoelastic wave equation with Balakrishnan–Taylor damping ($\sigma \neq 0$),

$$u_{tt} - \big(\xi_1 + \xi_2\|\nabla u\|^2 + \sigma(\nabla u, \nabla u_t)\big)\Delta u + \int_0^t h(t-s)\Delta u(s)\,ds = f(u), \tag{1.6}$$

Zaraï and Tatar [35] studied (1.6) with $f(u) = |u|^p u$. They proved the global existence and the polynomial decay of the problem. Exponential decay and blow up of solution to the problem were established in Tatar and Zaraï [36]. If $f(u) = 0$ in (1.6), Park [33] established a general decay result of the problem without imposing the usual relation between the relaxation function g and its derivative. Recently, Ha [12] investigated the following viscoelastic wave equation of the form

$$u_{tt} - \big(a + b\|\nabla u\|^2 + \sigma(\nabla u, \nabla u_t)\big)\Delta u + \int_0^t h(t-s)\Delta u(s)\,ds + g(u_t) = |u|^\rho u.$$

The author proved a general decay result of energy. For more results concerning wave equation with Balakrishnann–Taylor damping, one can refer to Clark [9], Kang [13], Ha [10], [11], Tatar and Zaraï [37], You [39] and Zaraï and Tatar [40] and so on.

In this paper we intend to consider (1.1)–(1.3) with both weak frictional damping and viscoelastic damping acting simultaneously and complementarily in the domain and also with a time-dependent coefficient $\eta(t)$. We establish a general decay rate for the energy without imposing any growth assumption near the origin on h and strongly weakening the usual assumptions on g. In other words, the result here holds for a larger class of functions g and h, from which the energy decay rates are not necessarily of exponential or polynomial types, and hence improve some previous related results.

The rest of this paper is as follows. In Section 2, we give some assumptions, state the main results and prove the global existence of solution. In Section 3, we establish the general decay result of the energy.

2. Assumptions and main results

In the following $L^q(\Omega)$ $(1 \le q \le \infty)$ and $H^1(\Omega)$ denote Lebesgue integral and Sobolev spaces. $\|\cdot\|_q$ is the norm in $L^q(\Omega)$. For simplicity we write $\|\cdot\|$ instead of $\|\cdot\|_2$. $c > 0$ is used to denote a generic constant. We give some assumptions used in this paper.

(A1) The constants ξ_1, ξ_2 and σ are positive constants. Let ρ be a constant satisfying

$$0 < \rho < \frac{2}{n-1} \quad \text{if } n \ge 3 \quad \text{and} \quad \rho > 0 \quad \text{if } n = 1, 2. \tag{2.1}$$

(A2) The function $\eta\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing C^1 function.

(A3) $a, b\colon \Omega \to \mathbb{R}^+$ are such that $a \in C^1(\overline{\Omega})$, $b \in L^\infty(\Omega)$ and

$$\operatorname{meas}\{x \in \partial\Omega : a(x) > 0\} > 0 \quad \text{and} \quad \inf_{x \in \overline{\Omega}} \{a(x) + b(x)\} = \beta > 0. \tag{2.2}$$

(A4) The relaxation function $g\colon [0, \infty) \to \mathbb{R}^+$ is a differentiable function such that

$$g(0) > 0, \quad \xi_1 - \|a\|_\infty \int_0^\infty g(s)\,ds = l > 0, \tag{2.3}$$

and there exists a positive function $H \in C^1(\mathbb{R}^+)$ and H is linear or strictly increasing and strictly convex C^2 function on $(0, r]$, $r < 1$, with $H(0) = H'(0) = 0$, such that

$$g'(t) \le -H(g(t)), \quad \text{for all } t > 0. \tag{2.4}$$

(A5) The function $h\colon \mathbb{R} \to \mathbb{R}$ is a nondecreasing C^0 function and there exist positive constants c_1 and c_2 such that

$$\begin{cases} c_1|s| \le |h(s)| \le c_2|s| & \text{if } |s| \ge r, \\ s^2 + h^2(s) \le H^{-1}(sh(s)), & \text{if } |s| \le r. \end{cases} \tag{2.5}$$

Remark 2.1. Assumption (A5), first introduced by Lasiecka and Tataru [16], gives us that $sh(s) \ge 0$ for all $s \ne 0$.

By using the Fadeo–Galerkin approximation and the contraction mapping theorem (see, for instance, Tatar and Zaraï [35], Berrimi and Messaoudi [5]) we can get the local existence of solution to problem (1.1)–(1.3).

Theorem 2.2. *Suppose* (A1)–(A5) *hold. Then, given* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *there exists* $T > 0$ *and a unique solution* $u(t)$ *of problem* (1.1)–(1.3) *such that*

$$u(t) \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)).$$

The energy functional associated with problem (1.1)–(1.3) is defined by

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{\xi_2}{4}\|\nabla u\|^4 \\ &\quad + \frac{1}{2}\int_\Omega \left(\xi_1 - a(x)\int_0^t g(s)\,ds\right)|\nabla u|^2\,dx + \frac{1}{2}(g \circ \nabla u) - \frac{1}{\rho+2}\|u\|_{\rho+2}^{\rho+2}, \end{aligned} \tag{2.6}$$

where

$$(g \circ v)(t) = \int_0^t g(t-s)\|v(t) - v(s)\|^2\,ds.$$

We can get the following lemma.

Lemma 2.3. *The energy functional* $E(t)$ *is nonincreasing and such that for any* $t \geq 0$,

$$\begin{aligned} E'(t) &= \frac{1}{2}(g' \circ \nabla u) - \frac{1}{2}g(t)\int_\Omega a(x)|\nabla u|^2\,dx \\ &\quad - \sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2\right)^2 - \eta(t)\int_\Omega b(x)u_t h(u_t)\,dx \leq 0. \end{aligned} \tag{2.7}$$

Proof. Multiplying (1.1) by u_t, and using integration by parts over Ω, we can easily get (2.7). By using assumption (A5), we know that $E'(t) \leq 0$, and hence $E(t)$ is nonincreasing. The proof is complete. □

To obtain the potential well, we define the following functions:

$$\begin{aligned} J(u(t)) &= \frac{\xi_2}{4}\|\nabla u\|^4 + \frac{1}{2}\int_\Omega \left(\xi_1 - a(x)\int_0^t g(s)\,ds\right)|\nabla u|^2\,dx \\ &\quad + \frac{1}{2}(g \circ \nabla u) - \frac{1}{\rho+2}\|u\|_{\rho+2}^{\rho+2}, \end{aligned}$$

and

$$I(u(t)) = \int_\Omega \left(\xi_1 - a(x)\int_0^t g(s)\,ds\right)|\nabla u|^2\,dx + (g \circ \nabla u) - \|u\|_{\rho+2}^{\rho+2}.$$

Lemma 2.4. *Suppose that* (A4) *holds. Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *such that*

$$\gamma := \frac{C_{\rho+2}^{\rho+2}}{l}\left(\frac{2(\rho+2)}{\rho l}E(0)\right)^{\rho/2} < 1 \quad \text{and} \quad I(u_0) > 0, \tag{2.8}$$

where $C_{\rho+2}$ *is an embedding constant of* $H_0^1(\Omega) \hookrightarrow L^{\rho+2}(\Omega)$. *Then* $I(u) > 0$ *for all* $t > 0$.

Proof. By using the continuity of $u(t)$ and noting that $I(u_0) > 0$, we can get that, for all $\widetilde{t} \in U(0)$, $I(u(\widetilde{t})) \geq 0$. We denote the maximal interval where the above inequality holds by $[0, t_{\max}]$. It follows that

$$
\begin{aligned}
(2.9)\quad J\big(u(\widetilde{t})\big) &= \frac{\rho}{2(\rho+2)}\bigg[\int_\Omega\bigg(\xi_1 - a(x)\int_0^t g(s)\,ds\bigg)\big|\nabla u(\widetilde{t})\big|^2\,dx + (g\circ\nabla u)(\widetilde{t})\bigg] \\
&\quad + \frac{\xi_2}{4}\big\|\nabla u(\widetilde{t})\big\|^4 + \frac{1}{\rho+2}I\big(u(\widetilde{t})\big) \\
&\geq \frac{\rho}{2(\rho+2)}\bigg[\int_\Omega\bigg(\xi_1 - a(x)\int_0^t g(s)\,ds\bigg)\big|\nabla u(\widetilde{t})\big|^2\,dx + (g\circ\nabla u)(\widetilde{t})\bigg],
\end{aligned}
$$

which, together with (2.3) and (2.7), gives us for all $t \in [0, t_{\max}]$

$$
\begin{aligned}
(2.10)\qquad \big\|\nabla u(\widetilde{t})\big\|^2 &\leq \int_\Omega\bigg(\xi_1 - a(x)\int_0^t g(s)\,ds\bigg)\big|\nabla u(\widetilde{t})\big|^2\,dx \\
&\leq \frac{2(\rho+2)}{\rho}\,J\big(u(\widetilde{t})\big) \leq \frac{2(\rho+2)}{\rho}\,E(\widetilde{t}) \leq \frac{2(\rho+2)}{\rho}\,E(0).
\end{aligned}
$$

By using (2.3), (2.8) and (2.10), we shall see that

$$
\begin{aligned}
\big\|u(\widetilde{t})\big\|_{\rho+2}^{\rho+2} &\leq C_{\rho+2}^{\rho+2}\|\nabla u(\widetilde{t})\|^{\rho+2} \\
&\leq \gamma l\big\|\nabla u(\widetilde{t})\big\|^2 < \int_\Omega\bigg(\xi_1 - a(x)\int_0^t g(s)\,ds\bigg)\big|\nabla u(\widetilde{t})\big|^2\,dx.
\end{aligned}
$$

Then we have $I(u(\widetilde{t})) > 0$ for every $t \in [0, t_{\max}]$. By repeating the procedure and using the fact that

$$
\lim_{\widetilde{t}\to t_{\max}} \frac{C_{\rho+2}^{\rho+2}}{l}\bigg(\frac{2(\rho+2)}{\rho l}\,E(\widetilde{t})\bigg)^{\rho/2} \leq \gamma < 1,
$$

$t_{\max}$ is extended to for all t. □

Remark 2.5. For Lemma 2.4, we can get that, for all $t > 0$,

$$
(2.11)\qquad 0 < I(u) \leq (\rho+2)J(u) \leq (\rho+2)E(t).
$$

Theorem 2.6. *Suppose* (A1)–(A5) *and* (2.8) *hold. If the initial data* (u_0, u_1) *in* $H_0^1(\Omega) \times L^2(\Omega)$ *satisfy* (2.8), *then the solution* $u(t)$ *of problem* (1.1)–(1.3) *is global in time.*

Proof. It remains to prove that $\|u_t\|^2 + \|\nabla u\|^2$ is bounded independently of t. It follows from (2.9) and Lemma 2.4 that

$$
\begin{aligned}
J(u(t)) &= \frac{\rho}{2(\rho+2)}\bigg[\int_\Omega\bigg(\xi_1 - a(x)\int_0^t g(s)\,ds\bigg)|\nabla u(t)|^2\,dx + (g\circ\nabla u)(t)\bigg] \\
&\quad + \frac{\xi_2}{4}\|\nabla u(t)\|^4 + \frac{1}{\rho+2}\,I(u(t)) \\
&> \frac{\rho}{2(\rho+2)}\bigg[\int_\Omega\bigg(\xi_1 - a(x)\int_0^t g(s)\,ds\bigg)|\nabla u(t)|^2\,dx + (g\circ\nabla u)(t)\bigg].
\end{aligned}
$$

Then we get

$$l\|\nabla u(t)\|^2 \leq \int_\Omega \left(\xi_1 - a(x)\int_0^t g(s)\,ds\right)|\nabla u(t)|^2\,dx < \frac{2(\rho+2)}{\rho}\,J(u(t)).$$

Therefore,

$$\frac{1}{2}\|u_t(t)\|^2 + \frac{\rho l}{2(\rho+2)}\|\nabla u(t)\|^2 < \frac{1}{2}\|u_t(t)\|^2 + J(u(t)) = E(t) \leq E(0).$$

Then there exists a constant $C > 0$ depending only on ρ and l such that

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 \leq CE(0). \qquad \square$$

Now we are in a position to state the stability result of energy to problem (1.1)–(1.3) given in the following theorem.

Theorem 2.7. *Suppose* (A1)–(A5) *and* (2.8) *hold. Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. *Then there exist positive constants* k_1, k_2, k_3 *and* ε_0 *such that the energy* $E(t)$ *defined by* (2.6) *satisfies for any* $t \geq 0$,

$$E(t) \leq k_3 H_1^{-1}\left(k_1\int_0^t \eta(s)\,ds + k_2\right), \tag{2.12}$$

and

$$H_1(t) = \int_t^1 \frac{1}{sH_0'(\varepsilon_0 s)}\,ds \quad \text{and} \quad H_0(t) = H(D(t))$$

and D is a positive C^1 function satisfying $D(0) = 0$, and H_1 is strictly increasing and convex C^2 function on $(0, r]$ with

$$\int_0^\infty \frac{g(s)}{H_0^{-1}(-g'(s))}\,ds < +\infty. \tag{2.13}$$

In addition, if we choose the function $D(t)$ satisfying $\int_0^1 H_1(t)\,dt < +\infty$, then we obtain

$$E(t) \leq k_3 G^{-1}\left(k_1\int_0^t \eta(s)\,ds + k_2\right), \quad \text{where } G(t) = \int_t^1 \frac{1}{sH'(\varepsilon_0 s)}\,ds. \tag{2.14}$$

In particular, this last estimate also holds for the special case $H(t) = ct^p$ with $1 \leq p < 3/2$.

Remark 2.8. (a) H_1 and G are strictly decreasing and convex on $(0, 1]$, with $\lim_{t\to 0} H_1(t) = \lim_{t\to 0} G(t) = +\infty$. Therefore,

$$\text{if} \quad \int_0^\infty \eta(s)\,ds = +\infty, \quad \text{then} \quad \lim_{t\to+\infty} E(t) = 0.$$

(b) The following Jensen's inequality is critical to prove our main result. Let F be a convex increasing function on $[a, b]$, $f\colon \Omega \to [a, b]$ and m are integrable functions on Ω such that $m(x) \geq 0$ and $\int_\Omega m(x)\,dx = k > 0$, then Jensen's inequality states that

$$\int_\Omega F^{-1}(f(x))m(x)\,dx \leq kF^{-1}\bigg[\frac{1}{k}\int_\Omega f(x)m(x)\,dx\bigg].$$

(c) It should be pointed out that the restriction that $p \in [1, 3/2)$ in Theorem 2.7 is not optimal. In fact the polynomial decay can be pushed up to $p < 2$, one can refer to Lasiecka and Wang [17].

(d) In the absence of Balakrishnan–Taylor damping in (1.1), i.e. $\sigma = 0$, we can get the same results.

(e) It follows from (A1) that $\lim_{t\to+\infty} g(t) = 0$. Similarly, assuming the existence of the limit, we can get $\lim_{t\to+\infty}(-g'(t)) = 0$. Then there exists some $t_1 \geq 0$ large enough such that

$$\max\{g(t), -g'(t)\} < \min\{r, H(r), H_0(r)\}, \quad \text{for all } t \geq t_1. \tag{2.15}$$

Since H is a positive continuous function and g is positive nonincreasing continuous function, we can get for every $t \in [0, t_1]$, $0 < g(t_1) \leq g(t) \leq g(0)$. Therefore there exist positive constants a and b such that, $a \leq H(g(t)) \leq b$, which yields, for every $t \in [0, t_1]$,

$$g'(t) \leq -H(g(t)) \leq -\frac{a}{g(0)}g(0) \leq -\frac{a}{g(0)}g(t). \tag{2.16}$$

Then for some positive constant μ,

$$g'(t) \leq -\mu g(t), \quad \text{for all } t \in [0, t_1]. \tag{2.17}$$

(f) If there exist two different functions H_1 and H_2 satisfying (A4)–(A5) such that $g'(t) \leq -H_1(g(t))$ and $s^2 + h^2(s) \leq H_2^{-1}(sh(s))$, then there is some $r < \min\{r_1, r_2\}$ so small that $H_1(t) \leq H_2(t)$ on $(0, r]$. Then, for all $t \geq t_1$, the function $H(t) = H_1(t)$ satisfies (A4) and (A5).

We end this section by giving an example in which a much larger class of relaxation functions g guarantee that the uniform decays are not necessarily of exponential or polynomial decay: one can find it in Mustafa and Messaoudi [31].

Example 2.9. Let $g(t) = ae^{-t^q}$ for $0 < q < 1$ and a chosen so that g satisfies (2.3), then $g'(t) = -H(g(t))$, with, for $t \in (0, r]$, $r < a$,

$$H(t) = \frac{qt}{[\ln(a/t)]^{1/q-1}},$$

which satisfies (A4). In addition, taking $D(t) = t^\varsigma$, (2.13) is satisfies for any $\varsigma > 1$. Therefore, if h satisfies (A5) with this function H, then we can use Theorem 2.7 to get the energy decays at the rate

$$E(t) \leq c\exp\bigg[-k\bigg(\int_0^t \eta(s)\,ds\bigg)^q\bigg].$$

3. General decay

In this section we shall study the general decay of energy to problem (1.1)–(1.3) to prove Theorem 2.7.

3.1. Technical lemmas. First we introduce a function $\alpha(x)$. Since $a(x)$ is continuous and $\operatorname{meas}\{x \in \partial\Omega : a(x) > 0\} > 0$, there exists an open set $V \subset \Omega$ and $\varepsilon_0 > 0$ such that $\operatorname{meas}(\partial V \cap \partial\Omega) > 0$ and $\inf_{x\in\overline{V}} a(x) \geq \varepsilon_0$. Set $d = \min\{\varepsilon_0, \beta\} > 0$, where β is defined in (2.2), and let $\alpha \in C^1(\overline{\Omega})$ be such that $0 \leq \alpha(x) \leq a(x)$ and

$$\begin{aligned} \alpha(x) &= 0, && \text{if } a(x) \leq \frac{d}{4}, \\ \alpha(x) &= a(x), && \text{if } a(x) \geq \frac{d}{2}. \end{aligned}$$

The same arguments as in Mustafa and Abusharkh [30], we can get the following lemma.

LEMMA 3.1. *The function $\alpha(x)$ is not identically zero and satisfies*

$$\inf_{x\in\overline{\Omega}}\{\alpha(x) + b(x)\} \geq \frac{d}{2}. \tag{3.1}$$

LEMMA 3.2. *Let u be the solution of problem* (1.1)–(1.3). *Then there exists $c > 0$ such that*

$$\int_\Omega \alpha(x)\left|\int_0^t g(t-s)(u(t)-u(s))\,ds\right|^2 dx \leq c(g\circ\nabla u)(t). \tag{3.2}$$

PROOF. Let $S_a = \{x \in \Omega : a(x) \geq d/4\}$. We note that $\operatorname{supp}\alpha \subset S_a$, and noting the definition of d, $\partial V \cap \partial\Omega \subset \partial S_a \cap \partial\Omega$, then we know that $\operatorname{meas}(\partial S_a \cap \partial\Omega) > 0$. It follows from Hölder's inequality and Poincaré's inequality that

$$\begin{aligned} &\int_\Omega \alpha(x)\left|\int_0^t g(t-s)(u(t)-u(s))\,ds\right|^2 dx \\ &\quad= \int_{\operatorname{supp}\alpha} \alpha(x)\left|\int_0^t g(t-s)(u(t)-u(s))\,ds\right|^2 dx \\ &\quad\leq c\int_0^t g(s)\,ds \int_{\operatorname{supp}\alpha}\int_0^t g(t-s)(u(t)-u(s))^2\,ds\,dx \\ &\quad\leq c\int_{S_a}\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx \end{aligned}$$

which, recalling the definition of S_a, implies

$$\begin{aligned} &\int_\Omega \alpha(x)\left|\int_0^t g(t-s)(u(t)-u(s))\,ds\right|^2 dx \\ &\qquad\leq c\int_{S_a}\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx \leq c(g\circ\nabla u)(t). \end{aligned}$$

The proof is complete. □

Remark 3.3. (a) We use a variant of the Poincaré's inequality in the proof of Lemma 3.2, see, for example, Cavalcanti and Oquendo [8]: Let Ω_1, Ω_2, and Ω be subsets of $\mathbb{R}^n$ with positive measure and such that $\overline{\Omega}_1 \subset \Omega_2$, $\Omega_2 \subset \Omega$. Then, assuming that Ω is bounded and moreover that meas $(\partial\Omega_2 \cap \partial\Omega) \neq 0$, we have

$$\int_{\Omega_1} |\omega|^2\,dx \leq c\int_{\Omega_2} |\nabla\omega|^2\,dx \quad \text{for all } \omega \in H_0^1(\Omega),$$

where c is a positive constant.

(b) Since $\operatorname{supp}|\nabla\alpha| \subset \operatorname{supp}\alpha$, then (3.2) also holds with the right-hand side $c(g\circ\nabla u)$ if α in the left-hand side is replaced by $|\nabla\alpha|$.

Now we define the following functionals:

$$\phi(t) = \int_\Omega u(t)u_t(t)\,dx + \frac{\sigma}{4}\|\nabla u(t)\|^4,$$

$$\psi(t) = -\int_\Omega \alpha(x)u_t(t)\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx.$$

Lemma 3.4. *Under the assumptions of Theorem* 2.7, *the functional* $\phi(t)$ *satisfies that for any* $t \geq 0$,

$$\begin{aligned}\phi'(t) \leq -\frac{l(1-\gamma)}{4}\|\nabla u(t)\|^2 + \|u_t(t)\|^2 - \xi_2\|\nabla u(t)\|^4 \\ + c(g\circ\nabla u)(t) + c\int_\Omega b(x)h^2(u_t)\,dx.\end{aligned} \tag{3.3}$$

Proof. It follows from (1.1) that

$$\begin{aligned}\phi'(t) &= \int_\Omega u_{tt}(t)u(t)\,dx + \|u_t\|^2 + \sigma\|\nabla u\|^2(\nabla u, \nabla u_t) \\ &= \|u_t\|^2 - \big(\xi_1 + \xi_2\|\nabla u\|^2\big)\|\nabla u\|^2 + \int_0^t g(s)\,ds\int_\Omega a(x)|\nabla u|^2\,dx \\ &\quad + \int_\Omega a(x)\nabla u(t)\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))\,ds\,dx \\ &\quad - \eta(t)\int_\Omega b(x)uh(u_t)\,dx + \int_\Omega |u|^{\rho+2}\,dx \\ &\leq \|u_t\|^2 - l\|\nabla u\|^2 - \xi_2\|\nabla u\|^4 - \eta(t)\int_\Omega b(x)uh(u_t)\,dx \\ &\quad + \int_\Omega |u|^{\rho+2}\,dx + \int_\Omega a(x)\nabla u(t)\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))\,ds\,dx.\end{aligned} \tag{3.4}$$

By using Young's inequality and Hölder's inequality, we can obtain that

$$\begin{aligned}\int_\Omega a(x)\nabla u(t)\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))\,ds\,dx \\ \leq \frac{l(1-\gamma)}{2\|a\|_\infty}\int_\Omega a(x)|\nabla u|^2\,dx\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&+\frac{\|a\|_\infty}{2l(1-\gamma)}\int_\Omega a(x)\bigg(\int_0^t g(t-s)(\nabla u(s)-\nabla u(t))\,ds\bigg)^2\,dx\\
&\le \frac{l(1-\gamma)}{2}\|\nabla u\|^2+\frac{c}{2l(1-\gamma)}(g\circ\nabla u),
\end{aligned}
$$

$$
-\eta(t)\int_\Omega b(x)uh(u_t)\,dx \le \frac{l(1-\gamma)}{4}\|\nabla u\|^2+\frac{c}{l(1-\gamma)}\int_\Omega b(x)h^2(u_t)\,dx, \tag{3.6}
$$

and

$$
\begin{aligned}
\int_\Omega |u|^{\rho+2}\,dx &\le C_{\rho+2}^{\rho+2}\bigg(\int_\Omega |\nabla u|^2\,dx\bigg)^{(\rho+2)/2}\\
&\le C_{\rho+2}^{\rho+2}\bigg(\int_\Omega |\nabla u|^2\,dx\bigg)^{\rho/2}\|\nabla u\|^2\\
&\le C_{\rho+2}^{\rho+2}\bigg(\frac{2(\rho+2)}{\rho l}E(0)\bigg)^{\rho/2}\|\nabla u\|^2\le l\gamma\|\nabla u\|^2.
\end{aligned} \tag{3.7}
$$

Inserting (3.5)–(3.7) into (3.4), we get the desired estimate (3.3). □

Lemma 3.5. *Under the assumptions of Theorem* 2.7*, the functional* $\psi(t)$ *satisfies that for any* $\delta>0$ *and for any* $t\ge 0$*,*

$$
\begin{aligned}
\psi'(t)\le &-\bigg(\int_0^t g(s)\,ds-\delta\bigg)\int_\Omega \alpha(x)u_t^2(t)\,dx+\delta\|\nabla u(t)\|^2\\
&+\frac{4\sigma^2\delta}{l}E(0)\bigg(\frac12\frac{d}{dt}\|\nabla u(t)\|^2\bigg)^2+\frac{c}{\delta}(g\circ\nabla u)(t)\\
&-\frac{c}{\delta}(g'\circ\nabla u)(t)+\delta\int_\Omega b(x)h^2(u_t)\,dx.
\end{aligned} \tag{3.8}
$$

Proof. By using (1.1) and integration by parts, we can get

$$
\begin{aligned}
\psi'(t)=&\big(\xi_1+\xi_2\|\nabla u\|^2+\sigma(\nabla u,\nabla u_t)\big)\int_\Omega \alpha\nabla u\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&+\big(\xi_1+\xi_2\|\nabla u\|^2+\sigma(\nabla u,\nabla u_t)\big)\int_\Omega \nabla\alpha\nabla u\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx\\
&-\int_\Omega \nabla\alpha\bigg(\int_0^t a(x)g(t-s)(\nabla u(s)-\nabla u(t))\,ds\bigg)\\
&\times\bigg(\int_0^t g(t-s)(u(t)-u(s))\,ds\bigg)\,dx\\
&-\int_\Omega \nabla\alpha\bigg(\int_0^t a(x)g(t-s)\nabla u(t)\,ds\bigg)\bigg(\int_0^t g(t-s)(u(t)-u(s))\,ds\bigg)\,dx\\
&-\int_\Omega \alpha\bigg(\int_0^t a(x)g(t-s)(\nabla u(s)-\nabla u(t))\,ds\bigg)\\
&\times\bigg(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\bigg)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&- \int_\Omega \alpha\bigg(\int_0^t a(x) g(t-s)\nabla u(t)\,ds\bigg)\bigg(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\bigg)\,dx \\
&+ \int_\Omega \eta(t) b(x) h(u_t) \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&- \int_\Omega \alpha |u|^\rho u \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&- \int_\Omega \alpha u_t \int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx - \int_0^t g(s)\,ds \int_\Omega \alpha u_t^2\,dx.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\psi'(t) = {} & \int_\Omega \alpha\bigg(\xi_1 - a(x)\int_0^t g(s)ds + \xi_2\|\nabla u\|^2\bigg)\nabla u \\
&\times \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&+ \int_\Omega \nabla\alpha\bigg(\xi_1 - a(x)\int_0^t g(s)\,ds + \xi_2\|\nabla u\|^2\bigg)\nabla u \\
&\times \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&+ \sigma(\nabla u, \nabla u_t)\int_\Omega \alpha\nabla u \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&+ \sigma(\nabla u, \nabla u_t)\int_\Omega \nabla\alpha\nabla u \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&+ \int_\Omega \alpha a(x)\bigg(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\bigg)^2 dx \\
&+ \int_\Omega \nabla\alpha a(x)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds \\
&\times \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&+ \int_\Omega \eta(t) b(x) h(u_t)\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&- \int_\Omega \alpha|u|^\rho u \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&- \int_\Omega \alpha u_t \int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx - \int_0^t g(s)ds\int_\Omega \alpha u_t^2\,dx.
\end{aligned} \tag{3.9}
$$

Noting that $E(t)$ is non-increasing we infer that

$$
\bigg(\xi_1 - \|a\|_\infty \int_0^\infty g(s)\,ds\bigg)\|\nabla u\|^2 \le \int_\Omega \bigg(\xi_1 - a(x)\int_0^t g(s)\,ds\bigg)|\nabla u|^2\,dx \le 2E(t),
$$

which, together with (2.3), gives us

$$
\|\nabla u\|^2 \le \frac{2}{l}E(0). \tag{3.10}
$$

Then by using Hölder's inequality, Young's inequality, (3.10) and (3.2), we see that, for any $\delta > 0$,

$$\begin{aligned}(3.11)\quad &\int_\Omega \alpha\Big(\xi_1 - a(x)\int_0^t g(s)\,ds\Big)\nabla u \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&\le \Big(\xi_1 + \frac{2\xi_2}{l}E(0)\Big)\int_\Omega |\alpha||\nabla u|\left|\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right|dx\\
&\le \frac{\delta}{3}\|\nabla u\|^2 + \frac{3}{4\delta}\Big(\xi_1 + \frac{2\xi_2}{l}\,E(0)\Big)^2\\
&\quad\times \int_\Omega \alpha^2\Big(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\Big)^2 dx\\
&\le \frac{\delta}{3}\|\nabla u\|^2 + \frac{c}{\delta}(g\circ\nabla u),\end{aligned}$$

$$\begin{aligned}(3.12)\quad &\sigma(\nabla u,\nabla u_t)\int_\Omega \alpha\nabla u\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&\le \sigma^2(\nabla u,\nabla u_t)^2\delta\|\nabla u\|^2\\
&\quad + \frac{1}{4\delta}\int_\Omega \alpha^2\Big(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\Big)^2 dx\\
&\le \sigma^2\frac{2\delta}{l}\,E(0)\Big(\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2\Big)^2 + \frac{c}{\delta}\,(g\circ\nabla u),\end{aligned}$$

and

$$(3.13)\quad \int_\Omega \alpha a(x)\Big(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\Big)^2 dx \le c(g\circ\nabla u).$$

Similarly, we can also get for any $\delta > 0$,

$$\begin{aligned}(3.14)\quad &\int_\Omega \nabla\alpha\Big(\xi_1 - a(x)\int_0^t g(s)\,ds + \xi_2\|\nabla u\|^2\Big)\nabla u\\
&\quad\times\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \le \frac{\delta}{3}\,\|\nabla u\|^2 + \frac{c}{\delta}\,(g\circ\nabla u),\end{aligned}$$

$$\begin{aligned}(3.15)\quad &\sigma(\nabla u,\nabla u_t)\int_\Omega \nabla\alpha\nabla u\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx\\
&\le \sigma^2\,\frac{2\delta}{l}\,E(0)\Big(\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2\Big)^2 + \frac{c}{\delta}\,(g\circ\nabla u),\end{aligned}$$

and

$$\begin{aligned}(3.16)\quad &\int_\Omega \nabla\alpha a(x)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\\
&\quad\times\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \le c(g\circ\nabla u).\end{aligned}$$

It follows from (A2) and Hölder's and Young's inequalities that, for any $\delta > 0$,

$$\int_\Omega \eta(t)b(x)h(u_t)\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \le \delta\int_\Omega b(x)h^2(u_t)\,dx + \frac{c}{\delta}(g\circ\nabla u), \tag{3.17}$$

$$-\int_\Omega \alpha u_t\int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx \le \delta\int_\Omega \alpha u_t^2\,dx - \frac{c}{\delta}(g'\circ\nabla u), \tag{3.18}$$

and, for any $\delta_1 > 0$,

$$\begin{aligned} -\int_\Omega \alpha|u|^\rho u\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx &\le \delta_1\int_\Omega |u|^{2\rho+2}\,dx + \frac{c}{\delta_1}(g\circ\nabla u) \\ &\le \delta_1 C_{\rho+2}^{2(\rho+2)}\bigg(\frac{2(\rho+2)}{\rho l}E(0)\bigg)^\rho\|\nabla u\|^2 + \frac{c}{\delta_1}(g\circ\nabla u). \end{aligned} \tag{3.19}$$

Taking

$$\delta_1 = \frac{\delta}{3C_{\rho+2}^{2(\rho+2)}\bigg(\dfrac{2(\rho+2)}{\rho l}E(0)\bigg)^\rho}$$

in (3.19) and then replacing (3.11)–(3.19) in (3.9), we can get (3.8). $\square$

Now we define the functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) := N_1E(t) + \phi(t) + N_2\psi(t),$$

where N_1 and N_2 are positive constants will be chosen later. It is easy to verify that for N_1 large, there exist $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$\beta_1E(t) \le \mathcal{L}(t) \le \beta_2E(t). \tag{3.20}$$

Lemma 3.6. *There exists a constant $m > 0$ such that for any $t \ge t_1$,*

$$\mathcal{L}'(t) \le -mE(t) + c(g\circ\nabla u)(t) + c\int_\Omega b(x)(u_t^2 + h^2(u_t))\,dx. \tag{3.21}$$

Proof. Let $g_1 = \int_0^{t_1} g(s)\,ds > 0$. Combining (2.7), (3.3) and (3.8), and taking $\delta = l/(4N_2)$, we can infer that, for any $t \ge t_1$,

$$\begin{aligned} \mathcal{L}'(t) \le &-\frac{l(1-\gamma)}{4}\|\nabla u\|^2 - \bigg(N_2g_1 - \frac{l}{4}\bigg)\int_\Omega \alpha u_t^2\,dx + \|u_t\|^2 - \xi_2\|\nabla u\|^4 \\ &+ \bigg(\frac{4c}{l}N_2^2 + c\bigg)(g\circ\nabla u) + \bigg(\frac{N_1}{2} - \frac{4c}{l}N_2^2\bigg)(g'\circ\nabla u) \\ &- (\sigma N_1 - \sigma^2E(0))\bigg(\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2\bigg)^2 + \bigg(\frac{l}{4} + c\bigg)\int_\Omega b(x)h^2(u_t)\,dx. \end{aligned} \tag{3.22}$$

Using (3.1), we have

$$\begin{aligned}
&-\left(N_2 g_1 - \frac{l}{4}\right)\int_\Omega \alpha u_t^2\,dx + \|u_t\|^2 \\
&\quad= -\left(N_2 g_1 - \frac{l}{4}\right)\int_\Omega \alpha u_t^2\,dx + \frac{2}{d}\int_\Omega \frac{d}{2}u_t^2\,dx \\
&\quad\le -\left(N_2 g_1 - \frac{l}{4}\right)\int_\Omega \alpha u_t^2\,dx + \frac{2}{d}\int_\Omega \big[\alpha(x)+b(x)\big]u_t^2\,dx \\
&\quad= -\left(N_2 g_1 - \frac{l}{4} - \frac{2}{d}\right)\int_\Omega \big[\alpha(x)+b(x)\big]u_t^2\,dx + \left(N_2 g_1 - \frac{l}{4}\right)\int_\Omega b(x)u_t^2\,dx \\
&\quad\le -\left(N_2 g_1 - \frac{l}{4} - \frac{2}{d}\right)\int_\Omega \frac{d}{2}u_t^2\,dx + \left(N_2 g_1 - \frac{l}{4}\right)\int_\Omega b(x)u_t^2\,dx,
\end{aligned}$$

which together with (3.22) implies that, for any $t \ge t_1$,

$$\begin{aligned}
\mathcal{L}'(t) \le &-\frac{l(1-\gamma)}{4}\|\nabla u\|^2 - \frac{d}{2}\left(N_2 g_1 - \frac{l}{4} - \frac{2}{d}\right)\|u_t\|^2 - \xi_2\|\nabla u\|^4 \\
&+\left(\frac{4c}{l}N_2^2 + c\right)(g\circ\nabla u) + \left(\frac{N_1}{2} - \frac{4c}{l}N_2^2\right)(g'\circ\nabla u) \\
&-\big(\sigma N_1 - \sigma^2 E(0)\big)\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2\right)^2 + \left(\frac{l}{4}+c\right)\int_\Omega b(x)h^2(u_t)\,dx \\
&+\left(N_2 g_1 - \frac{l}{4}\right)\int_\Omega b(x)u_t^2\,dx.
\end{aligned}$$

At this point we take $N_2 > 0$ so large that $N_2 g_1 - l/4 - 2/d > 0$, and then take $N_1 > 0$ large enough so that $N_1/2 - 4cN_2^2/l$ and $\sigma N_1 - \sigma^2 E(0) > 0$. Thus there exist a positive constant m such that, for any $t \ge t_1$,

$$\mathcal{L}'(t) \le -mE(t) + c(g\circ\nabla u)(t) + c\int_\Omega b(x)(u_t^2 + h^2(u_t))\,dx,$$

which completes the proof. □

3.2. Proof of Theorem 2.7. Following the same arguments as in Komornik [14], we consider

$$\Omega^+ = \{x \in \Omega : |u_t| \le r\} \quad\text{and}\quad \Omega^- = \{x \in \Omega : |u_t| > r\}.$$

It follows from (A2), (A5), (2.17) and (2.7) that, for any $t \ge t_1$,

$$\begin{aligned}
\eta(t)&\int_0^{t_1} g(s)\int_\Omega a(x)|\nabla u(t) - \nabla u(t-s)|^2\,dx\,ds \\
&+\eta(t)\int_{\Omega^-} b(x)(u_t^2 + h^2(u_t))\,dx \\
\le &-\frac{c}{\mu}\int_0^{t_1} g'(s)\int_\Omega a(x)|\nabla u(t) - \nabla u(t-s)|^2\,dx\,ds \\
&+c\eta(t)\int_{\Omega^-} b(x)u_t h(u_t)\,dx \le -cE'(t).
\end{aligned} \tag{3.23}$$

Here we denote $F(t) = \eta(t)\mathcal{L}(t) + cE(t)$. It is easy to derive that $F(t) \sim E(t)$. Multiplying (3.21) by $\eta(t)$ and using (3.23), we can conclude that for any $t \geq t_1$,

$$F'(t) \leq -m\eta(t)E(t) + c\eta(t)\int_{t_1}^{t} g(s)\int_{\Omega} a(x)|\nabla u(t) - \nabla u(t-s)|^2\,dx\,ds + c\eta(t)\int_{\Omega^+} b(x)(u_t^2 + h^2(u_t))\,dx. \tag{3.24}$$

In order to prove Theorem 2.7 we distinguish the following two cases .

3.2.1. Special case: $H(t) = ct^p$ *and* $1 \leq p < 3/2$. Hölder's inequality yields that

$$\begin{aligned}\eta(t)\int_{\Omega^+} b(x)(u_t^2 + h^2(u_t))\,dx &\leq c\eta(t)\int_{\Omega^+} b(x)[u_t h(u_t)]^{1/p}\,dx \\ &\leq c\eta(t)\bigg(\int_{\Omega^+} b(x)u_t h(u_t)\,dx\bigg)^{1/p} \leq c\eta^{(p-1)/p}(t)\big[-E'(t)\big]^{1/p}.\end{aligned} \tag{3.25}$$

Case 1. $p = 1$. It follows from (3.24)–(3.25) that, for any $t \geq t_1$,

$$F'(t) \leq -m\eta(t)E(t) - c\eta(t)(g\circ\nabla u) - cE'(t) \leq -m\eta(t)E(t) - cE'(t). \tag{3.26}$$

We define $\mathcal{J}(t) = F(t) + cE(t)$, which is equivalent to $E(t)$, and can get from (3.26) that $\mathcal{J}'(t) \leq -c\eta(t)\mathcal{J}(t)$, for any $t \geq t_1$, which implies for any $t \geq t_1$,

$$\mathcal{J}(t) \leq \mathcal{J}(t_1)\exp\bigg(-c\int_{t_1}^{t}\eta(s)\,ds\bigg).$$

Using the fact $\mathcal{J}(t) \sim E(t)$, we have

$$E(t) \leq c\exp\bigg(-c\int_0^t \eta(s)\,ds\bigg) = c\,G_1^{-1}\bigg(c\int_0^t \eta(s)\,ds\bigg).$$

Case 2. $1 < p < 3/2$. As in Messaoudi [21], we know that, for any $\delta_0 < 2-p$, $\int_0^\infty g^{1-\delta_0}(s)\,ds < +\infty$. Choosing t_1 so large as needed and using (2.7), we can obtain that, for any $t \geq t_1$,

$$\begin{aligned}\pi(t) &:= \int_{t_1}^{t} g^{1-\delta_0}(s)\int_{\Omega} a(x)|\nabla u(t) - \nabla u(t-s)|^2\,dx\,ds \\ &\leq c\int_{t_1}^{t} g^{1-\delta_0}(s)\int_{\Omega}\big(|\nabla u(t)|^2 + |\nabla u(t-s)|^2\big)^2 dx\,ds \\ &\leq cE(0)\int_{t_1}^{t} g^{1-\delta_0}(s)\,ds < 1.\end{aligned} \tag{3.27}$$

From Jensen's inequality, (2.7), (A4) and (3.27) it follows that, for any $t \geq t_1$,

$$
\begin{aligned}
(3.28)\quad & \int_{t_1}^{t} g(s) \int_\Omega a(x)|\nabla u(t) - \nabla u(t-s)|^2\,dx\,ds \\
&= \int_{t_1}^{t} g^{\delta_0}(s) g^{1-\delta_0}(s) \int_\Omega a(x)|\nabla u(t) - \nabla u(t-s)|^2\,dx\,ds \\
&= \int_{t_1}^{t} g^{(p-1+\delta_0)\delta_0/(p-1+\delta_0)}(s) g^{1-\delta_0}(s) \\
&\quad \times \int_\Omega a(x)|\nabla u(t) - \nabla u(t-s)|^2\,dx\,ds \\
&\leq \pi(t)\biggl(\frac{1}{\pi(t)} \int_{t_1}^{t} g^{(p-1+\delta_0)}(s) g^{1-\delta_0}(s) \\
&\quad \times \int_\Omega a(x)|\nabla u(t) - \nabla u(t-s)|^2\,dx\,ds\biggr)^{\delta_0/(p-1+\delta_0)} \\
&\leq \biggl(\int_{t_1}^{t} g^{p}(s) \int_\Omega a(x)|\nabla u(t) - \nabla u(t-s)|^2\,dx\,ds\biggr)^{\delta_0/(p-1+\delta_0)} \\
&\leq \biggl(\int_{t_1}^{t} (-g'(s)) \int_\Omega a(x)|\nabla u(t) - \nabla u(t-s)|^2\,dx\,ds\biggr)^{\delta_0/(p-1+\delta_0)} \\
&\leq c\bigl[-E'(t)\bigr]^{\delta_0/(p-1+\delta_0)}.
\end{aligned}
$$

Taking $\delta_0 = 1/2$ in (3.28), we can obtain from (3.24) that, for any $t \geq t_1$,

$$
(3.29)\quad F'(t) \leq -m\eta(t)E(t) + c\eta(t)\bigl[-E'(t)\bigr]^{1/(2p-1)} + c\eta^{(p-1)/p}(t)\bigl[-E'(t)\bigr]^{1/p}.
$$

Multiplying (3.29) by $E^{2p-2}(t)$ and using (2.7), we can get

$$
\begin{aligned}
(FE^{2p-2})' &\leq F'E^{2p-2} \\
&\leq -m\eta E^{2p-1} + c\eta E^{2p-2}(-E')^{1/(2p-1)} + c\eta^{(p-1)/p}E^{2p-2}(-E')^{1/p},
\end{aligned}
$$

which, applying Young's inequality, implies for any $\varepsilon > 0$ and $\delta > 0$,

$$
(FE^{2p-2})' \leq -(m-\varepsilon)\eta E^{2p-1} + C_\varepsilon \eta(-E'(t)) + \delta\eta E^{2p} + C_\delta(-E').
$$

Noting $E^{2p} \leq E(0)E^{2p-1}$ and choosing $\varepsilon + \delta E(0) < m$, we can derive that

$$
F_0'(t) \leq -c\eta(t)E^{2p-1}(t) \quad \text{with } F_0 = FE^{2p-2} + cE \sim E.
$$

Hence $F_0'(t) \leq -c\,\eta(t)F_0^{2p-1}(t)$, which gives us

$$
(3.30)\qquad E(t) \leq c\biggl(c_1 \int_0^t \eta(s)\,ds + c_2\biggr)^{-1/(2p-2)}.
$$

Recalling $p < 3/2$ and using (3.30), we know that $\int_0^\infty \eta(t)E(t)\,dt < +\infty$. Noting that

$$
\eta(t)\int_0^t \int_\Omega a(x)|\nabla u(t) - \nabla u(t-s)|^2\,dx\,ds \leq c\int_0^t \eta(s)E(s)\,ds,
$$

then we see that

$$\eta(t)\int_0^t g(s)\int_\Omega a(x)|\nabla u(t)-\nabla u(t-s)|^2\,dx\,ds=\eta(t)(g^{p\cdot(1/p)}\circ\nabla u)$$
$$\le c\eta^{1/p}(t)\big[(g^p\circ\nabla u)\big]^{1/p}\le c\eta^{(p-1)/p}(t)\big[(-g'\circ\nabla u)\big]^{1/p}$$
$$\le c\eta^{(p-1)/p}(t)\big[-E'(t)\big]^{1/p},$$

which, together with (3.24), yields

$$F'(t)\le -m\eta(t)E(t)+c\eta^{(p-1)/p}(t)\big[-E'(t)\big]^{1/p}. \tag{3.31}$$

Multiplying (3.31) by E^{p-1} and using the same arguments as before, we can obtain that

$$E(t)\le c\bigg(c_1\int_0^t\eta(s)\,ds+c_2\bigg)^{-1/(p-1)}=c\,G^{-1}\bigg(c_1\int_0^t\eta(s)\,ds+c_2\bigg).$$

3.2.2. General case. First we define $I(t)$ by

$$I(t):=\int_{t_1}^t\frac{g(s)}{H_0^{-1}(-g'(s))}\int_\Omega a(x)|\nabla u(t)-\nabla u(t-s)|^2\,dx\,ds,$$

where H_0 is such that (2.13) satisfied. In view of (3.27) we get that $I(t)$ satisfies, for any $t\ge t_1$,

$$I(t)<1. \tag{3.32}$$

Without loss of generality, we assume that $I(t)>0$ for all $t\ge t_1$, otherwise (3.24) implies an exponential decay. In addition, we define $\lambda(t)$ by

$$\lambda(t)=-\int_{t_1}^t g'(s)\,\frac{g(s)}{H_0^{-1}(-g'(s))}\int_\Omega a(x)|\nabla u(t)-\nabla u(t-s)|^2\,dx\,ds.$$

From (A4) and the properties of H_0 and D it follows that, for some $k_0>0$,

$$\frac{g(s)}{H_0^{-1}(-g'(s))}\le\frac{g(s)}{H_0^{-1}(H(g(s)))}=\frac{g(s)}{D^{-1}(g(s))}\le k_0.$$

Choosing t_1 so large as needed and using (2.7), we can obtain that for any $t\ge t_1$,

$$\begin{aligned}\lambda(t)&\le -k_0\int_{t_1}^t g'(s)\int_\Omega a(x)|\nabla u(t)-\nabla u(t-s)|^2\,dx\,ds\\&\le -cE(0)\int_{t_1}^t g'(s)\,ds\le cg(t_1)E(0)\le\frac12\min\{r,H(r),H_0(r)\}.\end{aligned} \tag{3.33}$$

Since $H_0(t)$ is strictly convex on $(0,r]$ and $H_0(0)=0$, then $H(\nu x)\le\nu H(x)$, provided $0\le\nu\le1$ and $x\in(0,r]$. By using (A4), (2.15), (3.32)–(3.33) and

Jensen's inequality, we can obtain

$$\begin{aligned}\lambda(t) &= \frac{1}{I(t)}\int_{t_1}^{t} I(t)H_0\big[H_0^{-1}(-g'(s))\big]\frac{g(s)}{H_0^{-1}(-g'(s))}\\ &\quad\times\int_\Omega a(x)|\nabla u(t)-\nabla u(t-s)|^2\,dx\,ds\\ &\geq \frac{1}{I(t)}\int_{t_1}^{t} H_0\big[I(t)H_0^{-1}(-g'(s))\big]\frac{g(s)}{H_0^{-1}(-g'(s))}\\ &\quad\times\int_\Omega a(x)|\nabla u(t)-\nabla u(t-s)|^2\,dx\,ds\\ &\geq H_0\bigg(\frac{1}{I(t)}\int_{t_1}^{t} I(t)H_0^{-1}(-g'(s))\frac{g(s)}{H_0^{-1}(-g'(s))}\\ &\quad\times\int_\Omega a(x)|\nabla u(t)-\nabla u(t-s)|^2\,dx\,ds\bigg)\\ &= H_0\bigg(\int_{t_1}^{t} g(s)\int_\Omega a(x)|\nabla u(t)-\nabla u(t-s)|^2\,dx\,ds\bigg),\end{aligned}$$

which gives us

$$\int_{t_1}^{t} g(s)\int_\Omega a(x)|\nabla u(t)-\nabla u(t-s)|^2\,dx\,ds \leq H_0^{-1}(\lambda(t)). \tag{3.34}$$

It remains to estimate the last integral in (3.24). We first assume that r is small enough such that, for any $|s|\leq r$,

$$sh(s)\leq \frac{1}{2}\min\{r, H(r), H_0(r)\}. \tag{3.35}$$

Define $S(t)$ by

$$S(t) := \frac{1}{\|b\|_{L^1(\Omega^+)}}\int_{\Omega^+} b(x)\,u_t h(u_t)\,dx.$$

Using (A5) and Jensen's inequality, we have

$$H^{-1}(S(t)) \geq c\int_{\Omega^+} b(x)H^{-1}(u_t h(u_t))\,dx \geq \int_{\Omega^+} b(x)\big(u_t^2+h^2(u_t)\big)\,dx. \tag{3.36}$$

Combining (3.34) and (3.36) with (3.24), we see that, for any $t\geq t_1$,

$$F'(t)\leq -m\eta(t)E(t)+c\eta(t)H_0^{-1}(\lambda(t))+c\eta(t)H^{-1}(S(t)).$$

Using the properties of H, D and H_0 and the fact that

$$H_0^{-1}(S(t)) = D^{-1}(H^{-1}(S(t))),\quad D^{-1}(0)=0\quad\text{and}\quad H^{-1}(S(t))\leq r,$$

we can infer that $H^{-1}(S(t))\leq cH_0^{-1}(S(t))$, which implies

$$\begin{aligned}F'(t)&\leq -m\eta(t)E(t)+c\eta(t)H_0^{-1}(\lambda(t))+c\eta(t)H_0^{-1}(S(t))\\ &\leq -m\eta(t)E(t)+c\eta(t)H_0^{-1}(\lambda(t)+S(t)).\end{aligned} \tag{3.37}$$

Now, for $\varepsilon_0 < r$ and $c_0 > 0$, we define the function $\mathcal{K}_1(t)$ by

$$\mathcal{K}_1(t) = H_0'\bigg(\varepsilon_0 \frac{E(t)}{E(0)}\bigg) F(t) + c_0 E(t),$$

which is equivalent to $E(t)$, i.e. there exist two positive constants a_1 and a_2 such that $a_1\mathcal{K}_1(t) \le E(t) \le a_2\mathcal{K}_1(t)$. Noting that $E'(t) \le 0$, $H_0' > 0$ and $H_0'' > 0$, we obtain for any $t \ge t_1$,

$$\begin{aligned} \mathcal{K}_1'(t) = &\,\varepsilon_0 \frac{E'(t)}{E(0)} H_0''\bigg(\varepsilon_0 \frac{E(t)}{E(0)}\bigg) F(t) + H_0'\bigg(\varepsilon_0 \frac{E(t)}{E(0)}\bigg) F'(t) + c_0 E'(t) \\ \le &- m\eta(t) E(t) H_0'\bigg(\varepsilon_0 \frac{E(t)}{E(0)}\bigg) + c_0 E'(t) \\ &+ c\eta(t) H_0'\bigg(\varepsilon_0 \frac{E(t)}{E(0)}\bigg) H_0^{-1}(\lambda(t) + S(t)). \end{aligned} \tag{3.38}$$

Now by H_0^* we denote the Fenchel conjugate function of the convex function H_0 (see, for example, Arnold [2]), i.e. $H_0^*(s) = \sup_{t\in\mathbb{R}^+} (st - H_0(t))$. Then

$$H_0^*(s) = s(H_0')^{-1}(s) - H_0[(H_0')^{-1}(s)],$$

is the Legendre transform of H_0, which satisfies $AB \le H_0^*(A) + H_0(B)$. With $A = H_0'(\varepsilon_0 E(t)/E(0))$ and $B = H_0^{-1}(\lambda(t) + S(t))$, and noting the fact $H_0^*(s) \le s(H_0')^{-1}(s)$, and using (2.7), (3.33) and (3.35), we see that

$$\begin{aligned} \mathcal{K}_1'(t) \le &- m\eta(t) E(t) H_0'\bigg(\varepsilon_0 \frac{E(t)}{E(0)}\bigg) + c\eta(t) H_0^*\bigg(H_0'\bigg(\varepsilon_0 \frac{E(t)}{E(0)}\bigg)\bigg) \\ &+ c\eta(t)(\lambda(t) + S(t)) + c_0 E'(t) \\ \le &- m\eta(t) E(t) H_0'\bigg(\varepsilon_0 \frac{E(t)}{E(0)}\bigg) \\ &+ c\,\varepsilon_0 \eta(t) \frac{E(t)}{E(0)} H_0'\bigg(\varepsilon_0 \frac{E(t)}{E(0)}\bigg) - cE'(t) + c_0 E'(t). \end{aligned} \tag{3.39}$$

For a suitable choice of ε_0 and c_0, we get that, for some constant $k > 0$,

$$\mathcal{K}_1'(t) \le -k\eta(t) \frac{E(t)}{E(0)} H_0'\bigg(\varepsilon_0 \frac{E(t)}{E(0)}\bigg) = -k\eta(t) H_2\bigg(\frac{E(t)}{E(0)}\bigg), \tag{3.40}$$

where $H_2(t) = tH_0'(\varepsilon_0 t)$.

Denote $R(t) = a_1\mathcal{K}_1(t)/E(0)$. It is easy to verify that

$$R(t) \sim E(t). \tag{3.41}$$

Since $H_2'(t) = H'(\varepsilon_0 t) + \varepsilon_0 t H''(\varepsilon_0 t)$, then, using the strict convexity of H on $(0, r]$, we know that $H_2'(t), H_2(t) > 0$ on $(0, 1]$. By (3.40), we can obtain for some $k_1 > 0$ and for all $t \ge t_1$,

$$R'(t) \le -k_1 \eta(t) H_2(R(t)). \tag{3.42}$$

Integrating (3.42) over (t_1, t), we have

$$\int_{t_1}^{t} \frac{-R'(s)}{H_2(R(s))}\, ds \geq k_1 \int_{t_1}^{t} \eta(s)\, ds,$$

then

$$k_1 \int_{t_1}^{t} \eta(s)\, ds \leq \int_{t_1}^{t} (H_1(R))'(s)\, ds = H_1(R(t)) - H_1(R(t_1)),$$

which, noting H_1 is strictly decreasing on $(0, r]$ and $\lim\limits_{t\to\infty} H_1(t) = +\infty$, gives us

$$R(t) \leq H_1^{-1}\biggl(k_1 \int_{t_1}^{t} \eta(s)\, ds + k_2\biggr), \tag{3.43}$$

where $H_1(t) = \int_t^1 1/H_2(s)\, ds$ and $k_2 > 0$ is a constant. Then (2.12) follows from (3.41) and (3.43) and continuity and boundedness of E and η.

In addition, if $\int_0^1 H_1(t)\, dt < +\infty$, then $\int_0^\infty H_1^{-1}(t)\, dt < +\infty$, and so, by (2.12), $\int_0^\infty E(t)\, dt < +\infty$. Then

$$\int_{t_1}^{t} \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\, dx\, ds \leq c \int_0^t E(s)\, ds < +\infty.$$

Hence we can repeat the same procedures with

$$I_{(t)} := \int_{t_1}^{t} \int_\Omega a(x) |\nabla u(t) - \nabla u(t-s)|^2\, dx\, ds,$$

and

$$\lambda(t) := -\int_{t_1}^{t} g'(s) \int_\Omega a(x) |\nabla u(t) - \nabla u(t-s)|^2\, dx\, ds,$$

to establish (2.14). □

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